ZERO SET OF SOBOLEV FUNCTIONS WITH NEGATIVE POWER OF INTEGRABILITY

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1. Introduction

Let $\Omega \subset \mathbb{R}^n$ ($n \geq 1$) be an open set, $p \geq 1$, we consider functions $u \in W^{1,p}(\Omega)$, such that, for some $\alpha > 0$

$$\int_{\Omega} |u|^{-\alpha} < \infty.$$  

(1.1)

When $p = 1$, it is also natural to consider $u \in BV(\Omega)$ which satisfies (1.1).

For any $u \in W^{1,p}(\Omega)$, $p \geq 1$ or $u \in BV(\Omega)$, we define its zero set by

$$\Sigma = \left\{ x \in \Omega : \lim_{r \to 0^+} \frac{1}{|B_r(x)|} \int_{B_r(x)} |u| \text{ exists, and is equal to 0.} \right\}.$$  

(1.2)

Our main theorem is the following:

**Theorem 1.** $\mathcal{H}^s(\Sigma) = 0$, where $s = \max\{0, n - p + \frac{p^2}{p + \alpha}\}$.

We note that, when $n - p + \frac{p^2}{p + \alpha} \leq 0$, one has in particular that $\Sigma$ is empty. This latter fact follows also from the Sobolev imbedding theorem: $u \in W^{1,p}(\Omega)$, $p > n$, implies $u \in C^{\beta}(\Omega)$, $\beta = 1 - \frac{n}{p}$. Indeed, if $u(x_0) = 0$ for some $x_0 \in \Omega$, then

$$|u(x)| \equiv |u(x) - u(x_0)| \leq c(u) |x - x_0|^{\beta}.$$  

Thus

$$\int_{\Omega} |u|^{-\alpha}(x) dx \geq c_2(u, \alpha) \int_{\Omega} \frac{1}{|x - x_0|^{\alpha \beta}} = +\infty$$  

whenever $\alpha \beta \geq n$. That is equivalent to $n - p + \frac{p^2}{p + \alpha} \leq 0$. In other words, under the additional assumption $\int_{\Omega} |u|^{-\alpha} < \infty$, one would have $u$ never vanish if $n - p + \frac{p^2}{p + \alpha} \leq 0$.

We also note that, for a $W^{1,p}(\Omega)$ function $u$, the set

$$\Sigma^* = \left\{ x \in \Omega : \lim_{r \to 0^+} \frac{1}{|B_r(x)|} \int_{B_r(x)} |u| \text{ doesn’t exist, or is equal to } \infty \right\}.$$  

(1.3)

is of Hausdorff dimension at most $n - p$, by a theorem of Federer-Ziemer [3]. And for functions of bounded variation, we have $H^{n-1}(\Sigma^*) = 0$, see section 5.9 of [1] for more information on fine properties of BV functions. Hence our result concerning $\Sigma$, the Lebesgue set of $u$ of the value zero makes sense because $s > n - p$.

In order to show Theorem 1, we will need a Poincaré type inequality which will be proven in the next section.

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We will concentrate our proof on the cases that \( n \geq 2 \) and \( p > 1 \). When \( n = 1 \), any function in \( W^{1,p}(\Omega) \) is Hölder continuous for \( p > 1 \) and absolutely continuous when \( p = 1 \). So it is easy to check that our theorem is valid in this case. In the case \( p = 1 \), since \( W^{1,1}(\Omega) \subset BV(\Omega) \), the theorem can be proved in the same manner as in the case \( p > 1 \), with the help of Theorem 4.

Our motivation for studying the zero set of general Sobolev function comes from considerations on the so-called rupture set of thin films, see [5]. Indeed, we consider a nonnegative solution \( u \) of

\[
\Delta u - \frac{1}{u^\alpha} + h(x) = 0,
\]

in \( \Omega \subset \mathbb{R}^n \), where \( \alpha > 1 \), and \( h \) is a smooth function in \( \Omega \). The value \( u \) represents the thickness of the thin films, and the zero set of \( u \) represents the ruptures. Naturally one is interested in how big can such rupture sets be.

We say \( u \geq 0 \) is a finite energy solution of (1.4), if \( u \) is nonnegative and continuous, satisfying (1.4) in \( \{ x \in \Omega : u(x) > 0 \} \), and such that

\[
\int_\Omega \left[ \frac{1}{\alpha} - \frac{1}{\alpha - 1} |u|^{-\alpha + 1} \right] dx
\]
is of finite value.

Applying Theorem 1, we have

**Corollary 1.** \( \mathcal{H}^{\mu_1}(\Sigma_1) = 0 \), where \( \Sigma_1 = \{ x \in \Omega : u(x) = 0 \} \), and \( \mu_1 = n - 2 + \frac{4}{\alpha + 2} \).

Alternatively, we say \( u \geq 0 \) is a weak solution of (1.4) in \( \Omega \), if \( u \in L^1(\Omega) \), \( u^{-\alpha} \in L^1(\Omega) \), and (1.4) holds in the sense of distribution Then we have the following:

**Theorem 2.** \( u \in H^{1,1}_{\text{loc}}(\Omega) \). Moreover, for any \( K \subset \subset \Omega \), there is a constant \( C > 0 \), such that for any \( x \in K \), \( 0 < r < \frac{1}{2} \text{dist}(K, \partial \Omega) \), we have

\[
\int_{B_r(x)} |\nabla u|^2 (y) dy \leq Cr^{n-2}.
\]

Furthermore, \( \mathcal{H}^{\mu}(\Sigma) = 0 \), for \( \mu = n - 2 + \frac{4}{\alpha + 2} \), where \( \mu \) is defined in (1.2).

The above estimates on the zero set of weak solutions of (1.4) is probably the first of its kind. However, we expect better estimates may be valid. The reason is that very little information of \( u \) being a weak solution of (1.4) is used in the proof of Theorem 2.

The paper is organized as follows. We will prove several Poincaré type inequalities in section 2. Then we prove Theorem 1 in section 3, and then discuss its application to (1.4) in the last section.

**2. Poincaré Type Inequality**

If \( p > n \) and \( u \in W^{1,p}(B_R) \), then \( u \) is Hölder continuous, and hence we have the following classical Poincaré lemma:

**Proposition 1.** Let \( p > n \), and \( B_R \) be any ball in \( \mathbb{R}^n \) with radius \( R \). Then for any \( u \in W^{1,p}(B_R) \) such that \( u(x) = 0 \) for some point \( x \in B_R \), we have

\[
\int_{B_R} u^p \leq c(n,p) R^p \int_{B_R} |\nabla u|^p.
\]
If $1 < p \leq n$, then $u(x) = 0$ for some point $x \in B_R$ is not well defined. However, we still have Poincaré inequality if the zero set is large enough.

**Theorem 3.** Let $n \geq 2$, $1 < p \leq n$, and $n - p < s \leq n$, let $B_R$ be any ball in $\mathbb{R}^n$ with radius $R$, and $T \subset B_R$ be a $\mathcal{H}^s$-measurable set, such that

$$\mathcal{H}^s(T) \geq \theta_1 R^s,$$

and that for any $x \in \mathbb{R}^n$, and $r > 0$,

$$\mathcal{H}^s(T \cap B_r(x)) \leq \theta_2 r^s$$

holds. Then for any $u \in W^{1,p}(B_R)$ such that $T \subset \Sigma$, where $\Sigma$ is defined in (1.2), we have

$$\int_{B_R} u^p \leq c(n,p,s) \frac{\theta_p}{\theta_1} R^p \int_{B_R} |\nabla u|^p.$$

**Proof.** After a scaling, we can always assume $R = 1$.

Step I: Let $\mu = \mathcal{H}^p|_T$, then $\mu$ is a Radon measure supported on $B_1$, such that

$$\mu(B_1) \geq \theta_1,$$

and for any $B_r(x) \subset \mathbb{R}^n$,

$$\mu(B_r(x)) \leq \theta_2 r^s.$$

By applying theorem 4.7.5 in [6], we have $\mu \in (W^{1,p}(\mathbb{R}^n))^*$, furthermore,

$$\|\mu\|_{(W^{1,p}(\mathbb{R}^n))^*} \leq c(n,p) \left( \int_{\mathbb{R}^n} \left( \frac{\mu(B_r(y))}{r^{n-p}} \right)^{\frac{p}{p-1}} dr d\mu(y) \right)^{\frac{p-1}{p}}$$

$$\leq c(n,p,s) \theta_2.$$

where in the last inequality, we used the fact that $\mu$ is supported on $B_1$ and $\mu(B_r(y)) \leq \theta_2 \min\{1,r^s\}$. Since every function $u \in W^{1,p}(B_1)$ can be extended to $\tilde{u}$ defined on whole $\mathbb{R}^n$, so that

$$\|\tilde{u}\|_{W^{1,p}(\mathbb{R}^n)} \leq c(n,p)\|u\|_{W^{1,p}(B_1)},$$

we can define for any $u \in W^{1,p}(B_1)$, $\mu(u) = \mu(\tilde{u})$, where $\tilde{u} \in W^{1,p}(\mathbb{R}^n)$ is an extension of $u$. Thus $\mu$ can be viewed as a member of $(W^{1,p}(B_1))^*$, with

$$\|\mu\|_{(W^{1,p}(B_1))^*} \leq c(n,p,s) \theta_2.$$

Step II: Applying lemma 4.1.4 in [6], we have

$$\int_{B_1} |u - \mu(u)|^p \leq c(n,p,s) \frac{\|\mu\|^p_{(W^{1,p}(B_1))^*}}{(\mu(1))^p} \int_{B_1} |\nabla u|^p$$

$$\leq c(n,p,s) \frac{\theta_p^2}{\theta_1^2} \int_{B_1} |\nabla u|^p.$$

Step III: Finally, we need to show $\mu(u) = 0$ under our assumption. To see this, let $\tilde{u}$ be the extension of $u$, and $\tilde{u}^M$ be the cutoff function of $\tilde{u}$ so that $|\tilde{u}^M| \leq M$, then $\tilde{u}^M \to \tilde{u}$ in $W^{1,p}(\mathbb{R}^n)$ as $M \to \infty$. Let $\tilde{u}_\varepsilon^M$ be the standard mollification of $\tilde{u}^M$, then $\tilde{u}_\varepsilon^M \to 0$, $\mu$-a.e. as $\varepsilon \to 0^+$. Hence the Lebesgue dominated convergence theorem implies

$$\int_{\mathbb{R}^n} \tilde{u}_\varepsilon^M d\mu \to 0.$$
as $\varepsilon \to 0$. On the other hand, since $\tilde{u}_\varepsilon^M \to \tilde{u}^M$ in $W^{1,p}(\mathbb{R}^n)$, and $\mu$ is a bounded operator on $W^{1,p}(\mathbb{R}^n)$, we have as $\varepsilon \to 0$,

$$\int_{\mathbb{R}^n} \tilde{u}_\varepsilon^M d\mu \to \mu(\tilde{u}^M).$$

Therefore, one has $\mu(\tilde{u}^M) = 0$. Letting $M \to \infty$, we deduce $\mu(u) = \mu(\tilde{u}) = 0$. $\square$

When $u$ is a function of bounded variation, we have a similar Poincaré type inequality:

**Theorem 4.** Let $n \geq 2$, $n-1 \leq s \leq n$, and $B_R$ be any ball in $\mathbb{R}^n$ with radius $R$, and $T \subset B_R$ be a $\mathcal{H}^s$-measurable set, such that

$$\mathcal{H}^s(T) \geq \theta_1 R^s,$$

and for any $x \in \mathbb{R}^n$, and $r > 0$,

$$\mathcal{H}^s(T \cap B_r(x)) \leq \theta_2 r^s$$

holds. Then for any $u \in BV(B_R)$ such that $T \subset \Sigma$, where $\Sigma$ is defined in (1.2), we have

$$\int_{B_R} |u| \leq c(n) R \left( \int_{B_R} \frac{u^p}{|\nabla u|^p} \right)^{\frac{n-1}{n}} \leq c(n) \frac{\theta_2}{\theta_1} R \int_{B_R} |\nabla u|.$$

**Proof.** We assume $R = 1$. Let $\mu = \mathcal{H}^s|_T$, then $\mu$ is a Radon measure supported on $B_1$, such that

$$\mu(B_1) \geq \theta_1,$$

and for any $B_r(x) \subset \mathbb{R}^n$,

$$\mu(B_r(x)) \leq \theta_2 r^s.$$

Since $s \geq n-1$, we have for any $B_r(x) \subset \mathbb{R}^n$,

$$\mu(B_r(x)) \leq \theta_2 r^{n-1}$$

when $r \leq 1$, and when $r > 1$,

$$\mu(B_r(x)) \leq \mu(B_1(0)) \leq \theta_2 \leq \theta_2 r^{n-1}.$$

So in either case, we always have

$$\mu(B_r(x)) \leq \theta_2 r^{n-1}.$$

The theorem then follows from theorem 5.12.7 in [6]. $\square$

**Corollary 2.** Let $p \geq 1$, $s = \max\{n - p + \frac{p^2}{p + \alpha}, 0\}$ for some $\alpha > 0$ and consider $u \in W^{1,p}(B_R)$ or $u \in BV(B_R)$ with $R \leq 1$. Suppose either $p > n$ and $u(x) = 0$ for some $x \in B_R$, or $1 \leq p \leq n$ and there exists $T$ such that $u$ satisfies conditions in theorem 3 or 4. Then under the assumption $\int_{B_R} |u|^{-\alpha} < \infty$, we have

$$\int_{B_R} |\nabla u|^p + \int_{B_R} |u|^{-\alpha} \geq c R^s.$$

where $c = c(n,p)$ if $p > n$ and $c = c(n,p,\alpha,\theta_1,\theta_2)$ if $1 \leq p \leq n$. 

Proof. Applying the Poincaré inequalities we just proved, we have

$$\int_{B_R} |\nabla u|^p + \int_{B_R} |u|^{-\alpha} \geq c R^{-p} \int_{B_R} |u|^p + \int_{B_R} |u|^{-\alpha} \geq c R^{n-p+\frac{2}{p+n}} \geq c R^s.$$  

Here we have used Young’s inequality.

3. Hausdorff dimension estimate for zero set

Proof of Theorem 1. We prove it by contradiction. Suppose that $H^s(\Sigma) > 0$ (possibly with infinite measure), then since $\Sigma$ is a Souslin set, theorem 5.6 and its proof in [2] says, there is a closed subset $T \subset \Sigma$, with $0 < H^s(T) < \infty$, and for some constant $\theta > 0$,

$$H^s(T \cap B_r(x)) \leq \theta r^s$$

holds for any $x \in \mathbb{R}^n$, $r > 0$.

For such $T$, the basic density lemma says that for $H^s$-a.e. $x \in T$,

$$\frac{1}{2^s} \leq \limsup_{r \to 0} \frac{H^s(B_r(x) \cap T)}{\alpha(s) r^s} \leq 1,$$

Let

$$T^* = \left\{ x \in T : \limsup_{r \to 0} \frac{H^s(B_r(x) \cap T)}{\alpha(s) r^s} \geq \frac{1}{2^s} \right\},$$

then for any $\delta > 0$ and for any $U$ open, such that $T^* \subset U$,

$$\left\{ B_r(x) : x \in T^*, 0 < r < \frac{1}{2}\delta, B_r(x) \subset U \text{ and } \frac{H^s(B_r(x) \cap T)}{\alpha(s) r^s} \geq \frac{1}{2^{s+1}} \right\}$$

is a fine covering of $T^*$. Hence, by Vitali covering lemma, there is a pairwise disjoint sub-collection $\{B_{r_k}(x_k)\}_{k=1}^{\infty}$, such that $T^* \subset \bigcup_{k=1}^{\infty} B_{5r_k}(x_k)$. Hence, applying corollary 2, we have

$$H_{5\delta}^s(T^*) \leq \sum_{k=1}^{\infty} \alpha(s)(5r_k)^s \leq c(n, p, s, \theta) \sum_{k=1}^{\infty} \int_{B_{5r_k}(x_k)} [||\nabla u||^p + |u|^{-\alpha}] \leq c(n, p, s, \theta) \int_{U} [||\nabla u||^p + |u|^{-\alpha}]$$

Since $H^s(T^*) < \infty$, we can choose $U$ with arbitrary small $H^n$-measure so that the right hand side of the inequality can be arbitrary small. Thus we would have $H_{5\delta}^s(T^*) = 0$. Let $\delta \to 0$, we conclude $H^s(T^*) = 0$, hence $H^s(T) = 0$, which gives the contradiction.

4. Rupture set of thin film model

Proof of Corollary 1. Since the energy is finite, we have $u \in W^{1,2}(\Omega)$ and $u^{-\alpha+1} \in L^1(\Omega)$, hence the result follows from Theorem 1.
Now we turn to the proof of Theorem 2. Actually, we would like to prove the theorem in a more general setting. Let $\Omega \subset \mathbb{R}^n$, $f \in L^1(\Omega)$, $f \geq 0$ in $\Omega$ and $g \in L^q(\Omega)$ for some $q \geq \frac{n}{2}$. We consider nonnegative solutions of
\begin{equation}
\Delta u = f + g \quad \text{in } \Omega.
\end{equation}
Since $f + g \in L^1(\Omega)$, classical elliptic theory implies $u \in W^{1,p}_{\text{loc}}(\Omega)$ for any $1 \leq p < \frac{n}{n-1}$. In our setting, we expect better results. First, we have

**Lemma 1.** Let $f, g, u$ as above with $q > \frac{n}{2}$. For any $B_{2R} \subset \Omega$, and for any $p > 1$ such that $\|u\|_{L^p(B_{2R})} < \infty$, we have
\[ \sup_{B_R} u \leq c(n, p, q) \left( R^{-\frac{2}{p}} \|u\|_{L^p(B_{2R})} + R^{2-\frac{n}{q}} \|g\|_{L^q(B_{2R})} \right). \]

**Proof.** This follows from the fact that $u$ is a subsolution of $\Delta u = g$. We could apply Theorem 8.17 in [4] directly if we have $u \in H^1_{\text{loc}}(\Omega)$. So naturally, we consider $u_\varepsilon$, the standard mollification of $u$, then we have
\[ \Delta u_\varepsilon = f_\varepsilon + g_\varepsilon. \]
Now $u_\varepsilon$ is smooth, so we can apply Theorem 8.17 in [4] to the mollified equation, and get
\[ \sup_{B_R} u_\varepsilon \leq c(n, p, q) \left( R^{-\frac{2}{p}} \|u_\varepsilon\|_{L^p(B_{2R})} + R^{2-\frac{n}{q}} \|g_\varepsilon\|_{L^q(B_{2R})} \right). \]
The lemma follows by letting $\varepsilon \to 0^+$. \hfill \square

Next, we need the following technical lemma:

**Lemma 2.** Let $f, g$ and $u$ be as above and $q = \frac{n}{2}$, then for any $x \in \Omega$, $0 < r \leq \min\{1, \text{dist}(x, \partial \Omega)\}$, we have
\[ r^{2-2n} \int_{B_{\frac{r}{2}}(x)} f(y) \, dy \leq c \left( \sup_{B_r} u + \|g\|_{L^2(B_r)} \right), \]
where $c = c(n)$.

**Proof.** For any $x \in \Omega$, $0 < r \leq \min\{1, \text{dist}(x, \partial \Omega)\}$, we have $B_r(x) \subset \Omega$. Now let $\varphi$ be the first eigenfunction of the Laplacian on the unit ball of $\mathbb{R}^n$:
\[ \begin{cases} -\Delta \varphi = \lambda_1 \varphi(x) & \text{in } B_1(0), \\ \varphi = 0 & \text{on } \partial B_1(0), \\ \varphi > 0 & \text{in } B_1(0), \end{cases} \]
and define $\varphi_r(x) = \varphi(\frac{x}{r})$. Now let $\eta_m \in C^\infty_0(B_r(x))$ be a smooth cutoff function such that $\eta_m = 1$ in $B_{\frac{2m-1}{m}}(x)$ and $\|
abla \eta_m\| \leq \frac{2m}{r}$. Using $\eta_m \varphi_r$ as a test function, we have
\[ -\int_{B_r(x)} \varphi_r \nabla u \cdot \nabla \eta_m + \int_{B_r(x)} u \nabla \varphi_r \cdot \nabla \eta_m = \int_{B_r(x)} u \Delta \varphi_r \eta_m + \int_{B_r(x)} \varphi_r \eta_m + \int_{B_r(x)} g \varphi_r \eta_m. \]
Now let $m \to \infty$, then $\eta_m \to \chi_{B_r(x)}$, and $\nabla \varphi_r \cdot \nabla \eta_m \to \frac{c_0}{r} H^{n-1}|\partial B_r(x)|$, where $c_0 = \frac{\partial \varphi_r}{\partial r} |_{r=1}$. Hence we deduce
\[ \lim_{m \to \infty} \int_{B_r(x)} u \nabla \varphi_r \cdot \nabla \eta_m = \frac{c_0}{r} \int_{\partial B_r(x)} u. \]
On the other hand, since $\varphi_r \nabla \eta_m$ is uniformly bounded in $B_r(x)$ and tends to 0 a.e., we have

$$\lim_{m \to \infty} \int_{B_r(x)} \varphi_r \nabla u \cdot \nabla \eta_m = 0.$$ 

Combining these limits, we have

$$r^{2-n} \int_{B_r(x)} f \varphi_r = c_0 r^{1-n} \int_{\partial B_r(x)} u - \lambda_1 r^{-n} \int_{B_r(x)} u \varphi_r + r^{2-n} \int_{B_r(x)} g \varphi_r.$$

\begin{align*}
&\leq c(n) \left( \|u\|_{L^\infty(B_r(x))} + r^{2-n} \|g\|_{L^1(B_r(x))} \right) \\
&\leq c(n) \left( \|u\|_{L^\infty(B_r(x))} + \|g\|_{L^\infty(B_r(x))} \right).
\end{align*}

Now we can present our regularity result:

**Theorem 5.** Let $f$, $g$, $u$ as above with $q > \frac{n}{2}$, then $u$ is locally bounded. Furthermore, $u \in H^1_{\text{loc}}(\Omega)$, and for any $B_{2R} \subset \Omega$ with $R \leq 1$, we have

$$R^{2-n} \int_{B_R} |\nabla u|^2 \, dx \leq c \sup_{B_R} u \left( \sup_{B_R} u + \|g\|_{L^\infty(B_R)} \right)$$

where $c = c(n)$.

**Proof.** Again, we consider the mollified equation

$$\Delta u_\varepsilon = f_\varepsilon + g_\varepsilon,$$

then we have

$$\Delta u_\varepsilon = 2 |\nabla u_\varepsilon|^2 + 2 u_\varepsilon (f_\varepsilon + g_\varepsilon).$$

Since $u_\varepsilon^2$ is locally bounded, applying lemma 2, we have

$$R^{2-n} \int_{B_R} \left( |\nabla u_\varepsilon|^2 + u_\varepsilon f_\varepsilon \right) \, dy \leq c \sup_{B_R} u_\varepsilon \left( \sup_{B_R} u_\varepsilon + \|g_\varepsilon\|_{L^\infty(B_R)} \right)$$

$$\leq c \sup_{B_R} u_\varepsilon \left( \sup_{B_R} u_\varepsilon + \|g_\varepsilon\|_{L^\infty(B_R)} \right).$$

Theorem is proved by letting $\varepsilon \to 0$. \hfill \qed

**Proof of Theorem 2.** Take $f = u^{-\alpha}$ and $g = -h$, then Theorem 5 says $u \in H^1_{\text{loc}}(\Omega)$, hence the result follows from Theorem 1 since we also have $u^{-\alpha} \in L^1(\Omega)$. \hfill \qed

**References**


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