1 Problem of the Points

1.1 Objectives

Read this section firstly with an eye toward

- Laplace’s definition of probability
- using trees to enumerate all cases
- how to count permutations and combinations
- the difference between distinguishable and indistinguishable objects

and secondly with an eye toward

- the concept of a probability model
- the overall procedure for investigating the imprecisely posed question

1.2 Problem Statement

We begin with a problem which was solved by Pascal and Fermat in 1654. This “problem of the points” concerns how to fairly divide the stakes of an unfinished game.

Consider a series of games between two people. The winner is the one who wins at least 4 of 7 rounds and will receive the entirety of some prize of money. In the event that the series can not be finished, how can the prize be fairly divided?

Certainly, in some cases, the answer is obvious. For example, if the score is tied, it is reasonable to split the stakes evenly.

What if the score is (3, 2)? It is unfair to the first player to split the stakes evenly, as he or she is ahead by luck or skill. On the other hand, division of the stakes at some arbitrary percentage isn’t a much better alternative.

It seems reasonable that the division of the stakes should reflect our confidence in who is going to win.

To answer the problem of the points, we need a model of the unplayed games which is sophisticated enough to be realistic yet simple enough to be analyzed.

It is unrealistic to believe we can determine who will win a series of games by precisely quantifying their skill, endurance, psychological strength, response to the weather, etc. For lack of something better, we choose to model the unplayed games as a sequence of fair coin tosses.
We will split the prize in accordance with the expected fraction of the time each player should win the series.

We are hence led to the question:

*If the game score is (3, 2) and each player is equally likely to win every following game, how likely is it that player 1 will win the best of seven?*

### 1.3 Simulation

An answer:

We can simulate the game with a computer or a coin and see how many times player 1 and player 2 win.

**Exercise 1.** Assume the score is (3, 2) and that each player is equally likely to win any unplayed games. Simulate the best of seven series 100 times and record how often players 1 and 2 win.

*What is a problem with this method of simulation?*

If we run such a simulation twice, we will likely get different results. There should be exactly one way to fairly split the stakes.

### 1.4 Outcome tree

Another answer:

Compare how many ways there are to win and how many ways there are to lose. We can do this by itemizing all outcomes.

\[
\begin{align*}
(3, 2) & \quad (4, 2) \\
(3, 3) & \quad (4, 3) \\
& \quad (3, 4)
\end{align*}
\]

We see there are two outcomes favorable to player 1 and only one outcome favorable to player 2. So this suggests that player 1 should get \( \frac{2}{3} \) of the prize.

*Do the simulation results agree that player 1 wins roughly 2/3 of the time?*

No. The simulations will likely be closer to having player 1 win \( \frac{3}{4} \) of the time. From the game tree, the first outcome, \((4, 2)\) is different from the other two as a total of only 6 rounds would have been played. In
a best of seven series, even though player 1 would win with a score of \((4, 2)\), the two could still play the seventh game without affecting the result of the series. In this case, the game tree looks like

\[
\begin{array}{c}
(5,3) \\
(4,2) \\
(4,3) \\
(3,2) \\
(4,3) \\
(3,3) \\
(3,4) \\
(4,3) \\
(5,3) \\
(4,3)
\end{array}
\]

We see that there are 3 of 4 outcomes favorable to player 1 and only 1 of 4 favorable to player 2. This suggests distributing the prize in a 75%/25% split, which is closer to our results from simulation.

We still do not have a complete answer to the problem of the points. So far, we have only proposed an answer in the case of a tie or the score \((3, 2)\).

What if only one game has been played, and the score is \((0, 1)\)?

The same method above would work. It says to model unplayed games as a sequence of 6 tosses of a fair coin.

- Make a tree that distinguishes all possible flips of 6 coins.
- Count the number of outcomes favorable to players 1 and 2.
- Split the prize in accordance to the ratio of favorable outcomes.

The tree of outcomes becomes an unwieldy tool with even six levels of branching, giving \(2^6 = 64\) different outcomes.

Hence we need a better method of bookkeeping for the problem of the points.

**Exercise 2 (Variant of Risk).** Suppose players 1 and 2 both roll a die. Player 2 wins if her roll is equal to or larger than Player 1’s roll. How many scenarios are possible? In how many does Player 1 win? In how many scenarios are the two dice equal? Can you figure out a way to use the answer to the third question to get the answer to the second? Can you think of a more compact way of enumerating all possible rolls than a tree?
1.5 Counting without full enumeration

In particular, we wish to count the number of outcomes where player 1 wins without enumerating every possibility.

When starting at a score of \((0, 1)\), we will flip a coin six times and wish to compute the number of possible outcomes with final scores \((4, 3)\), \((5, 2)\), and \((6, 1)\). These are the three scenarios under which player 1 could win. We thus pose

*With six flips of a coin, in how many ways can one get 4, 5, or 6 heads?*

More generally,

*In \(n\) flips of a coin, in how many ways can we get \(k\) heads?*

When posed with such a question, we would like to get a feeling for the answer by trying extreme cases.

Certainly, the only relevant values of \(k\) are \(0, 1, \ldots, n - 1,\) and \(n\).

For now, let's consider \(n = 6\).

If \(k = 0\), this means there are no heads. The only possible sequence of coin flips is thus

\[ TTTTTT \]

If \(k = 1\), there is exactly one head and \(1 = 5\) tails. This can happen in 6 ways:

\[
\begin{align*}
HTTTTT \\
THTTTT \\
TTHTTT \\
TTTHTT \\
TTTHTT \\
TTTTTH
\end{align*}
\]

If \(k = 2\), there are 15 possibilities

\[
\begin{align*}
TTTTTH & \quad TTHHTT & \quad HTTTTH \\
TTTHTH & \quad THTTTT & \quad HHTTHT \\
TTTHHT & \quad THTHTT & \quad HTHHTT \\
TTHTTH & \quad THTHTT & \quad HHTHTT \\
TTHTHT & \quad THTTTT & \quad HHTTTT
\end{align*}
\]

If \(k = 4, 5,\) or \(6\), the number is the same as if \(k = 0, 1,\) or \(2,\) respectively. This is because the number of ways to have, say 4 heads is the same as the number of ways to have 2 tails, which by symmetry must be the same as the number of ways to have 2 heads.

Let's try to explain the \(k = 2, n = 6\) case. As there must be two heads, the first head must occur in either the 1st, 2nd, 3rd, 4th, or 5th position. If the first head happens in the first position, there are 5 choices for the second head. If the first head occurs in the third position, there are only 3 choices for the second
heads. Thus there are a total of $5 + 4 + 3 + 2 + 1 = 15$ possibilities.

**In how many ways can we get $k = 2$ heads in $n$ flips of a coin?**

The first head can occur in any of the first $n - 1$ positions. If it occurs in position $i$, there are $n - i$ choices for the location of the second heads. Thus there are $\sum_{i=1}^{n-1} (n - i)$ total possibilities. This sum is

$$\sum_{i=1}^{n-1} n - \sum_{i=1}^{n-1} i = n(n - 1) - \frac{n(n - 1)}{2} = \frac{n(n - 1)}{2}$$

**In how many ways can we get $k = 3$ heads in $6$ flips of a coin?**

Along the same lines as $k = 2$, we see there are 4 possibilities for the location of the first heads. The number of ways that the first head is position $i$ is now the same as the number of ways to get 2 heads in $6 - i$ flips. Hence the number of possibilities is

$$\sum_{i=1}^{4} \frac{(6 - i)(6 - i - 1)}{2} = \frac{5 \cdot 4}{2} + \frac{4 \cdot 3}{2} + \frac{3 \cdot 2}{2} + \frac{2 \cdot 1}{2} = 10 + 6 + 3 + 1 = 20$$

While finding such a formula for $k = 3$ could proceed along the same lines, it is clear that this is still too cumbersome for general $n$ and $k$.

### 1.6 Counting via permutations

Another way to look at the problem is to count how many ways can one distinctly rearrange the letters, say $HHHTTT$, for the case $k = 3$, $n = 6$.

**In how many ways can one rearrange the letters $HHHTTT$?**

There is some complication due to the repeat letters. To start, lets answer the same question by with the letters $ABCD$.

**In how many ways can one rearrange the letters $ABCD$?**

Lets write them out in alphabetical order

$$\begin{align*}
ABCD & \quad BACD & \quad CABD & \quad DABC \\
ABDC & \quad BADC & \quad CADB & \quad DACB \\
ACBD & \quad BCAD & \quad CBAD & \quad DBAC \\
ACDB & \quad BCDA & \quad CBDA & \quad DBCA \\
ADBC & \quad BDAC & \quad CDAB & \quad DCAB \\
ADCB & \quad BDCA & \quad CDBA & \quad DCBA
\end{align*}$$

It is easy to count these. There are 4 choices for the first letter, then 3 choices for the next, then two for the third, forcing 1 choice for the last letter. There are $4 \cdot 3 \cdot 2 \cdot 1 = 24$ choices.
We can now answer

*In how many ways can one permute \(n\) distinguishable objects?*

We count the number of ways of filling each position. There are \(n\) choices for the first position, \(n - 1\) remaining choices for the 2nd, \(n - 2\) remaining for the 3rd, etc. until there are 2 choices for the \(n - 1\)st position and only 1 choice for the last position. Thus there are a total of

\[
\prod_{k=1}^{n} (n-k+1) = n! 
\]

permutations.

We now return to

*In how many ways can one rearrange the letters HHHT TT?*

We can not directly apply the factorial result as

- it would give us \(6! = 720\) choices when we know there are only 20. There are only \(2^6 = 64\) possible outcomes from 6 flips anyway
- the \(H\)'s and \(T\)'s are not distinguishable.

We wonder if we can make the \(H\)'s distinguishable, so long as we properly adjust the result. Let's give each of the \(H\)'s and \(T\)'s subscripts.

*In how many ways can one rearrange the symbols \(H_1H_2H_3T_1T_2T_3\)?*

In \(6! = 720\) ways.

*How do we undistinguish the \(H\)'s and \(T\)'s?*

Any permutation of \(HHHTTT\) can correspond to \(3! \cdot 3!\) permutations of \(H_1H_2H_3T_1T_2T_3\). The \(H\)'s can be labeled with 1, 2, or 3 in \(3!\) ways. For each such way, the \(T\)'s can be similarly labeled in \(3!\) ways. Hence \(3!^2\) of the permutations of \(H_1H_2H_3T_1T_2T_3\) correspond to each permutation of \(HHHTTT\). Thus there are \(\frac{6!}{3!3!} = \frac{720}{36} = 20\) permutations of \(HHHTTT\).

Finally, we can now answer

*In how many ways can one get \(k\) heads in a sequence of \(n\) flips of a coin?*

This is the same as the number of permutations of the letters \(\underbrace{HH\cdots H}_{k\text{ times}} \underbrace{TT\cdots T}_{n-k\text{ times}}\). Each \(H\) can be distinguished with a subscript 1, 2, \ldots, \(k\) in \(k!\) ways. Each \(T\) can similarly be distinguished in \(n - k\) ways. Every rearrangement of the \(H\)'s and \(T\)'s thus corresponds to \((k!)\)\((n-k)!\) of the permutations of \(H_1H_2\cdots H_kT_1T_2\cdots T_{n-k}\). As there are \(n!\) arrangements of \(H_1\cdots T_{n-k}\), there are thus \(\frac{n!}{k!(n-k)!}\) distinct ways to permute \(k\) \(H\)'s and \(n - k\) \(T\)'s.

An alternative phrasology of the above question is

*In how many ways can one choose \(k\) items among \(n\) distinguishable items?*
We write \[ \frac{n!}{k!(n-k)!} = \binom{n}{k} \]
which is read “n choose k.”

We can now give a general solution to the problem of the points.

We add up the number of ways each person can win and divide by the total number of possibilities to determine the fraction of the prize each person should get.

For example, if the score is \((0, 1)\) in a best of seven series, the prize should be divided into fractions

\[ \frac{\binom{6}{4} + \binom{6}{5} + \binom{6}{6}}{2^6} = \frac{11}{32} \quad \text{and} \quad \frac{\binom{6}{0} + \binom{6}{1} + \binom{6}{2} + \binom{6}{6}}{2^6} = \frac{21}{32} \]

for players 1 and 2, respectively.

1.7 Binomial Coefficients and \( n \) choose \( k \)

Having seen one instance where \( \binom{n}{k} \) pops up, it is worth noting some properties.

What is the relation of \( n \) choose \( k \) to the binomial coefficients?

The binomial theorem states that

\[(a + b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}\]

with the understanding that 0! = 1. In particular,

\[(1 + t)^n = \sum_{k=0}^{n} \binom{n}{k} t^k\]

This equality can be related to our work on permutations of sequences of letters. For concreteness, let’s work with \( n = 6 \).

\[(1 + t)^6 = (1 + t)(1 + t)(1 + t)(1 + t)(1 + t)(1 + t)\]

which can be expanded into \( 2^6 \) terms

\[
\begin{align*}
1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \\
+ 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot t \\
+ 1 \cdot 1 \cdot 1 \cdot 1 \cdot t \cdot 1 \\
+ \cdots \\
+ t \cdot t \cdot t \cdot t \cdot t \cdot 1 \\
+ t \cdot t \cdot t \cdot t \cdot t \cdot t
\end{align*}
\]
It is now apparent that the coefficient of $t^k$ is the number of ways to choose $k$ of the $n = 6$ things being multiplied together.

This link between combinatorics and analysis can be used to derive that

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

We calculate the $i$th derivative, evaluated at zero, of both sides of $(1 + t)^n = \sum_{k=0}^{n} \binom{n}{k} t^k$ The $i$th derivative of the left hand is

$$\frac{d^i}{dt^i}(1 + t)^n \bigg|_{t=0} = n(n-1)(n-2)\cdots(n-i+1)(1+t)^{n-i} \bigg|_{t=0} = \frac{n!}{(n-i)!}$$

and the $i$th derivative of the right hand side is

$$\frac{d^i}{dt^i} \sum_{k=0}^{n} \binom{n}{k} t^k \bigg|_{t=0} = \binom{n}{i} i!$$

Thus

$$\frac{n!}{(n-i)!} = \binom{n}{i} i!$$

$$\frac{n!}{i!(n-i)!} = \binom{n}{i}$$

What is the relation of $n$ choose $k$ to Pascal’s triangle?

$\binom{n}{k}$ is the $k$th item in the $n$th row of Pascal’s triangle.

```
1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
```

For example, $\binom{4}{2} = 6$, $\binom{5}{4} = 5$, and $\binom{6}{4} = 15$.

Each item is the sum of those to its top left and top right. That is, $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$
Other identities, which can be proved by combinatorial reasoning include

- \( \binom{n}{k} = \binom{n}{n-k} \)
- \( \sum_{k=0}^{n} \binom{n}{k} = 2^n \)
- \( \binom{2n}{n} = \sum_{j=0}^{n} \left( \frac{n}{j} \right)^2 \)

1.8 Laplace’s Definition of Probability

**Definition** The *probability* of an event is the ratio of the number of outcomes in the event to the total number of outcomes in the state space.

**Definition** An *event* is a subset of the state space.

**Definition** The *state space* is the collection of all outcomes.

As an example, consider flipping a coin 3 times. The state space is the set

\[ \Omega = \{ HHH, HHT, HTH, HTT, THT, TTH, TTT \} \]

The event that exactly one heads occurred is the set

\[ A = \{ HTT, THT, TTH \} \subset \Omega \]

The probability of getting exactly one head in 3 tosses is thus

\[ \frac{|A|}{|\Omega|} = \frac{3}{8} \]