Pricing European Options in Realistic Markets

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Abstract
We investigate the relation between the fair price for European-style vanilla options and the probability of short-term returns on the underlying asset in the absence of transaction costs. If the asset’s future price has finite expectation, the option’s fair value satisfies a parabolic partial differential equation of the Black-Scholes type in the absence of arbitrage opportunities. However, the evolution in general is in the variance of the asset’s returns rather than in trading time. The variance of the asset’s returns when the European option expires is the only uncertainty in this case. By immunizing the portfolio against large-scale price fluctuations of the asset, the valuation of options is extended to the realistic case of assets whose short-term returns have finite variance but very large, or even infinite, higher moments. A dynamic Delta-hedged portfolio that is statically insured against exceptionally large fluctuations of the asset’s returns includes at least two different options. The fair value of an option in this case is determined by a drift function \( \alpha(x, v) \) that is common to all options on the asset. This drift is interpreted as the premium for an investment exposed to risk due to exceptionally large changes in the asset’s returns. It affects the option valuation like a cost-of-carry for the underlying would. The derived pricing formula for options in realistic markets is arbitrage free by construction. A simple model with constant drift \( \alpha > 0 \) qualitatively reproduces the often observed volatility skew and term structure.

1 Introduction
An important result of modern finance is that the fair (no-arbitrage) price \( V \) for a European-style option is the expected present value (PV) of its future payoff,

\[
V = \langle PV(\text{payoff}) \rangle_Q .
\] (1)

The expectation in Eq. (1) is with respect to a risk-neutral (martingale) measure \( Q \) on the space of price-paths of the underlying asset. The fundamental theorem of asset pricing ensures the existence of the risk-neutral measure \( Q \) in the absence of arbitrage opportunities, but does not explicitly relate it to the

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process for the underlying asset. In some cases of interest Eq. (1) furthermore is found to be valid only for options with a bounded payoff.

For stock prices that follow a diffusion process, the measure \( Q \) can in principle be found. The risk-neutral pricing formula Eq. (1) in this case is equivalent to the existence of a dynamic hedging strategy that (at least in principle) is without appreciable risk. The analysis of the fair value of European call and put options by Black and Scholes\(^3\) was based on a stochastic model in which the returns of the stock follow a random walk. The fair value \( C_{BS} \) of an European-style call in this model depends only on the current stock price \( S_0 \), the volatility \( \sigma \) and risk-free rate \( r \) (both assumed constant), time to exercise \( T \) and the option’s strike \( K \). Dimensional analysis requires that

\[
C_{BS}(S_0, \sigma, r, T; K) = S_0 c_{BS}(rT, \sigma^2T; K/S_0).
\]  

Assuming that the (mean) risk-free rate is known, Eq. (2) can be inverted to give the (implied) volatility \( \sigma_{BS}^{\text{implied}} \) with which the Black-Scholes model would reproduce the observed spot-price \( C \) of a call with time to expiration \( T \) and strike \( K \),

\[
\sigma_{BS}^{\text{implied}} = \tilde{\sigma}(C/S_0, rT, K/S_0) / \sqrt{T}.
\]

Limitations of the Black-Scholes option pricing formula are expressed by the fact that the implied volatility of European-type options generally is found to depend on the strike \( K \) and time to expiration \( T \). The volatility implied by calls that are in-the-money very often is higher than that implied by out-of-the-money calls. A graph of the implied volatility against the call’s strike therefore tends to ”smile” (somewhat crookedly) rather than frown. The effect is referred to as the volatility -smile or -skew. The dependence of the implied volatility on \( T \) is known as the volatility’s term structure.

The observed volatility skew has been traced to a number of causes. All of them are related to a higher probability for exceptionally large fluctuations in the returns than the random walk model allows. Stochastic models that simulate this effect have been considered\(^4\), but a simple quantitative explanation of the empirically observed fluctuations has only recently been proposed\(^5\). Since the observed large-scale fluctuations in returns are not quantitatively reproduced by a simple stochastic model it is of some interest to (re)examine the problem of option pricing assuming as few properties of the short-term transition probability as possible. The fair price of an option in fact does not depend on many details of the process for the underlying. Only the short-term transition probability is relevant for a fully dynamic hedging strategy. We will see that the hedging strategy distinguishes between three classes of processes for the underlying:
I. The asset’s expected future price is well-defined by the short-term transition probability of its returns.

II. The variance of the short-term returns on the asset is finite, but the asset’s future expected price diverges.

III. The variance of the short-term returns on the asset diverges.

A log-normal distribution of the short-term returns is an example for a process in the first class, but so is any distribution whose moments are Borel-summable. No-arbitrage arguments uniquely price European-style options if the process of the underlying belongs to this class. It is possible to construct a dynamic portfolio with just one kind of option (in addition to the asset) that is without appreciable risk.

The other two classes sub-divide assets with sub-exponential return distributions. The processes for equities\(^1\), indices\(^6\) and commodities\(^7\) historically fall in the second class and this case will therefore concern us most. It turns out that one still can construct a dynamic portfolio that is without appreciable risk, but the portfolio in this case includes at least two different options on the underlying. A portfolio with just one option (and the asset) cannot be insured against large price fluctuations of the underlying and is therefore not without risk. Although the risk-neutral measure \(Q\) of Eq. (1) exists for options with bounded payoffs, it no longer is uniquely related to the stochastic process for the underlying.

Very little can be said about the third possibility, the Paretian case. The construction of a risk-free dynamic portfolio from options on the underlying is no longer possible. Indeed, the notion that the variance of the returns is a measure of risk has to be reexamined and Eq. (1) may not be very meaningful. Since the variance of returns on assets on which vanilla options can be drawn apparently is finite\(^1,6,7\), the Paretian case will not concern us in great detail.

We proceed as follows. Using the variance of the asset’s returns as the evolution parameter, the Black-Scholes analysis is extended to all processes of class I in the next section. In section 3 we then extend the analysis to include options on assets that follow processes in class II (the realistic case). Section 4 summarizes and discusses some aspects of the results.

2 A Variation on the Black-Scholes Analysis of Option Prices

It is useful to slightly generalize the Black-Scholes analysis to the case where the volatility of the underlying can be an arbitrary function of time. Note that for dimensional reasons the valuation (2) of a European-style call depends only
on the final variance $v_f = \sigma^2 T$, and the integrated discount factor $rT$, rather than separately on the volatility $\sigma$, risk-free rate $r$ and time to expiration $T$.

We will use the variance $v$ of the asset’s returns rather than a (continuous) trading time to parameterize the evolution of an option’s fair value. The variance is a monotonically increasing quantity; a trading- or calendar- time $t$, can be viewed as defining an instantaneous volatility $\sigma(t)$:

$$\sigma^2(t) := \frac{\partial v}{\partial t} \geq 0.$$ 

(4)

On any realized volatility path $\{\sigma(t); 0 \leq t \leq T\}$ there is a one-to-one correspondence between the variance $v$ and the "time" $t$.

$$v(t) = \int_0^t d\xi \sigma^2(\xi).$$ 

(5)

[The origin of the time-scale here is chosen to coincide with the moment of vanishing uncertainty in the asset’s price.] Eq. (5) enables us to formally consider any evolution in "time" as an evolution in the variance of the underlying’s returns, assuming that their volatility is finite.

To compensate for the time value of money, all prices will be stated as multiples of the price for an actively traded risk-free bond that matures when the European-style option expires. The spot price of the bond is $S_B(t)$ and its nominal value $N_B = S_B(\text{maturity})$. Since the transition probability is for the returns rather than for the price of the underlying, it is convenient to convert to the dimensionless variables,

$$x(t) := \ln[S(t)/S_B(t)], \quad k := \ln[K/N_B].$$ 

(6)

Changes in $x$ are the return on the underlying relative to the return on the bond and $k$ is the strike value of $x$. We assume that the fair call price $C$ at any moment depends only on the strike of the European-type call, the price of the underlying and the price of the bond. The fair call price in multiples of $S_B$ expressed in dimensionless quantities is,

$$c_k(x, v) := C(S(t), S_B(t), t; K)/S_B(t).$$ 

(7)

2.1 Generic Properties of the Short-Term Transition Probability

The dynamic hedging strategy of Black and Scholes that assigns a fair value to a European call depends on the existence of a very simple portfolio that is without appreciable risk for a sufficiently short period of time.
Let the current log-price of the stock be $x$ and the probability that the stock will have an excess return between $y - x$ and $y + dy - x$ a short time from now be described by the transition probability density,

$$p_h(y|x, \ldots) = p_h(y|x, v).$$

(8)

A small variance $h$ of the transition probability corresponds to a short time interval. The ellipses denote all additional quantities on which the transition probability may depend, such as market- and economic- indicators, the political environment, etc. In effect, the transition probability for the returns depends on the current time $t$, respectively on the variance $v$. One fortunately does not require detailed knowledge of $p_h(y|x, v)$ to value an option on the asset.

The transition probability Eq. (8) furthermore depends only on the excess return $y - x$ rather than on $x$ and $y$ individually. This financially plausible proposition can be cast in the form of a denominational argument: the probability for a certain change of the asset’s price should not depend on whether a single bond with value $S_B(t)$ or a package of two, three, or for that matter 6.378 bonds is used as a price reference. With the definition (6), this freedom in changing the reference denomination implies that the transition probability is invariant under (global) translations $z$ of all log-prices,

$$p_h(y - z|x - z, v) = p_h(y|x, v) = p_h(y - x|0, v), \quad \forall z,$$

(9)

where the latter expression is obtained by setting $z = x$. It is important that neither the variance $h$ of the short-term returns nor the variance $v$ of the overall returns are affected by this translation. Because the short-term transition probability density $p_h$ depends only on the difference $y - x$, the variance of the overall returns is additive: $v$ increases to $v + h$ after the short time interval we are considering.

Since $h \to 0_+$ as the time interval is shortened, the transition probability density has to approach Dirac’s distribution in this limit,

$$\lim_{h \to 0_+} p_h(y|x, v) = \delta(y - x).$$

(10)

Due to Eq. (10) the mean log-price of the stock,

$$\bar{y}(h; v) = x + \mu(h; v) := \int_{-\infty}^{\infty} dy y p_h(y|x, v),$$

(11)

approaches $x$ and $\mu(h; v)$ must become arbitrarily small for $h \to 0_+$

$$\lim_{h \to 0_+} \mu(h; v) = 0.$$

(12)
\( \mu(h; v) \) denotes the expected excess return on the asset at a given point in time when the variance of the asset’s returns increases by a small amount \( h \sim 0 \). It may appear financially reasonable to assume that \( \mu(h; v) \) for \( h \sim 0 \) has the expansion \( \mu(h; v) = a(v)h + b(v)h^2 + \ldots \). However, neglecting transaction costs, very short-term stock investments could have a higher expected return than long-term ones. To avoid the financially unstable situation that the expected return becomes absolutely certain for very short-term investments, it is sufficient to require that,

\[
|\mu(h \sim 0; v)| < a(v)h^\gamma, \quad \text{with } \gamma > \frac{1}{2} \text{ for } h \sim 0. \tag{13}
\]

The expected return in other words should not outstrip the width of the distribution of short-term returns.

The second moment of the transition probability, by definition, is given by its variance \( h \) and \( \bar{y}(h; v) \)

\[
\langle y^2 \rangle_{p_h} = h + \bar{y}(h; v)^2 := \int_{-\infty}^{\infty} dy y^2 p_h(y|x, v). \tag{14}
\]

Somewhat surprisingly perhaps, one does not require detailed knowledge of the higher moments of the distribution of short-term returns.

2.2 The Black-Scholes Valuation of a European Call

Emulating the analysis of Black and Scholes\(^3\), the fair value of a European-type call is found by constructing a portfolio that is without appreciable risk for sufficiently small \( h \). Consider a portfolio \( P \) of one European-type call with strike \( K \) and \( -\Delta \) of the underlying\(^a\). When the portfolio is set up at a log-price \( x \) for the asset, the value of this position is,

\[
V_P(x, v) = c_k(x, v) - \Delta(x, v)e^x \tag{15}
\]
bonds at \( S_B(v) \). If the hedge ratio \( \Delta(x, v) \) is not changed, the value of this portfolio when the variance of the asset’s returns has increased by \( h \) is,

\[
V_P(y, v + h) = c_k(y, v + h) - \Delta(x, v)e^y, \tag{16}
\]
if the stock’s excess return over this period is \( y - x \). To avoid arbitrage, the value of the position when the hedge is set up should be its expected future

\(^a\)We assume that the asset can be sold short and ignore transaction fees, dividends and costs-of-carry.
value discounted by a factor that accounts for the risk of investing in this portfolio. One thus quite generally comes to the conclusion that,

\[ V_P(x, v) = e^{-R_P(h, v)} \int_{-\infty}^{\infty} dy \, p_h(y|x, v)V_P(y, v + h) . \]  

(17)

The discount factor \( e^{-R_P(h, v)} \) is compensation for the excess risk associated with holding the portfolio \( P \) rather than the (risk-free) bonds [we have taken the time value of money into account by pricing relative to the bond]. \( R_P(h, v) \) depends not only on the perceived risk of the portfolio, but also on the valuation of this risk, which may explicitly depend on the circumstances, that is the time \( t \), respectively the variance \( v \). \( R_P(0, v) = 0 \) in order for Eq. (17) to be consistent. The absence of arbitrage opportunities requires that \( R_P(h, v) \geq 0 \) for all \( h \). If the portfolio is without appreciable risk over the interval \( h \), \( R_P(h, v) = 0 \) and Eq. (17) becomes the martingale hypothesis. Note that the general form of Eq. (17) is valid for finite \( h \) and could be used as a starting point to value hedge-slippage.

Since the transition probability \( p_h(y|x, v) \) is strongly peaked near \( y \sim x \) for \( h \to 0 \), one is led to expand the portfolio value (16) about \( y = x \) and \( h = 0 \). The first few terms of this expansion are,

\[ V_P(y, v + h) = V_P(x, v) + (y - x)[c_k'(x, v) - \Delta(x, v)e^x] + h\dot{c}_k(x, v) \]
\[ + \frac{1}{2}(y - x)^2[c_k''(x, v) - \Delta(x, v)e^x] + O(h(y - x), (y - x)^3, h^2) , \]

(18)

where the shorthand notation,

\[ \dot{\phi}(x, v) := \frac{\partial}{\partial v} \phi(x, v) \quad \text{and} \quad \phi'(x, v) := \frac{\partial}{\partial x} \phi(x, v) , \]

(19)

denotes partial derivatives of a function with respect to \( v \) and \( x \).

The term proportional to \( y - x \) in Eq. (18) vanishes for the particular hedge ratio

\[ \Delta(x, v) \to \Delta(x, v) = e^{-x}c_k'(x, v) , \]

(20)

and the corresponding portfolio will be denoted by \( \mathcal{P} \). Its value \( V_{\mathcal{P}} \) for \( y \sim x \) and \( h \sim 0 \) has the simplified expansion,

\[ V_{\mathcal{P}}(y, v + h) = V_{\mathcal{P}}(x, v) + h\dot{c}_k(x, v) + \frac{(y - x)^2}{2}[c_k''(x, v) - c'(x, v)] \]
\[ + O(h(y - x), (y - x)^3, h^2) . \]

(21)

We show below that the portfolio \( \mathcal{P} \) is without appreciable risk for sufficiently small \( h \) if the short-term return distribution of the asset belongs to class I. It is
important that this hedge depends only on the (observed) price \( x \) at the time it is entered into\(^b\).

Let us assume that the contribution to the integral of Eq. (17) from higher order terms in the expansion Eq. (18) becomes negligible for \( h \to 0_+ \) (see Appendix A). In this case the hedge (20) allows us to evaluate the RHS of Eq. (17) for sufficiently small \( h \) as,

\[
V_P(x, v) = e^{-R_P(h,v)} \int_{-\infty}^{\infty} dy p_h(y|x, v) V_P(y, v + h) \\
\sim V_P(x, v) - R_P(h, v) V_P(x, v) \\
+ h \dot{c}_k(x, v) + \frac{h + \mu^2(h;v)}{2} [c_k''(x, v) - c_k'(x, v)] ,
\]

(22)

where we have used that by Eqs. (11) and (14),

\[
h + \mu^2(h;v) = \int_{-\infty}^{\infty} dy (y - x)^2 p_h(y|x, v).
\]

(23)

If the expected return is bounded as in Eq. (13), \( \mu^2(h;v) \) becomes negligible compared to \( h \) for \( h \sim 0 \). Taking the limit \( h \to 0_+ \) of Eq. (22), the fair option price \( c_k(x, v) \) is seen to satisfy the partial differential equation,

\[
\dot{c}_k(x, v) + \frac{1}{2} [c_k''(x, v) - c_k'(x, v)] = R_P(v)[c_k(x, v) - c_k'(x, v)],
\]

(24)

with a mean excess portfolio return per unit of variance of,

\[
r_P(v) = \lim_{h \to 0_+} h^{-1} R_P(h;v).
\]

(25)

Note that \( c_{-\infty}(x, v) = e^x \), is a particular solution to Eq. (24), because a call with strike \( K = 0 \) has the same intrinsic value as the underlying. The mean return \( \mu(h;v) \) of the underlying does not enter Eq. (24) as long as it satisfies the bound (13).

Using the definitions (6) and (7), and parameterizing the evolution by a trading time instead of by the variance, Eq. (24) assumes the more familiar form,

\[
\left[ \frac{\partial}{\partial t} + \dot{r}_P(t) S \frac{\partial}{\partial S} + \frac{\sigma^2(t)}{2} S^2 \frac{\partial^2}{\partial S^2} \right] C(S, t; K) = \dot{r}_P(t) C(S, t; K).
\]

(26)

\(^b\)To verify that Eq. (20) is precisely the hedge of Black and Scholes, note that with definition (6), \( e^{-x} \frac{\partial}{\partial x} = S_B \frac{\partial}{\partial S} \)
The portfolio’s instantaneous overall return rate \( \tilde{r}_P(t) \) in Eq. (26) consists of two parts: the (risk-free) return rate of the bond \( r(t) = S^{-1}_B dS_B(t)/dt \) and the risk-premium of the portfolio,

\[
\tilde{r}_P(t) = r(t) + \sigma^2(t)r_P(v(t)).
\]  

Eq. (26) is the partial differential equation of Black and Scholes\(^3\) for the valuation of options with an in general time-dependent volatility and an option-dependent discount rate \( \tilde{r}_P(t) \).

We have yet to show that the portfolio \( P \) described by the hedge ratio (20) is (at least formally) without appreciable risk and that \( r_P(v) \) therefore vanishes in the absence of arbitrage opportunities. \( \tilde{r}_P(t) = r(t) \) in Eq. (26) then does not depend on the option and becomes the risk-free forward rate represented by the bond.

2.3 The Risk of Holding the Dynamically Hedged Portfolio \( P \)

The portfolio is without appreciable short-term risk compared to an investment in the asset alone, if the variance of the portfolio’s return decreases faster than the variance of the asset’s return, which is \( h \). One thus has to show that

\[
\lim_{h \to 0^+} h^{-1} \text{Var}[V_P(y, v + h)] = \frac{d}{dv} \text{Var}[V_P(y, v)] = 0
\]  

We continue to assume that the transition probability is sufficiently sharply peaked about \( y \sim x \) and again expand,

\[
V_P^2(y, v + h) = V_P^2(x, v) + 2V_P(x, v) \left\{ h \dot{c}_k(x, v) + \frac{(y - x)^2}{2} [c''_k(x, v) - c'_k(x, v)] \right\} + O(h(y - x), (y - x)^3, h^2).
\]

The expectation of \( V_P^2(y, v + h) \) to order \( h \) then is,

\[
\langle V_P^2 \rangle_{p_h} := \int_{-\infty}^{\infty} dy p_h(y|x, v) V_P^2(y, v + h)
\]

\[
= V_P^2(x, v) + 2V_P(x, v) \left\{ h \dot{c}_k(x, v) + \frac{h + \mu^2(h)}{2} [c''_k(x, v) - c'_k(x, v)] \right\} + O(h^{3/2})
\]

\[
= \langle V_P^2 \rangle_{p_h}^2 + O(h^{3/2}).
\]

The variance of the returns on portfolio \( P \) therefore is of order \( h^{3/2} \) and the risk of entering into this investment compared to an investment in the underlying
can theoretically be made as small as one wishes by rebalancing the portfolio sufficiently often. To avoid arbitrage, the discount rate therefore must be the risk-free one. The portfolio $P$ in this sense is perfectly hedged over the short-term and we should set $r_P(v) = 0$ in Eqs. (24) and (27).

### 2.4 A Comment on Stochastic Volatility

The parabolic partial differential equation (24) was derived without specifying a stochastic process for the asset’s price. Specific properties of the short-term returns enter the solution to Eq. (24) only through the boundary conditions. Integration of Eq. (24) for a European-style option requires knowledge of the payoff of the option and of the variance of the asset’s returns at exercise. The option payoff is readily expressed in terms of the value of a risk-free bond that matures when the option expires. For European-style options the only uncertainty thus is in the final variance $v_f$ of the asset’s returns when the option expires.

By construction, the price of a European call option does not depend on the volatility path. Stochastic processes with the same marginal probability distribution $q(v_f|T,\ldots)$ for the variance $v_f$ of the asset’s returns on the expiration date of the option lead to the same fair option price.

Since the fair value of an European option for a given final variance can be written as a risk-neutral expectation of the option payoff with the pdf $^c$,

$$p_{BS}(y|x, v_f) = (2\pi\sqrt{v_f})^{-1} \exp\left[-(y - x + v_f/2)^2/(2v_f)\right], \quad (31)$$

the risk-neutral valuation of European-like options by Eq. (1) is explicitly possible: the probability measure $Q$ for European-style options is described by the pdf

$$p_Q(y|x, T, \ldots) = \int_0^\infty dv_f q(v_f|T, \ldots)p_{BS}(y|x, v_f). \quad (32)$$

The pdf $^d q(v_f|T,\ldots)$ is the only ingredient that specifically depends on the process for the underlying. Since we do not have options on a particular stock for every strike $K$, the market is not complete. The distribution $q(v_f|T,\ldots)$ thus unfortunately cannot be uniquely inferred from observed option prices.

### 3 The Valuation of European Calls in Realistic Markets

In deriving Eq. (24) we tacitly assumed that the expectation in Eq. (17) is meaningful. Since the fair value of a call that is deep in-the-money approaches

$^c$With respect to the variance at expiration $v_f$, the distribution (31) solves the “backward” evolution equation that corresponds to (24).

$^d$The ellipses again denote any other pre-visible quantities.
$S - K$, we see that for finite $\Delta$, the fair value of the portfolios we have been considering essentially becomes proportional to the price of the underlying $S = S_B e^y$ for large values of $y$. The expectation in Eq. (17) for such portfolios is finite only if the price of the underlying has finite expectation,

$$\langle \frac{S}{S_B} \rangle_{pb} = \int_{-\infty}^{\infty} dy e^y p_h(y|x, v) < \infty.$$  

(33)

Together with the result of Appendix A that the contribution from higher moments becomes negligible in the limit $h \to 0_+$, we thus find that Eq. (24) holds for processes in class I only.

The historically observed processes for equities\(^1\), indices\(^6\) and commodities\(^7\) do not belong to this class. Empirically the probability densities for short-term returns have tails that fall off as a power in $x$ only. For time intervals between 5 minutes and three weeks the observed transition pdf's of the returns on equities are all shape-similar and well reproduced by\(^5\) a t-distribution with $f \sim 3$ degrees of freedom, mean $\bar{y}(h; v) = x + \mu(h; v)$ and variance $h$,

$$p_h^{\text{emp}}(y|x, \mu) \sim \frac{2h^{3/2}}{\pi((y - \bar{y})^2 + h)^{3/2}}.$$  

(34)

The integral in Eq. (33) diverges in this case and the valuation of the previous portfolios is all but meaningless: being long a call apparently becomes a very attractive position – unfortunately, the risk associated with this position also is not calculable. If the probability for relatively large fluctuations is sufficiently great, the expected future value of some portfolios is not determined by small fluctuations about $y \sim x$, even as $h \to 0_+$. The short-term expected value of the portfolios we have been considering mainly comes from large fluctuations, even though these are not the most frequent. Truncating the expansion (18) about $y = x$ in this case is inadequate and gives an inaccurate representation of the portfolio’s variation in price and Eq. (24) evidently is no longer correct.

The damage can be contained by considering only portfolios that are immune to large variations in the price of the underlying. It is sufficient to restrict to portfolios whose value is uniformly bounded by a finite constant $V_{\text{max}}(P)$,

$$|V_P(x, v)| < V_{\text{max}}(P), \quad \forall x, v < v_f.$$  

(35)

Examples of simple portfolios that are bounded in this sense are a vanilla put or a covered vanilla call. For class II processes one can select those portfolios from the above set that are bounded and without appreciable risk for sufficiently
short periods of time. Since this set is much smaller than in the case of class I processes, it is not surprising that the valuation of options on assets that follow class II processes becomes less constrained.

The simplest bounded dynamic portfolio that is without appreciable risk contains two different options on the underlying. We here discuss the case of a portfolio of two covered calls, \( \tilde{c}_1 \) and \( \tilde{c}_2 \) with the same expiration but strikes \( k_1 \) and \( k_2 \) respectively.

The portfolio’s fair value in bonds when the variance of the return distribution is \( v \) and the asset’s log-price is \( y \) can be written,

\[
V_P(y, v) = \Delta_1 \tilde{c}_1(y, v) + \Delta_2 \tilde{c}_2(y, v),
\]

where the fair price of a covered call is,

\[
\tilde{c}_i(y, h) = c_i(y, h) - e^y, \quad i = 1, 2
\]

The weights \( \Delta_1 \) and \( \Delta_2 \) of the two covered calls are chosen so that the portfolio’s price does not change appreciably for small variations of the asset’s price about its current log-price \( x \),

\[
\left. \frac{\partial}{\partial y} V_P(y, v) \right|_{y=x} = 0.
\]

The weights,

\[
\Delta_1 = \tilde{c}_2'(x, v) ; \quad \Delta_2 = -\tilde{c}_1'(x, v)
\]

are one possible solution to Eq. (38). When the variance increases by \( h \), the portfolio \( P \) with weights (39) assumes the value,

\[
V_P(y, v + h) = \tilde{c}_1(y, v + h)\tilde{c}_2'(x, v) - \tilde{c}_2(y, v + h)\tilde{c}_1'(x, v)
\]

\[
= \begin{bmatrix} \tilde{c}_1(y, v + h) & \tilde{c}_1'(x, v) \\ \tilde{c}_2(y, v + h) & \tilde{c}_2'(x, v) \end{bmatrix} .
\]

Near \( y = x \) and \( h = 0 \), \( V_P(y, v + h) \) has the expansion,

\[
V_P(y, v + h) = V_P(x, v) + h \left[ \begin{bmatrix} \tilde{c}_1(x, v) & \tilde{c}_1'(x, v) \\ \tilde{c}_2(x, v) & \tilde{c}_2'(x, v) \end{bmatrix} + \frac{(x-y)^2}{2} \begin{bmatrix} \tilde{c}_1''(x, v) & \tilde{c}_1'(x, v) \\ \tilde{c}_2''(x, v) & \tilde{c}_2'(x, v) \end{bmatrix} + O(h^2, (x-y)h, (x-y)^3) \right].
\]

\( ^e \)This is no longer possible for Pareto return distributions, with a divergent variance\(^8\).
Using that the value of the portfolio $P$ is bounded, its expected price with the pdf (34) for sufficiently short time intervals is,

$$
\int_{-\infty}^{\infty} dy \ p_{h}^{\text{emp}}(y|x, \mu)V_{P}(y, v + h) = \int_{y-1}^{y+1} dy \ p_{h}^{\text{emp}}(y|x, \mu)V_{P}(y, v + h) + O(h^{3/2})
$$

if the expected short-term return on the asset is bounded by Eq. (13). The determinant of the $3 \times 3$ matrix results from combining the expectations of the two determinants in Eq. (41). Because the portfolio value is immunized against large price fluctuations, the truncation of the transition probability in Eq. (42) induces an error of order $h^{3/2}$ only (see Appendix A for details). For class II short-term returns, the valuation of a bounded Delta-hedged portfolio thus effectively is reduced to the previous case.

In the limit $h \to 0_{+}$, the fair values of any two covered European-style calls on a class II asset thus satisfy,

$$
\begin{vmatrix}
-1 & 0 & 1 \\
\tilde{c}_1(x, v) & \tilde{c}_1'(x, v) & 1/2\tilde{c}_1''(x, v) \\
\tilde{c}_2(x, v) & \tilde{c}_2'(x, v) & 1/2\tilde{c}_2''(x, v)
\end{vmatrix}
= 0 .
$$

(43)

This is one partial differential equation for two unknown functions. However, since the determinant vanishes only when the corresponding system of linear equations is dependent, we can disentangle Eq. (43) into two linear partial differential equations for each covered call separately – at the cost of introducing a function $\alpha(x, v)$. Excluding the possibility that the value of a covered call does not depend on the stock price, Eq. (43) is equivalent to the set of linear equations,

$$
\begin{pmatrix}
-1 & 0 & 1 \\
\tilde{c}_1(x, v) & \tilde{c}_1'(x, v) & 1/2\tilde{c}_1''(x, v) \\
\tilde{c}_2(x, v) & \tilde{c}_2'(x, v) & 1/2\tilde{c}_2''(x, v)
\end{pmatrix}
\begin{pmatrix}
1 \\
\alpha(x, v) - 1/2 \\
0
\end{pmatrix}
= \begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix} .
$$

(44)

The first of these equations is true for any $\alpha(x, v)$. The latter two imply that call options on the asset satisfy the partial differential equation,

$$
\dot{c}(x, v) + (\alpha(x, v) - 1/2)c'(x, v) + 1/2c''(x, v) = \alpha(x, v)e^x .
$$

(45)
\[ \alpha(x, v) \] can be expressed in terms of the "Greeks" for any call on the underlying,

\[ \alpha(x, v) = -\frac{\dot{c}(x, v) - \frac{1}{2}\ddot{c}(x, v) + \frac{1}{2}\dddot{c}(x, v)}{\dot{c}(x, v)}. \] (46)

\( \alpha(x, v) \) thus is a universal function that in particular does not depend on the strike of the option.

These considerations of course also apply to transition probabilities that fall off more rapidly than (34). One recovers the partial differential equation (24) of Black and Scholes as the special case

\[ \alpha(x, v) = 0. \] (47)

As noted before, since a call with strike \( K = 0 \) is worth the stock at exercise, \( c_{-\infty}(x, v) = e^x \) must be a special solution to Eq. (45) that does not depend on \( \alpha(x, v) \). The inhomogeneous term in Eq. (45) for the valuation of call options therefore is a matter of consistency. Put-call parity implies that a European-style put with the same strike and expiration date as a call satisfies the homogeneous partial differential equation with the same \( \alpha(x, v) \).

The function \( \alpha(x, v) \neq 0 \), can be viewed as a risk-premium on a covered call (respectively a put). The reason for such a premium is evident from the derivation: it compensates for the cost of insuring a simple Delta-hedged portfolio with just one option against large fluctuations in the price of the underlying. Note that \( \alpha(x, v) \) enters the evolution equation for options with bounded payoffs as an effective cost-of-carry for the underlying asset would.

This interpretation of \( \alpha \) becomes evident if we consider the stochastic process whose generator \( \hat{A} \) is the evolution operator in Eq. (45),

\[ \hat{A}(v)\phi(w, v) = \left\{ \frac{1}{2} \frac{\partial^2}{\partial w^2} + (\alpha(w, v) - \frac{1}{2}) \frac{\partial}{\partial w} \right\} \phi(w, v). \] (48)

The corresponding stochastic process is

\[ dw = (\alpha(w, v) - \frac{1}{2})dv + dB_v \] (49)

where \( B_v \) denotes Brownian motion with zero mean and unit variance. The measure \( Q \) of Eq. (1) that corresponds to Eq. (49) is unique\(^6\) as long as the drift \( \alpha(w, v) \) is finite for all \( w, v \) and does not increase faster\(^7\) than \( |w| \) for \( |w| \sim \infty \). Assuming this to be the case, the fair price of a European-style

\(^{7}\)The interpretation of \( \alpha \) as an effective cost of carry makes this mathematical statement rather obvious: nobody will hold an asset whose cost of carry grows faster than its return.
option on an asset following a class II process is uniquely specified by $\alpha(w, v)$ and the marginal stopping distribution $q(v_f|T, \ldots)$.

[The $\frac{1}{2}$ in the drift-term of Eq. (49) does not appear in the corresponding stochastic process for $n(v) := e^{w(v)}$, which follows geometric Brownian motion

$$\frac{dn}{n} = \alpha(n, v)dv + dB_v,$$

with mean instantaneous drift $\alpha(w = \ln(n), v)$.]}

A constant effective risk premium on options was recently interpreted by Derman\textsuperscript{11} as due to a stock’s intrinsic time-scale generated by short-term speculators. Although our argument apparently is somewhat different, the very similar effect described here may have a common cause: large exceptional fluctuations in the short-term returns of the underlying perhaps can be traced to speculation.

It is difficult to compare a risk due to exceptionally large fluctuations to any risk arising from "normal" fluctuations that is described by the variance. How this risk is valued furthermore depends on the perception of investors. The function $\alpha(x, v)$ thus perhaps is specified only by the observed option prices themselves. In the absence of options to every strike and exercise date, the problem of calibrating $\alpha(x, v)$ to the market prices is not complete. Additional assumptions are required to uniquely specify $\alpha(x, v)$ – for instance that the relative entropy to the Black-Scholes model is maxima\textsuperscript{12}.

To better visualize the effect a non-vanishing drift has on option prices, let us consider constant $\alpha > 0$. One can explicitly solve Eq. (8) in this case and obtains that the Black-Scholes variance $v_{BS}(\tilde{k}, \alpha; v_f)$ implied by a European call is implicitly given by the relation,

$$\ln(v_{BS}/v_f) + \frac{(\tilde{k} + v_{BS}/2)^2}{v_{BS}} = \frac{(\tilde{k} + v_f(1/2 - \alpha))^2}{v_f}.$$

(51)

Here $v_f$ is the final variance at the time of exercise of the option and $\tilde{k} = k - x = \ln(KS_B/N_BS) = \ln \tilde{K}$ gives the discounted strike in terms of the spot price.

\textsuperscript{9}The mean drift $\alpha(n, v)$ in Eq. (50) should not be confused with the mean return of the asset. The two are not even related: the drift $\alpha(n, v)$ is due to large fluctuations in the price of the underlying, not due to its mean return. The stochastic process Eq. (49) is not the one followed by the log-price $x(v)$ of the asset.
4 Summary and a Discussion of the Results

For the purpose of option valuation, the class of assets with sub-exponential short-term return distributions can be divided into those with finite and infinite variance (classes II and III respectively). Class II assets (most financially
interesting instruments historically appear to belong to this class\textsuperscript{3,6,7}) still allow
the construction of a dynamic portfolio with negligible risk. Using the variance $v$ of the returns on the underlying instead of a "trading time", the evolution of the fair price of an option was found to be characterized by a drift $\alpha(x, v)$. Constant $\alpha > 0$ qualitatively reproduces the volatility smile and term-structure often observed in equity markets. For short-term return distributions that fall off exponentially or faster, $\alpha = 0$ and the diffusion reduces to the one of Black-Scholes, albeit evolving in $v$ instead of in a trading time.

One may object to considering sub-exponential return distributions for the underlying, since the price variations of an asset, although perhaps large, can be thought of as restricted to a finite range in the finite life-time of the option. There are at least two objections to this argument. Firstly, the value of equities does sometimes change considerably and fluctuations may exceed several standard deviations when a company is forced into bankruptcy or announces new patents or acquisitions. These scenarios are not so rare that they can be disregarded in the valuation of options (except by insiders). One realistically therefore might want to immunize a portfolio against large price-changes of the underlying. It furthermore is operationally and financially quite impossible to rebalance a portfolio arbitrarily often. If the tails of the probability distribution are sufficiently fat, higher moments can become relevant in the evaluation of Eq. (17) when the change in variance $\Delta v$ between updates is finite. If higher moments of the transition probability are sufficiently large (not necessarily infinite) it again is advisable to immunize the portfolio against large price fluctuations of the underlying asset, that is restrict the portfolio’s variation in value for large variations of the asset’s price. The same strategy was found to be useful in pricing options on class II assets: the portfolio is statically immunized against exceptionally large fluctuations of the underlying and dynamically hedged to make it insensitive to normal ones as well.

Thus, although it may in reality not be possible to distinguish sharply between assets of class I and II, the strategy employed here for assets of class II is the more realistic one. The two kinds of assets evidently can be continuously deformed into each other and it is gratifying that the evolution equations satisfied by the corresponding option prices can also be continuously deformed from $\alpha(x, v) \neq 0$ (class II) to $\alpha(x, v) = 0$ (class I).

The returns on financial assets in realistic markets fortunately have finite variance and the Pareto (class III) scenario therefore is quite academic. For options with bounded payoffs, the measure $Q$ in Eq. (1) exists for financial assets whose returns have finite variance. But $Q$ is uniquely determined by the process for the underlying only if the asset’s expected future price is finite (class I). In the more realistic case of assets in class II, the drift $\alpha(x, v)$ does not
vanish. \( \alpha(x,v) \) can be viewed as the investor’s compensation for the residual risk of a Delta-hedged portfolio due to exceptionally large fluctuations in the asset’s returns. As such, this drift is not explicitly related to the process for the underlying.

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A Estimates of the Remainders

To justify the estimates in the text we here show that the remainders in the expansions (21) and (41) are indeed negligible as \( h \to 0 \).

We first consider a pdf \( p_h(x) \) of class I with variance \( h \) and vanishing mean. The argument of \( p_h \) can always be shifted to obtain a distribution with non-vanishing mean. Being in class I implies that,

\[
\int_{-\infty}^{\infty} dy p_h(y) e^{\lambda y} < \int_0^{\infty} dy p_h(y) e^{y} + \text{Max}[p_h] \int_{-\infty}^{0} dy e^{\lambda y} < \infty, \quad \forall \lambda < 1, \tag{52}
\]

that is, the moment generating function is analytic about \( \lambda = 0 \) and all moments of \( p_h(y) \), in particular, are finite.

We are interested in integrals of the form

\[
\int_{-\infty}^{\infty} dy p_h(y) f(y), \tag{53}
\]

for functions \( f(y) \) that are analytic (almost) everywhere. The remainder \( R_N(y) \) in the McLaurin series

\[
f(y) = \sum_{n=0}^{N} \frac{y^n}{n!} f^{(n)}(0) + R_N(y), \tag{54}
\]

thus vanishes as \( N \to \infty \) for (almost) all \( y \). An expression for the remainder is,

\[
R_N(y) = \frac{y^{N+1}}{(N+1)!} f^{(N+1)}(y\xi) \tag{55}
\]

with \( 0 < \xi < 1 \). Changing the scale of the integration variable \( y \to y\sqrt{h} \) the expectation of \( R_N \) for small \( h \) is,

\[
\langle R_N \rangle_{p_h} := \int_{-\infty}^{\infty} dy p_h(y) R_N(y)
\]
\[
\frac{h^{(N+1)/2}}{(N+1)!} \int_{-\infty}^{\infty} dy \ y^{(N+1)} \{ \sqrt{h} p_h(y \sqrt{h}) \} f^{(N+1)}(y \xi \sqrt{h}) . \tag{56}
\]

The pdf \( \sqrt{h} p_h(y \sqrt{h}) \) has unit variance and the higher moments of the limiting pdf,

\[
p_1(y) := \lim_{h \to 0_+} \sqrt{h} p_h(y \sqrt{h}) , \tag{57}
\]

are finite. Using that \( f^{(N+1)}(x) \) is analytic about \( x = 0 \), the integral in Eq. (56) has a finite limit for \( h \to 0_+ \) and one concludes that,

\[
\langle R_N \rangle_{p_h} = O(h^{(N+1)/2}) . \tag{58}
\]

This estimate continues to hold when the function \( f(x) \) is expanded about a point \( x = \mu(h) \) that is bounded by Eq. (13). For analytic portfolio values, the estimates in Eqs. (22) and (30) thus are justified and neglected terms are of higher order in \( h \).

The valuation of bounded portfolios with pdf’s of class II can be reduced to the previous case if the error from truncating the pdf becomes negligible for \( h \to 0_+ \). To see this, consider a pdf with zero mean and variance \( h \) that for sufficiently small \( h \) is bounded by,

\[
p_h(y) \leq \frac{D}{\sqrt{h}} \left( \frac{h}{|y|^2} \right)^\nu \text{ for } |y| > 1, h < h_0 , \tag{59}
\]

where \( D > 0 \) and \( \nu \) are constants that do not depend on \( h \). Note that if the variance of \( p_h(y) \) is finite, Eq. (59) holds for some \( \nu > 3/2 \). The contribution of the tails of the distribution to the expectation of a bounded function \( |f(y)| \leq f_{\text{max}} \) in this case is,

\[
\int_{|y| > 1} dy \ p_h(y) f(y) \leq f_{\text{max}} \int_{|y| > 1} dy p_h(y) \leq 2D f_{\text{max}} h^{\nu-1/2} . \tag{60}
\]

The tails of any distribution in class II (with \( \nu > 3/2 \)) therefore give a sub-leading contribution to the expectation of a bounded function and can be cut off. For bounded functions, a pdf of class II effectively can be replaced by one that vanishes for \( |y| > 1 \). This pdf of class II with truncated tails is a pdf of class I up to a normalization factor. Using Eq. (60) with \( f(y) = 1 \), the normalization correction is of order \( h^{\nu-1/2} \) and is itself sub-leading. The estimate of the expectation of bounded functions for pdf’s of class II thus is reduced to the previous case of class I distributions. [Note that \( \nu = 2 \) for the realistic pdf of Eq. (34) – the error induced by cutting off the tails in this case is of the same order as that due to neglecting the remainder in the expansion]
of the portfolio’s value.] For short-term returns with a finite variance, the estimate of the order of the corrections in Eq. (42) is justified for portfolios with bounded values and a distribution of the returns on the asset that falls off like Eq. (34).


