1. you have 110 minutes for this exam.

2. Do not omit calculations that’re needed for your conclusions in part B.

1. (20%, True or false)

(a) ___: If a $5 \times 7$ matrix has rank $= 5$, then its null space has dimension 2.

(b) ___: If a matrix equation $Ax = b$ has two different solutions, then it has infinitely many solutions.

(c) ___: If the columns of a square matrix $A$ is an linearly independent set, then the rows of the same matrix is also an linearly independent set.

(d) ___: Suppose $UA = I_2$ for some $4 \times 2$ matrix $A$, $2 \times 4$ matrix $U$, then there also exists some $2 \times 4$ matrix $V$ such that $AV = I_4$.

(e) ___: If for any two vectors $u = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$, $v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$, we define $\langle u, v \rangle = u_1v_1 + u_2v_2$. Then $\langle u, v \rangle$ is an inner product on $\mathbb{R}^3$.

2. (20%, Systems of Linear Equations).

Let $A = \begin{bmatrix} -6 & -3 & 2 & 1 \\ 4 & -2 & -2 & 4 \\ 8 & 2 & 3 & -3 \end{bmatrix}$, and $b = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

(a) Find a basis for $\text{null}(A)$. (6%)

(b) Find a basis for $\text{col}(A)$. (6%)

(c) Solve the equation $Ax = b$. If there is more than one solution, write the general solution in parametric vector form. (8%)

3. (20%, Least-Square Problems ) Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \\ 2 & -1 \end{bmatrix}$, $b = \begin{bmatrix} 3 \\ -2 \\ 2 \end{bmatrix}$.

(a) Find the least square solution of $Ax = b$ by solving the normal equation. (10%)
(b) Find the orthogonal projection of \( b \) onto \( \text{Co}A \). (5%)
(c) What’s the least-square error, i.e, \( \| b - Ax \| \) ?

4. (20%, Orthogonal diagonalization). Let \( A = \begin{bmatrix} 6 & -3 & -3 \\ -3 & 6 & -3 \\ -3 & -3 & 6 \end{bmatrix} \).

(a) Find all the eigenvalues of \( A \). Hint: one of them is 0. (6%)
(b) Find an orthonormal basis consisting of eigenvectors. (8%)
(c) Find the spectral decomposition of \( A \). (6%)

5. (20%, Gram-Schmidt process; inner product spaces).
Define the inner product \( \langle \cdot, \cdot \rangle \) on the vector space \( C[-1,0] \) by \( \langle f, g \rangle = \int_{-1}^{1} f(t)g(t) \, dt \).
Now let \( W \) be the subspace spanned by the polynomials \( p_1(t) = 1, p_2(t) = t^2 \).

(a) Use the Gram-Schmidt process to find an orthogonal basis for \( W \). (10%)
(b) Find the best approximation to \( q(t) = t^4 \) by polynomials in \( W \). (10%)

6. (20%, QR factorization). Let \( A = \begin{bmatrix} 1 & 2 & 5 \\ -1 & 1 & -4 \\ -1 & 4 & -3 \\ 1 & -4 & 7 \end{bmatrix} \).

(a) Find an orthogonal basis for the column space of \( A \). (10%)
(b) Find a QR factorization of the matrix \( A \). (10%)