Density Problems for $W^{1,1}(M,N)$

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1 Introduction

Let M and N be smooth compact connected Riemannian manifolds without boundary. Assume N is isometrically embedded into $\mathbb{R}^{\bar{l}}$ for some large $\bar{l} \in \mathbb{N}$. For $1 \leq p < \infty$, we define

$$W^{1,p}(M,N) = \left\{ u \in W^{1,p}(M,\mathbb{R}^{\bar{l}}), u(x) \in N \text{ for } x \in M \ a.e. \right\}.$$

As a subset of $W^{1,p}(M,\mathbb{R}^{\bar{l}})$, it inherits the strong and weak topology. From the calculus of variations, the following two mapping spaces are of interest,

$$\begin{aligned} H_S^{1,p}(M,N) = & \text{ the strong closure of } C^\infty(M,N) \text{ in } W^{1,p}(M,N), \\ H_W^{1,p}(M,N) = & \Big\{ u \in W^{1,p}(M,N), \exists \text{ a sequence } u_j \in C^\infty(M,N) \\ & \text{such that } u_j \rightharpoonup u \text{ in } W^{1,p}(M,\mathbb{R}^{\bar{l}}) \Big\}. \end{aligned}$$

Clearly we have

$$H_S^{1,p}(M,N) \subseteq H_W^{1,p}(M,N) \subseteq W^{1,p}(M,N). \tag{1.1}$$

Lots of effort has been devoted to determine whether the above inclusions are strict or not and to give an exact description of these spaces. One should refer to [8] for a brief sketch of the history and detailed references. Especially, from [16], [4], [3], [8] and [9] we know for $1 \leq p < \dim M$, $H_S^{1,p}(M,N) = W^{1,p}(M,N)$ if and only if $\pi_{[p]}(N) = 0$ and M satisfies the $([p]-1)$-extension property with respect to N (see [9] for definition). In addition, when $1 \leq p < \dim M$, $p \notin \mathbb{Z}$, we have $H_W^{1,p}(M,N) = H_S^{1,p}(M,N)$. But not much is known for the case $p \in \mathbb{Z}$. One of the aim of this paper is to settle down this question for the case $p = 1$. More precisely we have

THEOREM 1.1 *Let M and N be smooth compact Riemannian manifolds without boundary, then $H_W^{1,1}(M,N) = H_S^{1,1}(M,N)$.*

Combine this result with the theorem on strong density property of smooth maps, we have

Communications on Pure and Applied Mathematics, Vol. 000, 0001–0012 (199X)
 CCC 0010–3640/98/000001-12

THEOREM 1.2 *Assume M and N are smooth compact Riemannian manifolds without boundary and $\dim M \geq 2$, then the following three conditions are equivalent,*

- $H_W^{1,1}(M,N) = W^{1,1}(M,N)$;
- $H_S^{1,1}(M,N) = W^{1,1}(M,N)$;
- $\pi_1(N) = 0$.

We shall see later that the key point for $p = 1$ is that for a weakly convergent sequence in L^1, no concentration could happen due to the Vitali-Hahn-Sacks theorem in measure theory. One may refer to Section 1.2.4 of [6] for more information.

In view of Theorem 1.2, when $\pi_1(N) \neq 0$ and $\dim M \geq 2$, smooth maps are not weakly sequentially dense in $W^{1,1}(M,N)$. But it was proved in [15] and [14] that for any smooth compact Riemannian manifolds M and N, smooth maps are dense in the sense of biting convergence for the gradient of maps in $W^{1,1}(M,N)$ (for the definition of biting convergence, one may refer to Section 1.2.7 of [6]). In proving this result, they first use maps with canonical singularity to approximate any map in the strong topology, then connect the singular sets of those approximating maps and modify them along the connecting set. This idea was used earlier in [2] to give an analytical description of $H_S^{1,2}(B^3,S^2)$. The other aim of the present paper is to take another approach, which is different from [15] and [14] but closely related to [4] and [7], to recover the above result.

THEOREM 1.3 *Let M and N be smooth compact Riemannian manifolds without boundary, then for any $u \in W^{1,1}(M,N)$, we may find a sequence $u_i \in C^\infty(M,N)$ such that $u_i \to u$ in $L^1(M)$, $|du_i|_{L^1(M)} \leq c(N)|du|_{L^1(M)}$ and $du_i \to du$ a.e. on M.*

Finally, we should mention that a more important reason to pursue Theorem 1.1 to 1.3 is that it shades lights on understanding the blowing up behavior for a sequence of smooth maps with uniformly bounded p energy for $2 \leq p < \dim M$, $p \in \mathbb{Z}$. We note that some important progress was made recently for the case $p = 2$ in Section 7 of [15], see also [10] for a generalization of weak sequential density result in [7] and further developments.

The paper is written as follows. In Section 2, we will prove Theorem 1.1. In Section 3, we will prove every map in $W^{1,1}(M,N)$ can be strongly approximated in $L^1(M,N)$ by a sequence of smooth maps with uniformly bounded energy and the differentials converge almost everywhere.

2 $H_W^{1,1}(M,N) = H_S^{1,1}(M,N)$

Before proving Theorem 1.1, we look at a special example.

EXAMPLE 2.1 *Assume $n \geq 2$, then $H_S^{1,1}(B_1^n, S^1) = H_W^{1,1}(B_1^n, S^1)$.*

To understand the situation more clearly, we present two proofs here.

FIRST PROOF OF EXAMPLE 2.1: We use the following characterization of $H_S^{1,1}(B_1, S^1)$ proved in [5]. That is for any $u \in W^{1,1}(B_1, S^1)$, u lies in $H_S^{1,1}(B_1, S^1)$ if and only if $d(u^1 du^2) = 0$ in distribution sense. If we are given a u in $H_W^{1,1}(B_1, S^1)$, then we may find a sequence $u_i \in C^\infty(\overline{B_1}, S^1)$ such that $u_i \rightharpoonup u$ in $W^{1,1}(B_1)$. From compactness we know $u_i \to u$ in $L^1(B_1)$. After passing to a subsequence we may assume $u_i \to u$ a.e.. It follows from Proposition 1 in [6], Section 1.2.4 that $u_i^1 du_i^2 \rightharpoonup u^1 du^2$ in L^1, hence $u_i^1 du_i^2 \to u^1 du^2$ in distribution sense and $0 = d(u_i^1 du_i^2) \to d(u^1 du^2)$, also in distribution sense. This implies $d(u^1 du^2) = 0$, and hence $u \in H_S^{1,1}(B_1, S^1)$. ∎

SECOND PROOF OF EXAMPLE 2.1: We use the methods of lifting in Sobolev spaces. Assume $u \in H_W^{1,1}(B_1, S^1)$, then we may find a sequence $u_j \in C^\infty(\overline{B_1}, S^1)$ such that $u_j \rightharpoonup u$ in $W^{1,1}(B_1)$. For each j, from the simply connectedness of $\overline{B_1}$, we may find a $\varphi_j \in C^\infty(\overline{B_1}, \mathbb{R})$ such that $u_j = e^{i\varphi_j}$ and $0 \leq \fint_{B_1} \varphi_j < 2\pi$. Since $du_j \rightharpoonup du$ in L^1, it follows from Theorem 2 in [6], Section 1.2.4 that du_j is equi-integrable and hence $d\varphi_j$ is equi-integrable too because $|d\varphi_j| = |du_j|$. On the other hand, it follows from $\int_{B_1} |d\varphi_j| = \int_{B_1} |du_j|$ that $\int_{B_1} |d\varphi_j|$ is uniformly bounded. From Theorem 2 in [6], Section 1.2.4, after passing to a subsequence we may find a 1 form $f^k dx_k$ such that $d\varphi_j \rightharpoonup f^k dx_k$ in $L^1(B_1)$. Since φ_j is uniformly bounded in $W^{1,1}(B_1)$, after passing to a subsequence, we may find a $\varphi \in L^1(B_1)$ such that $\varphi_j \to \varphi$ in L^1 and a.e., hence $d\varphi = f^k dx_k$ and this implies $\varphi \in W^{1,1}(B_1)$. Clearly we have $u = e^{i\varphi}$, this shows u lies in $H_S^{1,1}(B_1, S^1)$. ∎

To deal with the general situation, we need some techniques of generic slicing for Sobolev functions and topological information carried by a Sobolev mapping from Section 3 and Section 4 of [9].

Assume M is a n dimensional Riemannian manifold without boundary, the parameter space P is a m dimensional Riemannian manifold, T is a d dimensional Riemannian manifold without boundary and $D \subset T$ is a domain with compact closure and Lipschitz boundary, and the dimensions satisfy $m + d \geq n$. Given a map $H : \overline{D} \times P \to M$, we assume it satisfies

(H_1) $H \in Lip(\overline{D} \times P)$ and $[H(\cdot,\xi)]_{Lip(\overline{D})} \le c_0$ for any $\xi \in P$.

(H_2) There exists a positive number c_1 such that $J_H(x,\xi) \ge c_1$ $\mathcal{H}^{m+d}$ a.e. $(x,\xi) \in \overline{D} \times P$.

(H_3) There exists a positive number c_2 such that $\mathcal{H}^{m+d-n}(H^{-1}(y)) \le c_2$ for $\mathcal{H}^n$ a.e. $y \in M$.

We also denote $H^x(\xi) = H_\xi(x) = H(x,\xi)$. For convenience, we recall the following simple but useful fact from Lemma 3.3 in [9].

LEMMA 2.2 *Let M, P, T, D be the same as described above, given a map $H : \overline{D} \times P \to M$ satisfying $(H_1), (H_2)$ and (H_3), then for any Borel function $\chi : M \to \widetilde{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$, with $\chi \ge 0$, we have*

$$\int_P d\mathcal{H}^m(\xi) \int_D \chi(H_\xi(x)) d\mathcal{H}^d(x) \le c_1^{-1} c_2 \int_M \chi(y) d\mathcal{H}^n(y).$$

Especially for any Borel subset $E \subset M$, we have

$$\int_P \mathcal{H}^d(H_\xi^{-1}(E)) d\mathcal{H}^m(\xi) \le c_1^{-1} c_2 \mathcal{H}^n(E).$$

If in addition, $\mathcal{H}^n(E) = 0$, then $\mathcal{H}^d(H_\xi^{-1}(E)) = 0$ for $\mathcal{H}^m$ a.e. $\xi \in P$.

The crucial step in proving Theorem 1.1 is the following lemma. One should compare it with Proposition 4.1 and Theorem 4.1 in [9].

LEMMA 2.3 *Assume M and N are smooth compact Riemannian manifolds without boundary and N is isometrically embedded into $\mathbb{R}^{\bar{l}}$, $n = dim M$. $u_i \in W^{1,1}(M,N)$, $u \in W^{1,1}(M,N)$, $u_i \rightharpoonup u$ in $W^{1,1}(M)$. K is a finite rectilinear cell complex, P is a m dimensional Riemannian manifold, $H : |K| \times P \to M$ is a map such that for any $\Delta \in K$, $H|_{\Delta \times P}$ satisfies $(H_1), (H_2)$ and (H_3), then after passing to a subsequence, we have $\chi_{1,H,u_i} \to \chi_{1,H,u}$ a.e. on P.*

For reader's convenience, we recall that as defined in [9], for a generic $\xi \in P$, $\chi_{1,H,u}(\xi)$ is equal to the homotopy class $[u \circ H_\xi|_{|K^1|}]_{|K^1|,N}$.

PROOF OF LEMMA 2.3: We may find an $\varepsilon_0 > 0$ such that $V_{2\varepsilon_0}(N) = \{y \backslash y \in \mathbb{R}^{\bar{l}}, d(y,N) < 2\varepsilon_0\}$ is a tubular neighborhood of N. Let π be the nearest point projection.

Without losing of generality, we may assume $u_i \to u$ a.e. on M. We claim that for any $\alpha > 0$, when i is large enough, we may find a $P_{\alpha,i} \subset P$

such that $\mathcal{H}^m(P_{\alpha,i}) \leq \alpha$ and $\chi_{1,H,u_i} = \chi_{1,H,u}$ on $P\backslash P_{\alpha,i}$. This clearly implies the lemma.

Since $u_i \rightharpoonup u$ in $W^{1,1}(M)$, it follows from Theorem 2 in [6], Section 1.2.4, that for any $\varepsilon > 0$, we may find a $\delta > 0$ such that for any $E \subset M$ with $\mathcal{H}^n(E) \leq \delta$, we have $\int_E |du_i| d\mathcal{H}^n \leq \varepsilon$ for any $i \in \mathbb{N}$ and $\int_E |du| d\mathcal{H}^n \leq \varepsilon$. From Egorov theorem, we know for any $0 < \delta_1 < \delta$, there exists a closed subset $\mathcal{G} \subset M$ such that $u_i \to u$ uniformly on $\mathcal{G}$ and $\mathcal{H}^n(M\backslash\mathcal{G}) \leq \delta_1$, denote $\mathcal{B} = M\backslash\mathcal{G}$.

For any $\Delta \in K$, $d = \dim\Delta$, it follows from Lemma 2.2 that

$$\int_P \mathcal{H}^d(H_\xi^{-1}(\mathcal{B}) \cap \Delta) d\mathcal{H}^m(\xi) \leq c_1^{-1} c_2 \delta_1.$$

Hence for $\gamma > 0$, if we denote

$$P_\Delta^1 = \{\xi\backslash\xi \in P, \mathcal{H}^d(H_\xi^{-1}(\mathcal{B}) \cap \Delta) \geq \gamma\},$$

then $\mathcal{H}^m(P_\Delta^1) \leq c_1^{-1} c_2 \gamma^{-1} \delta_1$. It is clear that for $\xi \in P\backslash P_\Delta^1$, we have

$$\mathcal{H}^d(H_\xi^{-1}(\mathcal{B} \cap \Delta)) < \gamma, \tag{2.1}$$

we may assume $\gamma < \frac{1}{2}\mathcal{H}^d(\Delta)$, then for such an ξ, we have

$$\mathcal{H}^d(H_\xi^{-1}(\mathcal{G} \cap \Delta)) \geq \frac{1}{2}\mathcal{H}^d(\Delta) > 0. \tag{2.2}$$

On the other hand, it follows from Lemma 2.2 that

$$\begin{aligned}
&\int_P d\mathcal{H}^m(\xi) \int_\Delta |d(u_i \circ H_\xi)(x)| \chi_\mathcal{B}(H_\xi(x)) d\mathcal{H}^d(x) \\
&\leq c_0 \int_P d\mathcal{H}^m(\xi) \int_\Delta |(du_i)(H_\xi(x))| \chi_\mathcal{B}(H_\xi(x)) d\mathcal{H}^d(x) \\
&\leq c_0 c_1^{-1} c_2 \int_M |du_i(y)| \chi_\mathcal{B}(y) d\mathcal{H}^n(y) \\
&\leq c_0 c_1^{-1} c_2 \varepsilon.
\end{aligned}$$

If we denote

$$P_{\Delta,i} = \left\{\xi\backslash\xi \in P, \int_{H_\xi^{-1}(\mathcal{B})\cap\Delta} |d(u_i \circ H_\xi)(x)| d\mathcal{H}^d(x) \geq \gamma\right\},$$

then $\mathcal{H}^m(P_{\Delta,i}) \leq c_0 c_1^{-1} c_2 \gamma^{-1} \varepsilon$. Similarly, if we denote

$$P_\Delta^2 = \left\{\xi\backslash\xi \in P, \int_{H_\xi^{-1}(\mathcal{B})\cap\Delta} |d(u \circ H_\xi)(x)| d\mathcal{H}^d(x) \geq \gamma\right\},$$

then $\mathcal{H}^m(P^2_\Delta) \le c_0 c_1^{-1} c_2 \gamma^{-1} \varepsilon$. For i large enough, we have $|u_i - u|_{L^\infty(\mathcal{G})} \le \frac{\varepsilon_0}{4}$. If we set

$$P_{\alpha,i} = \bigcup_{\Delta \in K^1} \left(P^1_\Delta \cup P^2_\Delta \cup P_{\Delta,i} \right),$$

then for δ_1 and ε small enough, we have $\mathcal{H}^n(P_{\alpha,i}) \le \alpha$. Now for any $\Delta \in K^1$, if $\dim\Delta = 0$, then it follows from (2.2) that $|u_i(H_\xi(x)) - u(H_\xi(x))| \le \frac{\varepsilon_0}{2}$ for $x \in \Delta$, $\xi \in P \backslash P_{\alpha,i}$ and i large enough, if $\dim\Delta = 1$, then for $\xi \in P \backslash P_{\alpha,i}$, we have $\mathcal{H}^1(H_\xi^{-1}(\mathcal{G} \cap \Delta)) > 0$. In addition, for each interval $I \subset H_\xi^{-1}(\mathcal{B}) \cap \Delta$, it follows from the definition of $P_{\Delta,i}$ and P^2_Δ that $\mathrm{osc}_I u_i \le \gamma$ and $\mathrm{osc}_I u \le \gamma$. We may assume $\gamma \le \frac{\varepsilon_0}{8}$, then we clearly have $|u_i(H_\xi(x)) - u(H_\xi(x))| \le \frac{\varepsilon_0}{2}$ for any $x \in \Delta$. This clearly implies $u_i \circ H_\xi|_{|K^1|} \sim u \circ H_\xi|_{|K^1|}$. The claim follows. ∎

We remark that the key point in the above proof is that in the weak convergence in L^1 (not in L^p, $1 < p < \infty$), concentrations could not happen due to Theorem 2 in [6], Section 1.2.4.

Proof of Theorem 1.1: Assume $u \in H^{1,1}_W(M,N)$, then we may find a sequence $u_i \in C^\infty(M,N)$ such that $u_i \rightharpoonup u$ in $W^{1,1}(M)$. It follows from the proof of Lemma 2.3 that on "generic" 1 skeletons M^1 of M, $u|_{M^1}$ has a continuous extension to the whole M, then it follows from Remark 6.2 in [9] and [3] that u must be in $H^{1,1}_S(M,N)$. ∎

Proof of Theorem 1.2: It follows from Theorem 1.3 in [8] that $H^{1,1}_S(M,N) = W^{1,1}(M,N)$ if and only if $\pi_1(N) = 0$ (note that because N is path connected, M satisfies the 0-extension property with respect to N). Clearly Theorem 1.2 follows from this fact and Theorem 1.1. ∎

3 Density in other senses

In this section, we shall prove Theorem 1.3. Our method depends essentially on the following interesting lemma.

Lemma 3.1 *Let N be a smooth compact m dimensional Riemannian manifold without boundary, then for $0 < \varepsilon < \frac{1}{2}$, we may find an open subset $V_\varepsilon \subset N$ such that*

- *$\mathcal{H}^m(V_\varepsilon) \le c(N)\varepsilon$;*
- *There exists a Lipschitz map $\phi_\varepsilon : N \to N \backslash V_\varepsilon$ such that $\phi_\varepsilon|_{N\backslash V_\varepsilon} = id_{N\backslash V_\varepsilon}$ and $|d\phi_\varepsilon| \le \frac{c(N)}{\varepsilon}$;*

- *There exists an open subset $U_\varepsilon \subset N$ such that $N \backslash V_\varepsilon \subset U_\varepsilon$ and U_ε is diffeomorphic to the open unit ball.*

We shall give two proofs for this useful fact. The first one depends on some theorems about the relations between cut locus and the conjugate locus and the regularity of cut locus. But it is conceptually clearer. The second one is more elementary and uses the so called deformations associated with the dual skeletons as in [9].

FIRST PROOF OF LEMMA 3.1: If $m = 1$ or N is diffeomorphic to S^2, then the conclusion of Lemma 3.1 is trivial. Hence we may assume $m \geq 2$ and N is not diffeomorphic to S^2. It follows from a theorem of A. Weinstein in [17], which gave a negative answer to Raunch's conjecture, that we may find a point $p \in N$ and a Riemannian metric g on N such that on N_p, the cut locus and conjugate locus are disjoint. Without losing of generality, we may replace the original metric on N by this new one. For $e \in N_p$ with $|e| = 1$, let $\rho(e)$ be the positive number such that $\rho(e)e$ is the cut point along the direction in e. It follows from [11] that ρ is a Lipschitz function. Let $C = \{te \backslash e \in N_p, |e| = 1, 0 \leq t \leq \rho(e)\}$, then for any $\xi \in C$, $|(d(exp_p)_\xi)^{-1}| \leq c(N)$. We may define a map $\psi : C \to \overline{B_1^m}$ by $\psi(\xi) = \rho(\frac{\xi}{|\xi|})^{-1}\xi$. It is clear that ψ is a bi-Lipschitz map with $[\psi]_{Lip(C)}, [\psi^{-1}]_{Lip(\overline{B_1})} \leq c(N)$. For $0 < \varepsilon \leq \frac{1}{2}$, we define a function $\eta_\varepsilon : [0,1] \to [0,1]$ by

$$\eta_\varepsilon(t) = \begin{cases} t, & \text{for } 0 \leq t \leq 1 - \varepsilon; \\ \frac{(1-\varepsilon)(1-t)}{\varepsilon}, & \text{for } 1 - \varepsilon \leq t \leq 1. \end{cases}$$

Then we may define a Lipschitz map $S_\varepsilon : \overline{B_1} \to \overline{B_1}$ by setting $S_\varepsilon(\xi) = \eta_\varepsilon(|\xi|)\frac{\xi}{|\xi|}$. Clearly $[S_\varepsilon]_{Lip(\overline{B_1})} \leq \frac{c(m)}{\varepsilon}$. Now let $V_\varepsilon = \exp_p(\psi^{-1}(\overline{B_1} \backslash B_{1-\varepsilon}))$ and $\phi_\varepsilon : N \to N$ be defined by

$$\phi_\varepsilon(exp_p(\xi)) = exp_p(\psi^{-1}(S_\varepsilon(\psi(\xi)))) \quad \text{for any } \xi \in C.$$

It is clear that ϕ_ε is continuous. On the other hand it follows from the uniform estimates of $|(d(exp_p)_\xi)^{-1}|$ that ϕ_ε is Lipschitz on N with $|d\phi_\varepsilon| \leq \frac{c(N)}{\varepsilon}$. The existence of U_ε as needed in the lemma clearly follows from the construction above. ∎

SECOND PROOF OF LEMMA 3.1: Fix a smooth triangulation of N, namely $h : K \to N$. For each $\Delta \in K$, we pick up a point $y_\Delta \in \text{Int}(\Delta)$.

Now we are going to use the notations and results in Section 6 and Appendix B of [9]. Hence we have $|\cdot|_0, \phi_1^0, \phi^0(x,\varepsilon)$ for $0<|x|_0<1$ and $0<\varepsilon<1$. Fix a maximal tree T of K^1. For $0<\varepsilon\le\frac{1}{2}$, denote $X_\varepsilon=\{x\backslash x\in|K|,|x|_0\ge\varepsilon\}$, $Y_\varepsilon=T\cup X_\varepsilon$. X_ε has $\#|K^0|$ components, each component corresponds to one of the vortex in K^0 and is bi-Lipschitz equivalent to the close unit ball with the vortex sent to origin. This plus the fact that T is a tree shows for any $0<\varepsilon\le\frac{1}{2}$, we may find an open set $U_\varepsilon\supset h(Y_\varepsilon)$, such that U_ε is diffeomorphic to the open unit ball. Now let us look at the identity map $i:Y_{\frac{1}{2}}\to Y_{\frac{1}{2}}$, since i is homotopic to a constant map. It follows from the proof of Proposition 2.2 (Homotopy Extension Theorem) that we may find a Lipschitz map $\psi:|K|\to Y_{\frac{1}{2}}$ such that $\psi|_{Y_{\frac{1}{2}}}=i$. For $0<\varepsilon\le\frac{1}{2}$, we define two maps $G_\varepsilon:|K|\to|K|$, $H_\varepsilon:|K|\to|K|$ by

$$G_\varepsilon(x)=\begin{cases} x, & |x|_0=0\\ \phi^0(x,2^{-1}\varepsilon^{-1}|x|_0), & 0<|x|_0\le\varepsilon\\ \phi^0(x,1-\frac{1-|x|_0}{2(1-\varepsilon)}), & \varepsilon\le|x|_0\le1\end{cases}$$

and

$$H_\varepsilon(x)=\begin{cases} x, & |x|_0=0\\ \phi^0(x,2\varepsilon|x|_0), & 0<|x|_0\le\frac{1}{2}\\ \phi^0(x,1-2(1-\varepsilon)(1-|x|_0)), & \frac{1}{2}\le|x|_0\le1\end{cases}.$$

It follows easily from Appendix B in [9] that G_ε and H_ε are Lipschitz maps and inverse to each other, in addition, we have $|dG_\varepsilon|\le\frac{c(K)}{\varepsilon}$ and $|dH_\varepsilon|\le c(K)$. Let $\psi_\varepsilon(x)=H_\varepsilon(\psi(G_\varepsilon(x)))$ for $x\in|K|$, then ψ_ε is a Lipschitz map with $|d\psi_\varepsilon|\le\frac{c(K)}{\varepsilon}$. In addition, from the construction we know $\psi_\varepsilon(|K|)\subset Y_\varepsilon$ and for any $x\in Y_\varepsilon$, we have $\psi_\varepsilon(x)=x$. Now let $V_\varepsilon=h(\{x\backslash x\in|K|,|x|_0<\varepsilon\}\backslash T)$, $\phi_\varepsilon=h\circ\psi_\varepsilon\circ h^{-1}$, then the estimate of $\mathcal{H}^m(V_\varepsilon)$ follows from Appendix B in [9]. The property of ϕ_ε and the existence of U_ε follows from the arguments above. ∎

Proof of Theorem 1.3: Denote $n=\dim M$, $m=\dim N$ and assume N is isometrically embedded in $\mathbb{R}^{\bar{l}}$ for some large $\bar{l}\in\mathbb{N}$. Fix a small $\varepsilon_0>0$ such that $V_{2\varepsilon_0}(N)=\{y\backslash y\in\mathbb{R}^{\bar{l}},d(y,N)<2\varepsilon_0\}$ is a tubular neighborhood of N. Let π be the nearest point projection. We define a map $H:N\times B_{\varepsilon_0}^{\bar{l}}\to N$ by $H(y,\xi)=\pi(y+\xi)=H_\xi(y)=H^y(\xi)$. We also assume that ε_0 is small enough such that for each $\xi\in B_{\varepsilon_0}$, $H_\xi:N\to N$ is a diffeomorphism with $|dH_\xi|,|dH_\xi^{-1}|\le c(N)$, $J_{H^y}(\xi)\ge c(N)>0$ for $\xi\in B_{\varepsilon_0}$, $y\in N$, and $\mathcal{H}^{\bar{l}-m}((H^{y_1})^{-1}(y_2))\le c(N)$ for any $y_1,y_2\in N$.

Given a map $u \in W^{1,1}(M,N)$. For any $\varepsilon > 0$ small enough, we may find the corresponding V_ε, ϕ_ε and U_ε as in Lemma 3.1. For $\xi \in B_{\varepsilon_0}$, we denote $V_{\varepsilon,\xi} = H_\xi^{-1}(V_\varepsilon)$. We have the following

$$
\begin{aligned}
&\int_{B_{\varepsilon_0}} \mathcal{H}^n(u^{-1}(V_{\varepsilon,\xi}))d\mathcal{H}^{\bar{l}}(\xi) \\
&= \int_{B_{\varepsilon_0}} d\mathcal{H}^{\bar{l}}(\xi) \int_M \chi_{V_\varepsilon}(H_\xi(u(x)))d\mathcal{H}^n(x) \\
&= \int_M d\mathcal{H}^n(x) \int_{B_{\varepsilon_0}} \chi_{V_\varepsilon}(H^{u(x)}(\xi))d\mathcal{H}^{\bar{l}}(\xi) \\
&\le c(N) \int_M d\mathcal{H}^n(x) \int_{B_{\varepsilon_0}} \chi_{V_\varepsilon}(H^{u(x)}(\xi))J_{H^{u(x)}}(\xi)d\mathcal{H}^{\bar{l}}(\xi) \\
&= c(N) \int_M d\mathcal{H}^n(x) \int_N \chi_{V_\varepsilon}(y)\mathcal{H}^{\bar{l}-m}((H^{u(x)})^{-1}(y))d\mathcal{H}^m(y) \\
&\le c(M,N)\mathcal{H}^m(V_\varepsilon) \\
&\le c(M,N)\varepsilon,
\end{aligned}
\tag{3.1}
$$

and

$$
\begin{aligned}
&\int_{B_{\varepsilon_0}} d\mathcal{H}^{\bar{l}}(\xi) \int_{u^{-1}(V_{\varepsilon,\xi})} |du(x)|d\mathcal{H}^n(x) \\
&= \int_{B_{\varepsilon_0}} d\mathcal{H}^{\bar{l}}(\xi) \int_M |du(x)|\chi_{V_\varepsilon}(H_\xi(u(x)))d\mathcal{H}^n(x) \\
&= \int_M d\mathcal{H}^n(x) \int_{B_{\varepsilon_0}} |du(x)|\chi_{V_\varepsilon}(H_\xi(u(x)))d\mathcal{H}^{\bar{l}}(\xi) \\
&\le c(N) \int_M d\mathcal{H}^n(x) \int_{B_{\varepsilon_0}} |du(x)|\chi_{V_\varepsilon}(H_\xi(u(x)))J_{H^{u(x)}}(\xi)d\mathcal{H}^{\bar{l}}(\xi) \\
&= c(N) \int_M d\mathcal{H}^n(x) \int_N |du(x)|\chi_{V_\varepsilon}(y)\mathcal{H}^{\bar{l}-m}((H^{u(x)})^{-1}(y))d\mathcal{H}^m(y) \\
&\le c(N)\mathcal{H}^m(V_\varepsilon)|du|_{L^1(M)} \\
&\le c(N)|du|_{L^1(M)}\varepsilon.
\end{aligned}
\tag{3.2}
$$

It follows from (3.1), (3.2) and the mean value inequality that we may find a $\xi \in B_{\varepsilon_0}$ such that

$$
\mathcal{H}^n\left(u^{-1}\left(V_{\varepsilon,\xi}\right)\right) \le c(M,N)\varepsilon, \tag{3.3}
$$

and

$$
\int_{u^{-1}(V_{\varepsilon,\xi})} |du|d\mathcal{H}^n \le c(N)|du|_{L^1(M)}\varepsilon. \tag{3.4}
$$

Let $u_\varepsilon(x) = H_\xi^{-1}(\phi_\varepsilon(H_\xi(u(x))))$ for $x \in M$, then $u_\varepsilon = u$ on $M \backslash u^{-1}(V_{\varepsilon,\xi})$ and it follows from (3.3) and (3.4) that

$$\begin{aligned}\int_M |du_\varepsilon| &= \int_{M\backslash u^{-1}(V_{\varepsilon,\xi})} |du| + \int_{u^{-1}(V_{\varepsilon,\xi})} |du| \\ &\leq |du|_{L^1(M)} + \frac{c(N)}{\varepsilon} \int_{u^{-1}(V_{\varepsilon,\xi})} |du| \\ &\leq c(N)|du|_{L^1(M)}.\end{aligned}$$

Also we have

$$\begin{aligned}\int_M |u_\varepsilon - u| &= \int_{u^{-1}(V_{\varepsilon,\xi})} |u_\varepsilon - u| \\ &\leq c(N)\mathcal{H}^n\left(u^{-1}\left(V_{\varepsilon,\xi}\right)\right) \\ &\leq c(M,N)\varepsilon \to 0 \quad \text{as } \varepsilon \to 0^+.\end{aligned}$$

Because $\mathcal{H}^m(\{u_\varepsilon \neq u\}) \leq c(M,N)\varepsilon$, we know for a sequence of positive numbers $\varepsilon_i \to 0$, $du_{\varepsilon_i} \to du$ a.e.. On the other hand, since $\operatorname{im}(u_\varepsilon) \subset N\backslash V_{\varepsilon,\xi} \subset H_\xi^{-1}(U_\varepsilon)$, which is diffeomorphic to the open unit ball, it follows from the standard convolution argument that $u_\varepsilon \in H_S^{1,1}(M,N)$. Hence we may find a sequence $v_i \in C^\infty(M,N)$ such that $v_i \to u$ in $L^1(M)$, $|dv_i|_{L^1(M)} \leq c(N)|du|_{L^1(M)}$ and $dv_i \to du$ a.e.. ∎

At last we shall make some remarks on why we emphasize that the set U_ε in the conclusion of Lemma 3.1 need to be diffeomorphic to the open ball, instead of just being bi-Lipschitz equivalent. The following explanation about the continuity of composition operator in Sobolev spaces makes things clear.

Assume $1 \leq p < \infty$, $f \in Lip(\mathbb{R},\mathbb{R})$, $\Omega \subset \mathbb{R}^n$ is an open bounded subset with Lipschitz boundary, then for any $u \in W^{1,p}(\Omega,\mathbb{R})$, $f \circ u \in W^{1,p}(\Omega,\mathbb{R})$ and we have the following chain rule,

$$\partial_i(f \circ u)(x) = \begin{cases} f'(u(x))\partial_i u(x), & \text{if } f' \text{ exists at } u(x); \\ 0, & \text{otherwise.}\end{cases}$$

Define a map $F : W^{1,p}(\Omega,\mathbb{R}) \to W^{1,p}(\Omega,\mathbb{R})$ by setting $F(u) = f \circ u$. If $f \in C^1(\mathbb{R},\mathbb{R}) \cap Lip(\mathbb{R})$, then it is not hard to prove that F is a continuous map. On the other hand, it was proved in [12] that as long as f is Lipschitz, F is always continuous on $W^{1,p}(\Omega)$.

Things change essentially for vector valued case (see [13] and [1]). Assume $m \geq 2$, $f \in Lip(\mathbb{R}^m,\mathbb{R})$, $\Omega \subset \mathbb{R}^n$ is an open bounded subset with Lipschitz boundary, then for any $u \in W^{1,p}(\Omega, \mathbb{R}^m)$, we still have $f \circ u \in W^{1,p}(\Omega, \mathbb{R})$, but the chain rule is much more subtle, it was proved in [1] that if for $x \in \Omega$, we define an affine space $T_{u,x} = u(x) + du_x(\mathbb{R}^n)$, then we have for a.e. $x \in \Omega$, $f|_{T_{u,x}}$ is differentiable at $u(x)$ and

$$d(f \circ u)_x = d(f|_{T_{u,x}})_{u(x)} \circ du_x.$$

Of course when $f \in C^1(\mathbb{R}^m, \mathbb{R}) \cap Lip(\mathbb{R}^m)$, we still have $d(f \circ u)_x = df_{u(x)} \circ du_x$ for a.e. $x \in \Omega$. The main subtle thing for general case is if f is only Lipschitz, then it is only differentiable a.e., and a large part (especially, a measure positive part) of Ω may be sent into the bad set by u. Define a map $F : W^{1,p}(\Omega, \mathbb{R}^m) \to W^{1,p}(\Omega, \mathbb{R})$ by setting $F(u) = f \circ u$. If $f \in C^1(\mathbb{R}^m, \mathbb{R}) \cap Lip(\mathbb{R})$, then it is not hard to prove F is continuous. In general F need not be continuous if we only assume f to be Lipschitz. To see this we observe that by the usual extension theorem for Lipschitz functions, we easily deduce that there exists a $f \in Lip(\mathbb{R}^2, \mathbb{R})$ such that $[f]_{Lip(\mathbb{R}^2)} \leq 4$ and for any $j \in \mathbb{Z}$, $x \in \mathbb{R}$, $f(x, 2^{-j}) = 2^{-j} \sin(2^j x)$. Then $f(\cdot, 2^{-j}) \nrightarrow 0$ in $W^{1,1}_{loc}(\mathbb{R})$. In fact $\partial_1 f(\cdot, 2^{-j}) = \cos(2^j x) \nrightarrow 0$ in $L^1_{loc}(\mathbb{R})$. On the other hand, if we set $u_j(x) = (x, 2^{-j})$ for $j \in \mathbb{N}$, then $u_j \to (x, 0)$ in $C^\infty(\mathbb{R})$. The above fact just shows $f \circ u_j \nrightarrow f \circ u$ in $W^{1,1}_{loc}(\mathbb{R})$.

Acknowledgment. I would like to thank Professor Fanghua Lin for his very kind encouragement.

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