A REMARK ON THE EXISTENCE OF SUITABLE VECTOR FIELDS RELATED TO THE DYNAMICS OF SCALAR SEMILINEAR PARABOLIC EQUATIONS

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Let Ω be a bounded smooth domain in $\mathbb{R}^n$, ν be the outer normal direction of $\partial\Omega$ and $f \in C^\infty\left(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n, \mathbb{R}\right)$. The infinite dimensional dynamical system defined by

$$\left\{ \begin{array}{l} u_t = \Delta u + f\left(x, u, \nabla u\right), \quad x \in \Omega, t > 0, \\ \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega, \end{array} \right. \tag{1}$$

on suitable Sobolev spaces has attracted much interest (see [A, H] and more recent references at the end of this note). Many efforts have been made to show the complexity of its dynamical behavior (see some survey papers and recent articles [DP, P1, P2, P3, Pr, PR, R] and the references therein). In particular, the following nice result was proven in [P2]: if there exists a smooth vector field Φ on $\overline{\Omega}$, $\Phi = (\phi_1, \cdots, \phi_n)$ such that

$$\left\{ \begin{array}{l} \operatorname{rank}\left(\Phi\left(x\right), \partial_1 \Phi\left(x\right), \cdots, \partial_n \Phi\left(x\right)\right) = n \text{ for all } x \in \overline{\Omega}, \\ \frac{\partial \Phi}{\partial \nu} = 0 \text{ on } \partial\Omega, \end{array} \right.$$

then for any smooth vector field X on $\mathbb{R}^n$, there exists a smooth function f, such that span$\{\phi_1, \cdots, \phi_n\}$ is invariant under (1) and for any integral curve of X, $c = c(t)$, $u = \sum_{i=1}^n c_i(t)\,\phi_i(x)$ is a solution to (1). Moreover, it was shown that such kind of vector field always exists on a starshaped domain. The main result of this short note is a classification of all the domains on which one may find this type of vector fields. More precisely, we have

Theorem 1. *Let $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$ be an open bounded smooth domain, then the necessary and sufficient condition for the existence of a smooth map $F : \overline{\Omega} \to \mathbb{R}^n$ with*

$$\left\{ \begin{array}{l} \operatorname{rank}\left(F\left(x\right), \partial_1 F\left(x\right), \cdots, \partial_n F\left(x\right)\right) = n \textit{ for any } x \in \overline{\Omega}, \\ \frac{\partial F}{\partial \nu} = 0 \textit{ on } \partial\Omega, \end{array} \right. \tag{2}$$

is that $\overline{\Omega}$ is diffeomorphic to $\overline{B}_1$ or $\overline{B}_2 \backslash B_1$.

Remark 1. *In fact, if $\overline{\Omega}$ is diffeomorphic to $\overline{B}_1$, then any solution to (2), F, must have exactly one zero in Ω. If $\overline{\Omega}$ is diffeomorphic to $\overline{B}_2 \backslash B_1$, then any solution to (2), F, does not vanish at all. These conclusions will follow from the arguments below.*

First we reduce the existence of such a vector field to the existence of vector field with less restrictions.

Lemma 1. *Let $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$ be an open bounded smooth domain, if there exists a smooth map $G : \overline{\Omega} \to \mathbb{R}^n$ such that*

$$\operatorname{rank}\left(G\left(x\right), \partial_1 G\left(x\right), \cdots, \partial_n G\left(x\right)\right) = n \textit{ for any } x \in \overline{\Omega}$$

and

$$\dim \operatorname{span}\left\{G(x), \operatorname{im}(G|_{\partial\Omega})_{*,x}\right\} = n \text{ for any } x \in \partial\Omega,$$

here $(G|_{\partial\Omega})_{*,x}$ *denotes the tangent map of* $G|_{\partial\Omega}$ *at* x*, then we may find a smooth map* $F : \overline{\Omega} \to \mathbb{R}^n$ *satisfying (2).*

Proof. Let $\varepsilon > 0$ be small enough such that the map

$$\begin{aligned} \phi &: \partial\Omega \times [0, 3\varepsilon] \to \left\{y \in \overline{\Omega} : \operatorname{dist}(y, \partial\Omega) \le 3\varepsilon\right\} \\ &: (x, t) \mapsto x - t\nu(x) \end{aligned}$$

is a diffeomorphism. Let $P = G \circ \phi$. For any $t \in [0, 3\varepsilon]$, let $P_t(x) = P(x, t)$ for $x \in \partial\Omega$, then we may assume ε is small enough such that for any $t \in [0, 3\varepsilon]$,

$$\dim \operatorname{span}\left\{P_t(x), \operatorname{im}(P_t)_{*,x}\right\} = n \text{ for all } x \in \partial\Omega.$$

Let $\eta : \mathbb{R} \to \mathbb{R}$ be a smooth function such that

$$\eta(t) = \begin{cases} \varepsilon, \text{ when } t \le \varepsilon/2; \\ t, \text{ when } t \ge 3\varepsilon/2; \end{cases}$$

and $\eta'(t) \ge 0$ for all t. Define

$$Q_t(x) = Q(x, t) = P(x, \eta(t)) \text{ for } x \in \partial\Omega,\ 0 \le t \le 3\varepsilon.$$

Then it is clear that for any $t \in [0, 3\varepsilon]$,

$$\dim \operatorname{span}\left\{Q_t(x), \operatorname{im}(Q_t)_{*,x}\right\} = n \text{ for any } x \in \partial\Omega.$$

Let

$$F(y) = \begin{cases} G(y), \text{ if } y \in \overline{\Omega}, \operatorname{dist}(y, \partial\Omega) \ge 2\varepsilon; \\ Q\left(\phi^{-1}(y)\right), \text{ if } y \in \overline{\Omega}, \operatorname{dist}(y, \partial\Omega) \le 3\varepsilon; \end{cases}$$

then it is clear F satisfies all the requirements. □

Corollary 1. *Assume* $\overline{\Omega}_1$ *is diffeomorphic to* $\overline{\Omega}_2$*, and for* Ω_1 *we may find a solution to (2), then we may find a solution to (2) for* Ω_2 *too.*

To derive the necessary condition for the existence of a vector filed satisfying (2), we will need

Lemma 2. *Let* $\Omega \subset \mathbb{R}^n$ $(n \ge 2)$ *be an open bounded smooth domain, if there exists a smooth map* $H : \overline{\Omega} \to S^{n-1}$ *such that*

$$\operatorname{rank}(\partial_1 H(x), \cdots, \partial_n H(x)) = n - 1 \text{ for any } x \in \overline{\Omega}$$

and

$$\dim \operatorname{im}(H|_{\partial\Omega})_{*,x} = n - 1 \text{ for any } x \in \partial\Omega,$$

then $\overline{\Omega}$ *is diffeomorphic to* $\overline{B}_2 \backslash B_1$*.*

Proof. First we claim that each path connected component of $\partial\Omega$ is diffeomorphic to S^{n-1}. This is clear when $n = 2$. If $n \ge 3$, since $H|_{\partial\Omega}$ has full rank everywhere and $\partial\Omega$ is compact, $H|_{\partial\Omega} : \partial\Omega \to S^{n-1}$ is a covering map (see [M]). Since S^{n-1} is simply connected, we see each path connected components of $\partial\Omega$ must be diffeomorphic to S^{n-1}. Indeed, the restriction of H to such a component serves as a diffeomorphism.

To proceed, we observe that from the assumption on H, it follows from implicit function theorem that for any $\xi \in S^{n-1}$, $H^{-1}(\xi)$ is a smooth one dimensional submanifold of $\overline{\Omega}$, moreover $H : \overline{\Omega} \to S^{n-1}$ is a smooth fiber bundle (see [M]). Fix a point $x_0 \in \partial\Omega$, let $\xi_0 = H(x_0)$ and $\Gamma = H^{-1}(\xi_0)$, then we have an exact sequence (see Theorem 6.7 of chapter VII in [B])

$$\pi_{n-1}(\Gamma, x_0) \to \pi_{n-1}\left(\overline{\Omega}, x_0\right) \to \pi_{n-1}\left(S^{n-1}, \xi_0\right) \to \pi_{n-2}(\Gamma, x_0).$$

If $n \geq 3$, then both $\pi_{n-1}(\Gamma, x_0)$ and $\pi_{n-2}(\Gamma, x_0)$ vanishes, this shows $\pi_{n-1}\left(\overline{\Omega}, x_0\right) \cong \mathbb{Z}$ and hence $\overline{\Omega}$ is diffeomorphic to $\overline{B}_2 \backslash B_1$. If $n = 2$, then since $\pi_1(\Gamma, x_0)$ vanishes and $\pi_0(\Gamma, x_0)$ is finite, we see $\pi_1\left(\overline{\Omega}, x_0\right)$ is again isomorphic to $\mathbb{Z}$, this shows $\overline{\Omega}$ must be diffeomorphic to $\overline{B}_2 \backslash B_1$. □

Now we are ready to prove the theorem.

Proof of theorem 1. First if $\Omega = B_1$ or $B_2 \backslash \overline{B}_1$, then $G(x) = x$ satisfies the assumption in the Lemma 1, half of the theorem follows from the lemma and Corollary 1. On the other hand, assume for some Ω, we may find a smooth map F satisfying (2). For $x \in \partial\Omega$, choose a base for the tangent space of $\partial\Omega$ at x, namely $e_1, \cdots, e_{n-1}$, then

$$\begin{aligned} & \operatorname{rank}\left(F(x), \partial_1 F(x), \cdots, \partial_n F(x)\right) \\ = & \operatorname{rank}\left(F(x), F_* e_1, \cdots, F_* e_{n-1}, F_* \nu\right) \\ = & \operatorname{rank}\left(F(x), F_* e_1, \cdots, F_* e_{n-1}\right) = n, \end{aligned}$$

we see $F(x) \neq 0$ on $\partial\Omega$. Moreover, it follows from the fact

$$\operatorname{rank}\left(F(x), \partial_1 F(x), \cdots, \partial_n F(x)\right) = n \text{ for any } x \in \overline{\Omega}$$

that the zeroes of F in Ω must be isolated, hence only finitely many, say $x_1, \cdots, x_m$. Then for $\varepsilon > 0$ small enough, let

$$U = \Omega \backslash \bigcup_{i=1}^{m} \overline{B_\varepsilon(x_i)},$$

we know

$$\operatorname{rank}\left(F(x), \partial_1 F(x), \cdots, \partial_n F(x)\right) = n \text{ for any } x \in \overline{U}$$

and

$$\dim \operatorname{span}\left\{F(x), \operatorname{im}\left(F|_{\partial U}\right)_{*,x}\right\} = n \text{ for any } x \in \partial U.$$

Let

$$H(x) = \frac{F(x)}{|F(x)|} \text{ for } x \in \overline{U},$$

then clearly

$$\operatorname{rank}\left(\partial_1 H(x), \cdots, \partial_n H(x)\right) = n - 1 \text{ for any } x \in \overline{U}$$

and

$$\dim \operatorname{im}\left(H|_{\partial U}\right)_{*,x} = n - 1 \text{ for any } x \in \partial U.$$

It follows from the Lemma 2 that $\overline{U}$ must be diffeomorphic to $\overline{B}_2 \backslash B_1$, hence $\overline{\Omega}$ must be diffeomorphic to either $\overline{B}_1$ or $\overline{B}_2 \backslash B_1$. □

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