

ON APPLICABILITY OF THE SPARSE GRID METHOD IN THE WORST CASE SETTING

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ABSTRACT. Let $T_k : W_p^{r_k} \rightarrow W_q^{s_k}$ be bounded linear operators. We provide several sufficient conditions for the validity of the inequality $\|\otimes_k T_k\| \leq \prod_k \|T_k\|$. These results can be applied to error and cost estimates for the sparse grid method.

Key words: Sparse grid method, Smolyak's algorithm, information-based complexity

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1. INTRODUCTION

The sparse grid method is a useful tool in the construction of algorithms for multivariate problems such as approximation and integration. This method is based on algorithms for univariate cases, and can be applied to different types of problems. It has several variations, including the Smolyak algorithm [4] and the weighted tensor product algorithm [8].

Let us describe the Smolyak algorithm. We adopt the following notations. The set of nonnegative integers is denoted by \mathbb{N} , and \mathbb{N}_+ denotes $\mathbb{N} - \{0\}$. If α is a vector, we denote its j -th component by α_j . Now we consider the problem of approximating the tensor product operator $S = \otimes_{k=1}^d S_k : \otimes_{k=1}^d F_k \rightarrow \otimes_{k=1}^d G_k$, where $S_k : F_k \rightarrow G_k$ are linear operators between normed linear spaces of functions. Assume that for every fixed k , the operator S_k is approximated by a sequence of algorithms A_k^i such that $\|A_k^i - S_k\| \rightarrow 0$ as $i \rightarrow \infty$. Let $q \in \mathbb{N}_+$ and $w = (w_1, \dots, w_d)$ for real numbers $w_k > 0$. Then we take

$$(1) \quad A_w(q, d) = \sum_{\alpha \in Q_w(q, d)} \Delta_1^{\alpha_1} \otimes \dots \otimes \Delta_d^{\alpha_d}$$

as an approximation for S . Here $\Delta_k^i = A_k^i - A_k^{i-1}$, where $A_k^0 = 0$, and $Q_w(q, d) = \{\alpha \in \mathbb{N}_+^d : \sum_{k=1}^d w_k \alpha_k \leq q\}$.

The sparse grid method has been applied to integration and approximation for multivariate functions with bounded mixed derivatives in many papers [1, 3, 5, 7, 8]. We are interested in the worst case performance such as the error and cost of the method; see Section 2 for relevant concepts. Many estimates on the error and cost depend crucially on the following inequality

$$(2) \quad \|T_1 \otimes \dots \otimes T_d\| \leq \prod_{k=1}^d \|T_k\|$$

for $T_k = \Delta_k^i = A_k^i - A_k^{i-1}$. In fact, if (2) is true, then we have the explicit error bound on the error of the algorithm $A_w(q, d)$ [2]:

$$e(A_w(q, d)) \leq C \cdot \exp(-q + a) \cdot \exp_d(q - a),$$

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where a and C are constants depending on A_k^i , and $\exp_d(a) = \sum_{k=0}^{d-1} \frac{1}{k!} a^k$. Here certain assumptions on $\|\Delta_k^i\|$ are needed for the estimate (see Section 4). Another consequence of the validity of (2) and these assumptions is as follows. Assume $T_1 = \dots = T_d$ and $w_1 = \dots = w_d = 1$. Then under the error demand $e(A_w(q, d)) \leq \varepsilon$, we have the explicit bound on the cost of the algorithm [7]:

$$\text{cost}(A_w(q, d)) \leq \beta_1 \left(\beta_2 + \beta_3 \frac{\ln 1/\varepsilon}{d-1} \right)^{\beta_4(d-1)} \varepsilon^{-\beta_5}.$$

Here, the β_i are constants determined by A_k^i .

The main aim of this article is to look for reasonable conditions under which (2) is satisfied. It is well known that the inequality is true when F_k and G_k are Hilbert spaces. Moreover, in [2, 3], (2) is verified for some very special function spaces. Here we look at the practically important case when $F_k = W_p^{r_k}(I_k)$ and $G_k = W_q^{s_k}(J_k)$. Here I_k and J_k are intervals; we assume $0 \in I_k \cap J_k$ without loss of generality.

For convenience, we use the following notations. Given $\alpha, \beta \in \mathbb{N}^d$, $|\alpha|$ denotes $\sum_{k=1}^d \alpha_k$, and $\alpha \leq \beta$ means $\alpha_k \leq \beta_k$ for each $1 \leq k \leq d$. If $f = f(x_1, \dots, x_d)$, then $f^{(\alpha)}$ denotes $\partial_{x_1}^{\alpha_1} \dots \partial_{x_d}^{\alpha_d} f$.

For $r \in \mathbb{N}^d$, we have

$$\otimes_{k=1}^d F_k = W_p^r(I_1 \times \dots \times I_d)$$

which is the completion of $C_c^\infty(I_1 \times \dots \times I_d)$ with respect to the norm

$$(3) \quad \|f\| = \begin{cases} \left(\sum_{\alpha \leq r} \|f^{(\alpha)}([x_\alpha])\|_{L_p(I_\alpha)}^p \right)^{1/p} & \text{if } p < \infty, \\ \max_{\alpha \leq r} \|f^{(\alpha)}([x_\alpha])\|_{L_\infty(I_\alpha)} & \text{if } p = \infty. \end{cases}$$

Here $[x_\alpha]$ denotes the vector whose j -th component is 0 or x_j according to whether α_j is less than r_j or equal to r_j , and I_α denotes the set $\{[x_\alpha] : x \in I_1 \times \dots \times I_d\}$. Obviously $W_p^0 = L_p$ for $p < \infty$ and $W_\infty^0 = C_0$, which is the space of continuous functions that vanish at infinity.

The concept of nonnegative algorithms plays an important role in deriving (2), and its definition will be given in Section 2. We emphasize that the nonnegativity we will use is essentially the same as the usual nonnegativity of operators from $L_p(X)$ to $L_q(Y)$ for some measure spaces X and Y , and many common algorithms do satisfy such conditions (see Section 4).

Now we may describe the main results of this paper. Assuming that $T_k : F_k \rightarrow G_k$ are bounded linear operators, we may define the operator $\otimes_{k=1}^d T_k : \otimes_{k=1}^d F_k \rightarrow \otimes_{k=1}^d G_k$ (see Section 2). Then under the norm (3), the inequality (2) holds in the following two cases (see Theorem 1):

- (i) $1 \leq p \leq \infty$, $1 \leq q \leq \infty$ and all T_k are nonnegative.
- (ii) $1 \leq p \leq q \leq \infty$ (and T_k are arbitrary).

Besides the norm (3), we have the following useful norm

$$(4) \quad \|f\| = \begin{cases} \left(\sum_{\alpha \leq r} \|f^{(\alpha)}\|_p^p \right)^{1/p} & \text{if } p < \infty, \\ \max_{\alpha \leq r} \|f^{(\alpha)}\|_\infty & \text{if } p = \infty. \end{cases}$$

We will show that (2) holds in the following two additional cases (see Proposition 1):

- (iii) $1 \leq p \leq q \leq \infty$ and the norm (4) is used for the domain and range spaces.
- (iv) $1 \leq p \leq q \leq \infty$, and we use the norm (4) for the domain space and the norm (3) for the range space (or vice versa).

Here, by saying that the norm (4) (or (3)) is used for the domain space, of course we also assume every F_k uses the same norm (4) (or (3)) with $d = 1$ and $r = r_k$. The same applies to the range space. The convention will be used for the rest of the paper.

The results above, that the inequality (2) holds in cases (i) to (iv), extend earlier works in [2, 3]. Indeed, in [2, 3], it was shown that (2) holds for the cases of $p = 1$ with $1 \leq q \leq \infty$, and $q = \infty$ with $1 \leq p \leq \infty$.

Section 2 of this paper contains basic concepts related to our results. In Section 3, we prove the main results. In Section 4, we discuss some applications.

2. SOME PREPARATIONS

In this section we recall some concepts from information-based complexity. We will also describe basics of partial orders on vector spaces and tensor products.

2.1. Information-based complexity. Following [6], let F and G be normed linear spaces of real functions of d variables, and $S : F \rightarrow G$ be a bounded linear operator. We are interested in algorithms approximating S . Of particular interests are the *function approximation* problem and the *multivariate integration* problem. The first problem deals with the approximation of the embedding operator $S(f) = f$. The second problem corresponds to the approximation of the integration operator $S(f) = \int_{[0,1]^d} f(\mathbf{x}) d\mathbf{x}$.

A linear operator $A : F \rightarrow G$ is called a (*nonadaptive*) *linear algorithm* if there exist $g_i \in G$ and $L_i \in F^*$, $i = 1, \dots, n$, such that

$$(5) \quad A(f) = \sum_{i=1}^n L_i(f)g_i \quad \text{for } f \in F.$$

Here n is called the *information cost*, or simply, the *cost* of A . The *worst case error* of A is

$$e(A) = \sup_{\|f\|_F \leq 1} \|A(f) - S(f)\| = \|A - S\|.$$

2.2. Partial orders and nonnegative operators. A partial order \geq_F on a linear space F is called *compatible* with the linear structure if the following holds: $f \geq_F g$ iff $f - g \geq_F 0$, and $f \geq_F 0$ implies $af \geq_F 0$ for $a \geq 0$. A natural partial order on $F = L_p(X)$ is defined as $f \in F, f \geq_F 0$ iff $f(x) \geq 0$ a.e. $x \in X$. A natural partial order on $F = C(X)$ is given by $f \in F, f \geq_F 0$ iff $f(x) \geq 0$ for every $x \in X$.

An operator $T : (F, \leq_F) \rightarrow (G, \leq_G)$ is called *nonnegative* if $f \geq_F 0$ implies $Tf \geq_G 0$. In particular, the algorithm A in (5) is nonnegative if L_i are nonnegative linear functionals and g_1, g_2, \dots, g_n from G are nonnegative elements.

2.3. Tensor products. Let F_k be linear spaces of real functions over sets X_k . The *algebraic tensor product* $F_1 \otimes_A \dots \otimes_A F_d$ is defined to be the subspace of the space of real functions over $X_1 \times \dots \times X_d$ spanned by $\{\prod_{k=1}^d f_k(x_k) : f_k \in F_k\}$. It is easy to see that $W_p^{r_1}(I_1) \otimes_A \dots \otimes_A W_p^{r_d}(I_d)$ is dense in $W_p^r(I_1 \times \dots \times I_d)$. Moreover, the norms (3) and (4) are *cross-norms*, that is, $\|\prod_{k=1}^d f_k(x_k)\| = \prod_{k=1}^d \|f_k(x_k)\|$.

Now let $T_k : F_k \rightarrow G_k$ be linear operators. We denote by $T_1 \otimes \dots \otimes T_d$ the linear operator from $F_1 \otimes_A \dots \otimes_A F_d$ to $G_1 \otimes_A \dots \otimes_A G_d$ such that $(T_1 \otimes \dots \otimes T_d)(\prod_{k=1}^d f_k(x_k)) = \prod_{k=1}^d T_k f_k$. The norm $\|\otimes_{k=1}^d T_k\|$ is then determined by the norms on $\otimes_{k=1}^d F_k$ and $\otimes_{k=1}^d G_k$. If $F_k = W_p^{r_k}(I_k)$, $G_k = W_q^{s_k}(J_k)$ and T_k are bounded, then $\otimes_{k=1}^d T_k$ can be uniquely extended to a bounded linear operator from $W_p^{r_1, \dots, r_d}(I_1 \times \dots \times I_d)$ to $W_q^{s_1, \dots, s_d}(J_1 \times \dots \times J_d)$.

3. MAIN RESULTS

First let us consider the norm (3) on the tensor product spaces. We will use the strategy of [2] to establish a natural isometry between W_p^r and an L_p or C_0 space (see Lemma 1 below). For the norm (4), an embedding into an L_p or C_0 space is established. These identifications reduce the problems for W_p^r spaces to L_p or C_0 spaces, which we will handle directly.

For convenience, we introduce the following notation that will be used for the rest of the paper:

$$LC_p(X) := \begin{cases} L_p(X) & \text{if } p < \infty, \\ C_0(X) & \text{if } p = \infty. \end{cases}$$

Lemma 1. *Assume the tensor product spaces are equipped with the norm (3). Let $d \in \mathbb{N}_+$ and $r_k \in \mathbb{N}$ for $k = 1, \dots, d$. Then we may find a locally compact space $M(I_1, \dots, I_d, r_1, \dots, r_d)$ with a Borel measure having the following properties:*

- (i) $M(I_1, r_1) \times \dots \times M(I_d, r_d) = M(I_1, \dots, I_d, r_1, \dots, r_d)$ as measure spaces and topological spaces.
- (ii) There is an isometry

$$u_{r_1, \dots, r_d, I_1, \dots, I_d} : W_p^{r_1, \dots, r_d}(I_1 \times \dots \times I_d) \rightarrow LC_p(M(I_1, \dots, I_d, r_1, \dots, r_d))$$

that is natural, in the sense that

$$u_{r_1, I_1} \otimes \dots \otimes u_{r_d, I_d} = u_{r_1, \dots, r_d, I_1, \dots, I_d} \quad \text{on } W_p^{r_1}(I_1) \otimes_A \dots \otimes_A W_p^{r_d}(I_d).$$

Proof. Let $Z_k = \{n_0, \dots, n_{r_k-1}\}$ be a set with r_k elements. We assume Z_k is disjoint from I_k , and is equipped with the discrete topology and counting measure. For $S \subset U_d = \{1, \dots, d\}$, let $M(S) = A_1 \times \dots \times A_d$, where $A_k = I_k$ if $k \in S$ and $A_k = Z_k$ if $k \notin S$. We define

$$M = M(I_1, \dots, I_d, r_1, \dots, r_d) = \bigcup_{S \subset U_d} M(S).$$

Note that the union is disjoint. Each $M(S)$ is equipped with the product topology and product measure. The topology and measure on M is defined in an obvious way. It is easy to see that (i) is true.

For (ii), let $f \in W_p^{r_1, \dots, r_d}(I_1 \times \dots \times I_d)$. We must define $u(f) = u_{r_1, \dots, r_d, I_1, \dots, I_d}(f)$ on M . It suffices to define $u(f)$ on $M(S)$ for every $S \subset U_d$. For that end, let

$$R_S = \{\alpha \in \mathbb{N}^d : \alpha_k < r_k \text{ for } k \notin S \text{ and } \alpha_k = r_k \text{ for } k \in S\}.$$

Let $z \in M(S)$. Then there exists a unique $\alpha \in R_S$ such that $z_k = n_{\alpha_k}$ for $k \notin S$. We take $u(f)(z) = f^{(\alpha)}([z_\alpha])$. Obviously u is linear. Now let $p < \infty$. Then $\|u(f)|_{M(S)}\|_p = (\sum_{\alpha \in R_S} \|f^{(\alpha)}([z_\alpha])\|_{L_p(I_\alpha)}^p)^{1/p}$. Notice that $\{\alpha \in \mathbb{N}^d : |\alpha| \leq r\} = \cup_{S \subset U_d} R_S$, and that the union is disjoint. Thus $\|u(f)\|_p = (\sum_{S \subset U_d} \|u(f)|_{M(S)}\|_p^p)^{1/p} = \|f\|$. The same holds for $p = \infty$. Thus u is an isometric embedding, and hence the image of u is a closed subspace of $LC_p(M)$. It is easy to check that $u(\prod_k f_k(x_k)) = \prod_k u_{r_k, I_k}(f_k)$, i.e., u is natural.

It remains to show that u is surjective. We first show that u_{r_k, I_k} is surjective for every k . Let any $g \in LC_p(M(I_k, r_k))$. We define a function $f_g \in W_p^{r_k}(I_k)$ as follows. If $r_k > 0$, we take

$$f_g(x_k) = \sum_{j=0}^{r_k-1} g(n_j) x_k^j / j! + \int_0^{x_k} \frac{(x_k - t_k)^{r_k-1}}{(r_k-1)!} g(t_k) dt_k,$$

and if $r_k = 0$, $f_g(x_k)$ is defined to be $g_k(x_k)$. It is easy to check that $u_{r_k, I_k}(f_g) = g$. This shows that u_{r_k, I_k} is surjective. Now we have

$$\begin{aligned} u(W_p^{r_1}(I_1) \otimes_{\mathbb{A}} \cdots \otimes_{\mathbb{A}} W_p^{r_d}(I_d)) &= u_{r_1, I_1}(W_p^{r_1}(I_1)) \otimes_{\mathbb{A}} \cdots \otimes_{\mathbb{A}} u_{r_d, I_d}(W_p^{r_d}(I_d)) \\ &= LC_p(M(I_1, r_1)) \otimes_{\mathbb{A}} \cdots \otimes_{\mathbb{A}} LC_p(M(I_d, r_d)), \end{aligned}$$

which is dense in $LC_p(M)$. It follows that the image of u is also dense in $LC_p(M)$. Being closed and dense in $LC_p(M)$, the image of u is $LC_p(M)$, i.e., u is surjective. \square

Next we introduce a partial order on W_p^r that is compatible with the linear structure.

Definition 1. *The isometry $u_{r_1, \dots, r_d, I_1, \dots, I_d}$ from Lemma 1 defines a compatible partial order \geq_F on the space $F = W_p^{r_1, \dots, r_d}(I_1 \times \cdots \times I_d)$, namely*

$$(6) \quad f \in F, f \geq_F 0 \text{ iff } \begin{cases} f^{(\alpha)}([x_\alpha]) \geq 0 \text{ for a.e. } x_\alpha, \alpha \leq r, & \text{if } p < \infty, \\ f^{(\alpha)}([x_\alpha]) \geq 0 \text{ for } x_\alpha \in I_\alpha, \alpha \leq r, & \text{if } p = \infty. \end{cases}$$

Note that $u_{r_1, \dots, r_d, I_1, \dots, I_d}$ is an order-preserving isometry.

Regarding the norm (4), we have the following lemma.

Lemma 2. *Assume that the tensor product spaces are equipped with the norm (4). Let $d \in \mathbb{N}_+$ and $r_k \in \mathbb{N}$ for $k = 1, \dots, d$. Then we may find a locally compact space $N(I_1, \dots, I_d, r_1, \dots, r_d)$ with a Borel measure having the following properties:*

- (i) $N(I_1, r_1) \times \cdots \times N(I_d, r_d) = N(I_1, \dots, I_d, r_1, \dots, r_d)$ as measure spaces and topological spaces.
- (ii) *There is an isometric embedding*

$$v_{r_1, \dots, r_d, I_1, \dots, I_d} : W_p^{r_1, \dots, r_d}(I_1 \times \cdots \times I_d) \rightarrow LC_p(N(I_1, \dots, I_d, r_1, \dots, r_d))$$

that is natural, in the sense that

$$v_{r_1, I_1} \otimes \cdots \otimes v_{r_d, I_d} = v_{r_1, \dots, r_d, I_1, \dots, I_d} \quad \text{on } W_p^{r_1}(I_1) \otimes_{\mathbb{A}} \cdots \otimes_{\mathbb{A}} W_p^{r_d}(I_d).$$

Proof. Let $N(I_1, \dots, I_d, r_1, \dots, r_d) = I_1 \times Z_{r_1} \times \cdots \times I_d \times Z_{r_d}$, where $Z_m = \{0, 1, \dots, m\}$, which is equipped with the discrete topology and counting measure. Then (i) is clearly true under this construction. For $f \in W_p^{r_1, \dots, r_d}(I_1 \times \cdots \times I_d)$ and $(x_1, \alpha_1, \dots, x_d, \alpha_d) \in N(I_1, \dots, I_d, r_1, \dots, r_d)$, we define

$$v_{r_1, \dots, r_d, I_1, \dots, I_d}(f)(x_1, \alpha_1, \dots, x_d, \alpha_d) = D^\alpha f(x_1, \dots, x_d), \text{ where } \alpha = (\alpha_1, \dots, \alpha_d).$$

Then (ii) can be easily verified. \square

Definition 2. *A Borel measure on a locally compact space X is called normal if the measure of any nonempty open set of X is nonzero.*

The product and (disjoint) union measures of two normal measures are normal. Hence the measures on $M(I_1, \dots, I_d, r_1, \dots, r_d)$ and $N(I_1, \dots, I_d, r_1, \dots, r_d)$ are normal. Notice that if dx is a normal measure on X then $\|f\|_{C_0} = \|f\|_\infty$ (with respect to dx) for $f \in C_0(X)$. Thus

$$(7) \quad \|f\|_{LC_p} = \|f\|_p \quad \forall p : 1 \leq p \leq \infty \quad \forall f \in LC_p(X).$$

Now we proceed to study some inequalities in L_p spaces. First, we prove a version of the generalized Minkowski inequality.

Lemma 3. *Let (X, dx) and (Y, dy) be two measure spaces. Assuming $1 \leq p \leq q \leq \infty$, then for any measurable functions $f : X \times Y \rightarrow \mathbb{R} \cup \{\infty, -\infty\}$ we have*

$$\| \|f\|_{L_{p,x}} \| \|f\|_{L_{q,y}} \leq \| \|f\|_{L_{q,y}} \| \|f\|_{L_{p,x}},$$

with equality holding if $p = q$.

Proof. Without loss of generality, we may assume f is nonnegative.

The case $p = 1$ is the usual generalized Minkowski inequality. The case $p = q$ follows easily from the Fubini theorem. Hence we assume $1 < p < q \leq \infty$. Then

$$\begin{aligned} \|\|f\|_{L_{p,x}}\|_{L_{q,y}} &= \|\|f^p\|_{L_{1,x}}^{1/p}\|_{L_{q,y}} = \|\|f^p\|_{L_{1,x}}\|_{L_{q/p,y}}^{1/p} \\ &\leq \|\|f^p\|_{L_{q/p,y}}\|_{L_{1,x}}^{1/p} = \|\|f\|_{L_{q,y}}^p\|_{L_{1,x}}^{1/p} = \|\|f\|_{L_{q,y}}\|_{L_{p,x}}, \end{aligned}$$

where we have used the usual generalized Minkowski inequality in the third step. \square

Next we derive a useful inequality under the nonnegativity assumption.

Lemma 4. *Let X, Y and Z be locally compact spaces with Borel measures dx, dy and dz , respectively. Let $1 \leq p \leq \infty, 1 \leq q \leq \infty$ and $1 \leq r \leq \infty$. Assume that $T : LC_p(X) \rightarrow LC_q(Y)$ is a nonnegative bounded linear operator, and that f is a measurable function over $X \times Z$ such that*

$$\|\|f\|_{p,x}\|_{r,z} < \infty.$$

Then

$$\|T(f(\cdot, z))(y)\|_{r,z} \leq T(\|f(\cdot, z)\|_{r,z})(y) \text{ a.e. } y \in Y.$$

Proof. Let r' be the conjugate power of r , that is, $\frac{1}{r} + \frac{1}{r'} = 1$. Let $u(y, z) = T(f(\cdot, z))(y)$. Then $\|\|u\|_{q,y}\|_{r,z} \leq \|T\| \|\|f\|_{p,x}\|_{r,z} < \infty$, which implies that $u(y, z) \in L_r Z$ for a.e. $y \in Y$. Thus

$$\|u(y, z)\|_{r,z} = \sup_{\|\psi(z)\|_{r',1}=1} \int_Z u(y, z)\psi(z) dz \text{ a.e. } y \in Y.$$

For any nonnegative function $\varphi \in L_1(Y) \cap L_\infty(Y)$, we have

$$\begin{aligned} &\int_Y \|T(f(\cdot, z))(y)\|_{r,z} \varphi(y) dy \\ &= \int_Y \left(\sup_{\|\psi\|_{r',1}=1} \int_Z T(f(\cdot, z))(y)\psi(z) dz \right) \varphi(y) dy \\ &= \int_Y \sup_{\|\psi\|_{r',1}=1} T\left(\int_Z f(\cdot, z)(y)\psi(z) dz \right) \varphi(y) dy \\ &\leq \int_Y T\left(\sup_{\|\psi\|_{r',1}=1} \int_Z f(\cdot, z)(y)\psi(z) dz \right) \varphi(y) dy \\ &= \int_Y T(\|f(\cdot, z)\|_{L_{r,z}}) \varphi(y) dy. \end{aligned}$$

The inequality above follows from the nonnegativity of T . This yields that $\|T(f(\cdot, z))(y)\|_{r,z} \leq T(\|f(\cdot, z)\|_{r,z})(y)$ a.e. $y \in Y$. \square

Now we may state and prove our main theorem.

Theorem 1. *Let $1 \leq p \leq \infty$ and $1 \leq q \leq \infty$. Let r_k and s_k be nonnegative integers, and T_k be bounded linear operators from $F_k = W_p^{r_k}(I_k)$ to $G_k = W_q^{s_k}(J_k)$ for $k = 1, \dots, d$. Assume the norm (3) is used in this theorem.*

(i) *If at least $d - 1$ operators of T_1, \dots, T_d are nonnegative then*

$$\|T_1 \otimes \cdots \otimes T_d\| = \prod_{k=1}^d \|T_k\|.$$

(ii) If $p \leq q$ then for arbitrary T_k we have

$$\|T_1 \otimes \cdots \otimes T_d\| = \prod_{k=1}^d \|T_k\|.$$

Proof. Since the norms on the domain and range spaces are cross-norms (see Section 2), it is easy to show that

$$\|T_1 \otimes \cdots \otimes T_d\| \geq \prod_{k=1}^d \|T_k\|.$$

To obtain the opposite inequality, we need Lemma 1. Let $X_k = M(I_k, r_k)$ and $Y_k = M(J_k, s_k)$. Using the isometries u_{r_k, I_k} and u_{s_k, J_k} , we define an operator U_k from $LC_p(X_k)$ to $LC_q(Y_k)$ as $U_k = u_{s_k, J_k} T_k u_{r_k, I_k}^{-1}$. Obviously, $\|U_k\| = \|T_k\|$. Let $T = \otimes_{k=1}^d T_k$ and $U = \otimes_{k=1}^d U_k$. Then the property (ii) in Lemma 1 implies that $T u_{r_1, \dots, r_d, I_1, \dots, I_d} = u_{s_1, \dots, s_d, J_1, \dots, J_d} U$. It follows that $\|T\| = \|U\|$ and thus $\|T\| \leq \prod_k \|T_k\|$ iff $\|U\| \leq \prod_k \|U_k\|$. Moreover, for each k , T_k is nonnegative iff U_k is nonnegative since u_{r_k, I_k} and u_{s_k, J_k} are order preserving.

The arguments above show that it suffices to prove the opposite inequality for tensor products of operators from LC_p to LC_q . Now assume T_k are bounded linear operators from $F_k = LC_p(X_k)$ to $G_k = LC_q(Y_k)$ for some locally compact spaces X_k and Y_k with normal Borel measures, for $k = 1, \dots, d$. Notice that $LC_\infty = C_0$, which is not L_∞ in general. However, according to (7), we have the following formulas for the norms with $1 \leq p \leq \infty$ and $1 \leq q \leq \infty$:

$$\|T_k\| = \sup_{0 \neq f_k \in F_k} \|T_k(f_k)\|_q / \|f_k\|_p$$

and

$$\|\otimes_{k=1}^d T_k\| = \sup_{0 \neq f \in F_1 \otimes_A \cdots \otimes_A F_d} \|(\otimes_{k=1}^d T_k)(f)\|_q / \|f\|_p.$$

For (i), we only need to handle the case $d = 2$, the general case follows from this by a simple induction argument. Without loss of generality, we may assume T_2 is nonnegative. For each $f \in F_1 \otimes_A F_2$ we define a function \bar{f} on $X_1 \times Y_2$ as

$$(8) \quad \bar{f}(x_1, y_2) = T_2(f(x_1, \cdot))(y_2).$$

Then

$$(9) \quad (T_1 \otimes T_2)(f)(y_1, y_2) = T_1(\bar{f}(\cdot, y_2))(y_1).$$

For any $f \in F_1 \otimes_A F_2$, we have

$$\begin{aligned} & \| (T_1 \otimes T_2)(f)(y_1, y_2) \|_{L_{q, y_1, y_2}} \\ &= \| \| T_1(\bar{f}(\cdot, y_2))(y_1) \|_{L_{q, y_1}} \|_{L_{q, y_2}} \\ &\leq \| T_1 \| \| \| \bar{f}(x_1, y_2) \|_{L_{p, x_1}} \|_{L_{q, y_2}} \\ &= \| T_1 \| \| \| T_2(f(x_1, \cdot))(y_2) \|_{L_{p, x_1}} \|_{L_{q, y_2}} \\ &\leq \| T_1 \| \| T_2 (\| f(x_1, \cdot) \|_{L_{p, x_1}}) (y_2) \|_{L_{q, y_2}} \quad (\text{by Lemma 4}) \\ &\leq \| T_1 \| \| T_2 \| \| \| f(x_1, x_2) \|_{L_{p, x_2}} \|_{L_{p, x_1}} \\ &= \| T_1 \| \| T_2 \| \| f \|_{L_{p, x_1, x_2}}. \end{aligned}$$

Hence $\|T_1 \otimes T_2\| \leq \|T_1\| \|T_2\|$.

For (ii), let $p \leq q$. Again, we only need to deal with the case $d = 2$, from which the general case follows by induction. Let $f \in F_1 \otimes_A F_2$. Using (8) and (9) we have

$$\begin{aligned}
& \| (T_1 \otimes T_2)(f)(y_1, y_2) \|_{L_{q, y_1, y_2}} \\
&= \| \| T_1(\bar{f}(\cdot, y_2))(y_1) \|_{L_{q, y_1}} \|_{L_{q, y_2}} \\
&\leq \| T_1 \| \| \| \bar{f}(x_1, y_2) \|_{L_{p, x_1}} \|_{L_{q, y_2}} \\
&\leq \| T_1 \| \| \| \bar{f}(x_1, y_2) \|_{L_{q, y_2}} \|_{L_{p, x_1}} \quad (\text{by Lemma 3}) \\
&= \| T_1 \| \| \| T_2(f(x_1, \cdot))(y_2) \|_{L_{q, y_2}} \|_{L_{p, x_1}} \\
&\leq \| T_1 \| \| T_2 \| \| \| f(x_1, x_2) \|_{L_{p, x_2}} \|_{L_{p, x_1}} \\
&= \| T_1 \| \| T_2 \| \| \| f \|_{L_{p, x_1, x_2}}.
\end{aligned}$$

Hence $\| T_1 \otimes T_2 \| \leq \| T_1 \| \| T_2 \|$. \square

Now we consider the norm (4) and cases when both (3) and (4) are used.

Proposition 1. *Assume that F_k, G_k and T_k are the same as in Theorem 1 for $k = 1, \dots, d$, and that $1 \leq p \leq q \leq \infty$.*

- (i) *If the norm (4) is used for the domain and range spaces, then $\| T_1 \otimes \dots \otimes T_d \| = \prod_{k=1}^d \| T_k \|$.*
- (ii) *If we use the norm (4) for the domain space and (3) for the range space or the other way, then $\| T_1 \otimes \dots \otimes T_k \| = \prod_{k=1}^d \| T_k \|$.*

Proof. We need only to prove $\| \otimes_{k=1}^d T_k \| \leq \prod_{k=1}^d \| T_k \|$ as both (3) and (4) are cross-norms. By Lemmas 1 and 2, we see that under either of the assumptions in (i) and (ii), we may identify F_k and G_k as Banach subspaces of $LC_p(X_k)$ and $LC_q(Y_k)$, respectively, for some locally compact spaces X_k and Y_k with normal Borel measures. As the identifications are natural, $\otimes_{k=1}^d F_k$ and $\otimes_{k=1}^d G_k$ can be identified as Banach subspaces of $LC_p(X_1 \times \dots \times X_d)$ and $LC_p(Y_1 \times \dots \times Y_d)$, respectively. Under these identifications, let us assume $d = 2$ and $f \in F_1 \otimes F_2$. Then we can use the exactly same proof for Part (ii) of Theorem 1 to show that

$$\| (T_1 \otimes T_2)(f) \|_{L_q} \leq \| T_1 \| \| T_2 \| \| \| f \|_{L_p}.$$

Thus $\| T_1 \otimes T_2 \| \leq \| T_1 \| \| T_2 \|$, and the general case follows from this by a simple induction argument. \square

4. SOME APPLICATIONS

In this section, we discuss how to efficiently use the results from Section 3 for estimating the error and cost of the sparse grid method for approximating $S = \otimes_{k=1}^d S_k : W_p^{r_1, \dots, r_d}(I_1 \times \dots \times I_d) \rightarrow W_q^{s_1, \dots, s_d}(J_1 \times \dots \times J_d)$, where $S_k : W_p^{r_k}(I_k) \rightarrow W_q^{s_k}(J_k)$ are bounded linear operators. By Theorem 1 and Proposition 1, we have satisfactory conclusions for $1 \leq p \leq q \leq \infty$. For $p > q$, we shall focus on nonnegative algorithms, which implies that the norm (3) will be used. For simplicity of the presentation, we assume $p < \infty$ and $s_k = 0$. Other cases may be treated in a similar way.

The following simple statement shows there are many nonnegative algorithms.

Lemma 5. *Let I and J be intervals ($0 \in I \cap J$), $r \in \mathbb{N}_+$ and $A : F = W_p^r(I) \rightarrow L_q(J)$ be a linear algorithm that uses function evaluation i.e., $A(f) = \sum_j f(x_j)g_j$ for some fixed $x_j \in I$ and $g_j \in L_q(J)$. If $g_j \geq 0$ a.e., then A is nonnegative.*

Proof. Let $f \in F$, $f \geq_F 0$. By Definition 1, we have $f^{(k)}(0) \geq 0$ for $k < r$, and $f^{(r)} \geq 0$ a.e.. It follows from

$$f^{(k)}(x) = f^{(k)}(0) + \int_0^x f^{(k+1)}(t)dt$$

and a simple induction argument that $f(x) \geq 0$ for $x \in I$. Thus $A(f) = \sum_j f(x_j)g_j \geq 0$ a.e.. This implies the lemma. \square

The Smolyak algorithm $A_w(q, d)$ in (1) makes use of the tensor product operator $\Delta_1^{\alpha_1} \otimes \dots \otimes \Delta_d^{\alpha_d}$ with $\Delta_k^i = A_k^i - A_k^{i-1}$ and $A_k^0 = 0$ for $k = 1, \dots, d$. Now we discuss conditions under which the algorithms Δ_k^i are nonnegative.

For convenience, for fixed k , we call a sequence of algorithms $(A_k^i)_{i=1}^\infty$ from $W_p^{r_k}(I_k)$ to $L_q(J_k)$ *nested* if there exists a sequence of information $(L_{k,j})_{j=1}^\infty$, a sequence of functions $(g_{k,j})_{j=1}^\infty$, and an increasing sequence $(s_k(i))_{i=1}^\infty$ of integers such that

$$A_k^i(f) = \sum_{j=1}^{s_k(i)} L_{k,j}(f)g_{k,j} \quad \text{for } f \in W_p^{r_k}(I_k).$$

It is easy to see that a sufficient condition for the algorithms Δ_k^i to be nonnegative for all $i \geq 1$ is that the sequence $(A_k^i)_{i=1}^\infty$ be nested with nonnegative $L_{k,j}$ and $g_{k,j}$. In this case, we have

$$\Delta_k^i(f) = \sum_{j=s_k(i-1)+1}^{s_k(i)} L_{k,j}(f)g_{k,j} \quad (s_k(0) = 0).$$

On the other hand, when applying the sparse grid method, the property

$$(10) \quad \|\Delta_k^i\| \leq C_k D_k^i \quad \text{for some constants } C_k > 0, D_k < 1 \text{ and for all } i \geq 1$$

is usually needed for every fixed k (see [2, 3, 7, 8]). The property (10) and the nonnegativeness of Δ_k^i are satisfied simultaneously if we generate the algorithms A_k^i as follows. Choose a nested sequence of nonnegative algorithms $(B_k(n))_{n=1}^\infty$ such that $\|B_k(n) - S_k\| = O(n^{-v_k})$ for some constant $v_k > 0$ as $n \rightarrow \infty$. Then we let

$$A_k^i = B_k(t(i)),$$

where $t(i) = c_1 a^i + c_2 \in \mathbb{N}_+$ ($\forall i \geq 1$) for some fixed constants $a > 1$, c_1 , and c_2 . For example, we may choose $t(i) = 2^{i-1}$, $t(i) = (3^i - 1)/2$, or $t(i) = (4^i - 1)/3$.

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