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TOPOLOGY OF SOBOLEV MAPPINGS IV

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Abstract. We will classify the path connected components of spaces of Sobolev maps between manifolds and study the strong and weak density of smooth maps in the spaces of Sobolev maps for the case the domain manifold has nonempty boundary and Dirichlet problems.

1. **Introduction.** This is a sequel to [6, 7, 8]. The main aim is to classify the path connected components of spaces of Sobolev maps between manifolds and study the density of smooth maps in the space of Sobolev maps similar to those in [7] for the case domain manifold has boundary and Dirichlet problems. We will emphasize those analytical techniques different from the no boundary cases. These techniques have their own interest in variational problems for maps between manifolds.

2. **Some preparations.** Let M be a smooth compact Riemannian manifold. At the beginning of section 2 of [7], we used Lipschitz triangulation to mean that one has a simplicial complex K and a bi-Lipschitz map $h : |K| \to M$. Sometime the name "Lipschitz triangulation" can be confused as "a triangulation which is Lipschitz". To avoid this, we will use the name "bi-Lipschitz triangulation" instead. Similar conventions apply to bi-Lipschitz cubeulation and bi-Lipschitz rectilinear cell decomposition. For basics about rectilinear cell complex and simplicial complex, we refer readers to [17] (appendix II) and [10]. For basics of homotopy theory, we refer readers to [16].

Let (X, A) be a pair of topological spaces, that is, X be a topological space and A be a subspace of X, Y be a topological space and $f \in C(A, Y)$, we write

$$C_f(X, Y) = \{u \in C(X, Y) : u|_A = f\}, \quad [X, Y; f]_{\mathrm{rel}.A} = C_f(X, Y) / \sim_{\mathrm{rel}.A} .$$

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For $u \in C_f(X,Y)$, we use $[u]_{\text{rel.}A}$ to denote the corresponding equivalence class in $[X,Y;f]_{\text{rel.}A}$.

If in addition, (X, X_0, A) is a triple of topological spaces i.e. $A \subset X_0 \subset X$, then we may define an equivalence relation $\sim_{X_0,\text{rel.}A}$ on $C_f(X,Y)$ by

$$u \sim_{X_0,\text{rel.}A} v \Longleftrightarrow u|_{X_0} \sim_{\text{rel.}A} v|_{X_0} \quad \text{for } u, v \in C_f(X,Y).$$

We write $\left[X|_{X_0}, Y; f\right]_{\text{rel.}A} = C_f(X,Y) / \sim_{X_0,\text{rel.}A}$. For $u \in C_f(X,Y)$, we denote $[u]_{X_0,\text{rel.}A}$ as the corresponding equivalence class of u in $\left[X|_{X_0}, Y; f\right]_{\text{rel.}A}$. It is clear that $[X,Y;f]_{\text{rel.}A} = [X|_X, Y; f]_{\text{rel.}A}$. Moreover, we have a natural injection

$$i : \left[X|_{X_0}, Y; f\right]_{\text{rel.}A} \to [X_0, Y; f]_{\text{rel.}A} : [u]_{X_0,\text{rel.}A} \mapsto \left[u|_{X_0}\right]_{\text{rel.}A},$$

here $u \in C_f(X,Y)$. We identify $\left[X|_{X_0}, Y; f\right]_{\text{rel.}A}$ as a subset of $[X_0, Y; f]_{\text{rel.}A}$ through this injection. In particular, for any $u \in C_f(X,Y)$, $[u]_{X_0,\text{rel.}A}$ is identified with $\left[u|_{X_0}\right]_{\text{rel.}A}$.

Definition 1. Let (X_1, X_2, A) be a triple of topological spaces, X, Y be topological spaces, $f \in C(A,Y)$ and $i : X_1 \to X$ be an embedding such that $(X, i(X_2))$ satisfies the homotopy extension property with respect to Y. Given an $\alpha \in \left[X_1|_{X_2}, Y; f\right]_{\text{rel.}A}$, if for any representative $u \in \alpha$, $u|_{X_2} \circ i^{-1}$ has a continuous extension to X, then we say α is extendible to X with respect to Y.

2.1. **k-extension property.** Let (X, X_0, A) be a triple of topological spaces such that X is compactly generated, X_0 and A are closed subsets of X, (X, X_0) and (X_0, A) satisfy the homotopy extension property. Then we know (X, A) satisfies the homotopy extension property too. Let Y be an arbitrary topological space.

At first we will define a map between homotopy classes which will be useful in Section 5.1. Assume $f, f' \in C(A,Y)$ and H is a homotopy from f to f' i.e. $H \in C(A \times [0,1], Y)$ such that $H(a,0) = f(a)$, $H(a,1) = f'(a)$ for $a \in A$; in another way, H is a continuous path in $C(A,Y)$ from f to f', we will define a map

$$\mu_H : \left[X|_{X_0}, Y; f\right]_{\text{rel.}A} \to \left[X|_{X_0}, Y; f'\right]_{\text{rel.}A}. \tag{1}$$

For any $u \in C_f(X,Y)$, let

$$u_H(x,t) = \begin{cases} u(x), & x \in X, t = 0; \\ H(x,t), & x \in A, 0 \le t \le 1, \end{cases}$$

then $u_H \in C((X \times \{0\}) \cup (A \times [0,1]), Y)$. By the homotopy extension property, we may find an extension of u_H, namely $w \in C(X \times [0,1], Y)$. Let $v(x) = w(x,1)$ for $x \in X$, then $v \in C_{f'}(X,Y)$. We let μ_H be the map which sends $[u]_{X_0,\text{rel.}A}$ to $[v]_{X_0,\text{rel.}A}$.

We have to show μ_H is well defined. For $i = 0, 1$, let $u_i \in C_f(X,Y)$ such that $u_0 \sim_{X_0,\text{rel.}A} u_1$. Then we may find a map $H_0 \in C(X_0 \times [0,1], Y)$ such that $H_0(x,0) = u_0(x)$, $H_0(x,1) = u_1(x)$ for $x \in X_0$ and $H_0(x,t) = f(x)$ for $x \in A$, $0 \le t \le 1$. We have

$$u_{i,H}(x,t) = \begin{cases} u_i(x), & x \in X, t = 0; \\ H(x,t), & x \in A, 0 \le t \le 1, \end{cases}$$

and $w_i \in C(X \times [0,1], Y)$, a continuous extension of $u_{i,H}$. Let $v_i(x) = w_i(x,1)$ for $x \in X$. We will show $v_0 \sim_{X_0,\text{rel}.A} v_1$. To achieve this, let

$$w_2(x,t) = \begin{cases} w_0(x, 1-3t), & x \in X_0, 0 \le t \le \frac{1}{3}; \\ H_0(x, 3t-1), & x \in X_0, \frac{1}{3} \le t \le \frac{2}{3}; \\ w_1(x, 3t-2), & x \in X_0, \frac{2}{3} \le t \le 1, \end{cases}$$

$$\overline{w}_3(x,t) = \begin{cases} v_0(x), & x \in X_0, t = 0; \\ f'(x), & x \in A, 0 \le t \le 1; \\ v_1(x), & x \in X_0, t = 1, \end{cases}$$

then we have $w_2|_{(X_0 \times \{0,1\}) \cup (A \times [0,1])} \sim \overline{w}_3$. In fact, the map

$$\Phi(x,t,s) = \begin{cases} v_0(x), & x \in X_0, t = 0, 0 \le s \le 1; \\ f'(x), & x \in A, 0 \le t \le \frac{s}{3}, 0 \le s \le 1; \\ H(x, s-3t+1), & x \in A, \frac{s}{3} \le t \le \frac{1}{3}, 0 \le s \le 1; \\ H(x,s), & x \in A, \frac{1}{3} \le t \le \frac{2}{3}, 0 \le s \le 1; \\ H(x, s+3t-2), & x \in A, \frac{2}{3} \le t \le 1 - \frac{s}{3}, 0 \le s \le 1; \\ f'(x), & x \in A, 1 - \frac{s}{3} \le t \le 1, 0 \le s \le 1; \\ v_1(x), & x \in X_0, t = 1, 0 \le s \le 1, \end{cases}$$

gives the needed homotopy. Using the homotopy extension theorem, we may find a continuous extension of $\overline{w}_3$, namely $w_3 \in C(X_0 \times [0,1], Y)$. w_3 is the needed homotopy for $v_0 \sim_{X_0,\text{rel}.A} v_1$.

If $f, f', f'' \in C(A,Y)$, H_1 is a homotopy from f to f' and H_2 is a homotopy from f' to f'', then we may define the homotopy from f to f'', namely $H_2 * H_1$ by

$$H_2 * H_1(a,t) = \begin{cases} H_1(a, 2t), & a \in A, 0 \le t \le \frac{1}{2}; \\ H_2(a, 2t-1), & a \in A, \frac{1}{2} \le t \le 1. \end{cases}$$

We have the following equality:

$$\mu_{H_2 * H_1} = \mu_{H_2} \circ \mu_{H_1}. \tag{2}$$

If $f, f' \in C(A,Y)$, H_0 and H_1 are two homotopies from f to f' such that $H_0 \sim H_1$ rel. $A \times \{0,1\}$, then we claim

$$\mu_{H_0} = \mu_{H_1} : \left[X|_{X_0}, Y; f\right]_{\text{rel}.A} \to \left[X|_{X_0}, Y; f'\right]_{\text{rel}.A}. \tag{3}$$

Indeed, let $H : A \times [0,1]^2 \to Y$ be a continuous map such that $H(a,t,0) = H_0(a,t)$, $H(a,t,1) = H_1(a,t)$ for $a \in A$, $0 \le t \le 1$; $H(a,0,s) = f(a)$, $H(a,1,s) = f'(a)$ for $a \in A$, $0 \le s \le 1$. Given $u \in C_f(X,Y)$, define

$$F(x,t,s) = \begin{cases} u(x), & x \in X, t = 0, 0 \le s \le 1; \\ H(x,t,s), & x \in A, 0 \le t, s \le 1. \end{cases}$$

It follows from homotopy extension property that F has a continuous extension to $X \times [0,1]^2$, namely $\overline{F} \in C\left(X \times [0,1]^2, Y\right)$. Then

$$\mu_{H_0}\left([u]_{X_0,\text{rel}.A}\right) = \left[\overline{F}(\cdot,1,0)\right]_{X_0,\text{rel}.A} = \left[\overline{F}(\cdot,1,1)\right]_{X_0,\text{rel}.A} = \mu_{H_1}\left([u]_{X_0,\text{rel}.A}\right).$$

The claim is verified.

It follows easily from (1) and (2) that if $f, f' \in C(A,Y)$ and H is a homotopy from f to f', let $\overline{H}(a,t) = H(a, 1-t)$, $a \in A$, $0 \le t \le 1$ be the homotopy from f' to f, then we have both $\mu_{\overline{H}} \circ \mu_H$ and $\mu_H \circ \mu_{\overline{H}}$ are identity maps. This implies μ_H is a bijection and $\mu_H^{-1} = \mu_{\overline{H}}$.

Next let us discuss some basics about pairs of spaces. Assume (X, A) is a pair of topological spaces. A continuous map from (X, A) to (Y, B) means a map $f \in C(X, Y)$ such that $f(A) \subset B$. A homotopy of maps f, g between pairs (X, A) and (Y, B) means a continuous map $H : (X \times [0,1], A \times [0,1]) \to (Y, B)$ such that $H(x, 0) = f(x)$, $H(x, 1) = g(x)$ for $x \in X$. Similar to lemma 2.1 of [7], we have

Lemma 1. *Let (X, A) and (Y, B) be pairs, $\phi_i : (X_i, A_i) \to (X, A)$ be homotopy equivalence between pairs for $i = 1, 2$. Here (X_i, A_i) is a relative CW complex. Assume $f, g : (X, A) \to (Y, B)$ are continuous maps, $k \in \mathbb{Z}$, $k \geq 0$. If $f \circ \phi_1|_{X_1^k}$ is homotopic to $g \circ \phi_1|_{X_1^k}$ as maps from (X_1^k, A_1) to (Y, B), then $f \circ \phi_2|_{X_2^k}$ is homotopic to $g \circ \phi_2|_{X_2^k}$ as maps from (X_2^k, A_2) to (Y, B). Here X_i^k means the k-skeleton of (X_i, A_i).*

Proof. Let $\psi_i : (X, A) \to (X_i, A_i)$ be a homotopy inverse of ϕ_i as maps of pairs. By the cellular approximation theorem (p77 of [16]) we may find a cellular map $\varphi : (X_2, A_2) \to (X_1, A_1)$ such that φ is homotopic to $\psi_1 \circ \phi_2$ rel. A_2. Considered as maps from (X_2^k, A_2) to (Y, B), we have

$$\begin{aligned} f \circ \phi_2|_{X_2^k} &\sim f \circ \phi_1 \circ \psi_1 \circ \phi_2|_{X_2^k} \sim f \circ \phi_1 \circ \varphi|_{X_2^k} \sim g \circ \phi_1 \circ \varphi|_{X_2^k} \\ &\sim g \circ \phi_1 \circ \psi_1 \circ \phi_2|_{X_2^k} \sim g \circ \phi_2|_{X_2^k}. \end{aligned}$$

□

In particular, if (X, A) is a relative CW complex and $f, g : (X, A) \to (Y, B)$ are continuous maps such that $f|_{X^k} \sim g|_{X^k}$ as maps between pairs (X^k, A) and (Y, B), then we say f and g are k-homotopic as pairs. It follows from Lemma 1 that this definition does not depend on the specific choice of the relative CW complex structure on (X, A). Similar to usual homotopy equivalence between pairs, we have k-homotopy equivalence between pairs which are relative CW complexes. In the same line as lemma 2.2 in [7], we have

Lemma 2. *Let (X, A) and (Y, B) be relative CW complexes, Z be a topological space and $g \in C(B, Z)$. Assume $k \in \mathbb{Z}$, $k \geq 0$, $\phi : (X, A) \to (Y, B)$ is a $(k+1)$-homotopy equivalence between pairs. If for any $v \in C(Y^{k+1}, Z)$ with $v|_B = g$, we may find $\overline{v} \in C(Y, Z)$ such that $\overline{v}|_{Y^k} = v|_{Y^k}$, then for any $u \in C(X^{k+1}, Z)$ with $u|_A = g \circ \phi|_A$, there exists a $\overline{u} \in C(X, Z)$ such that $\overline{u}|_{X^k} = u|_{X^k}$.*

Proof. We may find a continuous map $\psi : (Y, B) \to (X, A)$ such that $\psi \circ \phi$ is $(k+1)$-homotopic to id_X and $\phi \circ \psi$ is $(k+1)$-homotopic to id_Y, both as maps of pairs. By the cellular approximation theorem, we may assume both ϕ and ψ are cellular maps. Let i be the map from X^k to X^{k+1} such that $i(x) = x$ for any $x \in X^k$.

We claim that $\psi \circ \phi \sim i$ as maps of pairs from (X^k, A) to (X^{k+1}, A). Indeed since $\psi \circ \phi|_{X^{k+1}} \sim id_X|_{X^{k+1}}$ as maps of pairs from (X^{k+1}, A) to (X, A), we may find a continuous map $H_0 : (X^{k+1} \times [0,1], A \times [0,1]) \to (X, A)$ such that $H_0(x, 0) = \psi(\phi(x))$ and $H_0(x, 1) = x$ for $x \in X^{k+1}$. By the cellular approximation theorem, we may find a cellular map $H : (X^{k+1} \times [0,1], A \times [0,1]) \to (X, A)$ such that $H(x, 0) = \psi(\phi(x))$ and $H(x, 1) = x$ for $x \in X^{k+1}$. Since $H(X^k \times [0,1]) \subset X^{k+1}$, the claim follows. For any $u \in C(X^{k+1}, Z)$ with $u|_A = g \circ \phi|_A$, we may define $v(y) = u(\psi(y))$ for $y \in Y^{k+1}$. Then for $y \in B$, $v(y) = g(\phi(\psi(y)))$. Since $\phi \circ \psi|_B \sim id_B$, we see $v|_B \sim g$. By the homotopy extension property we may find a continuous extension of g to Y^{k+1}, namely w, such that w is homotopic to v. By assumption, we may find a $\overline{w} \in C(Y,Z)$ such that $\overline{w}|_{Y^k} = w|_{Y^k}$. Using homotopy extension property again we see there exists a $\overline{v} \in C(Y,Z)$ such that $\overline{v}|_{Y^k} = v|_{Y^k}$. Now it follows from the above claim that $\overline{v} \circ \phi|_{X^k} = u \circ \psi \circ \phi|_{X^k} \sim u|_{X^k}$ and hence $u|_{X^k}$ has a continuous extension to X. □

Now let us introduce the following

Definition 2. Let (X,A) be a relative CW complex, Y be a topological space, $f \in C(A,Y)$, $k \in \mathbb{Z}$, $k \geq 0$. If for any $u \in C_f\left(X^{k+1}, Y\right)$, $u|_{X^k}$ has a continuous extension to X, then we say $(X,A;f)$ satisfies the k-extension property with respect to Y.

It follows from Lemma 2 that the k-extension property does not depend on the specific choice of relative CW complex structure on (X,A).

Let (X,A) and (Y,B) be relative CW complexes, $\phi : (X,A) \to (Y,B)$ be a cellular map, Z be a topological space, $g \in C(B,Z)$ and $k \in \mathbb{Z}$, $k \geq 0$. We define a map

$$\nu_\phi : \left[Y^{k+1}|_{Y^k}, Z; g\right]_{\text{rel}.B} \to \left[X^{k+1}|_{X^k}, Z; g \circ \phi|_A\right]_{\text{rel}.A}$$

as follows : for any $v \in C_g\left(Y^{k+1}, Z\right)$, set $\nu_\phi\left([v]_{Y^k,\text{rel}.B}\right) = \left[v \circ \phi|_{X^{k+1}}\right]_{X^k,\text{rel}.A}$. It is easy to see that ν_ϕ is well defined. The next lemma will help us understanding the set of equivalence classes defined above.

Lemma 3. *Let (X,A) and (Y,B) be relative CW complexes, Z be a topological space and $g \in C(B,Z)$. Assume $k \in \mathbb{Z}$, $k \geq 0$ and $\phi : (X,A) \to (Y,B)$ is a cellular map which is a $(k+1)$-homotopy equivalence between pairs. Then*

$$\nu_\phi : \left[Y^{k+1}|_{Y^k}, Z; g\right]_{rel.B} \to \left[X^{k+1}|_{X^k}, Z; g \circ \phi|_A\right]_{rel.A}$$

is a bijection.

Proof. Let $\psi : (Y,B) \to (X,A)$ be a $(k+1)$-homotopy inverse of ϕ. By the cellular approximation theorem, we may assume ψ is cellular. Since $\phi \circ \psi$ is $(k+1)$-homotopic to id_Y as maps between pairs, we may find a continuous map $H \in C\left(Y^{k+1} \times [0,1], B \times [0,1]\right) \to (Y,B)$ such that $H(y,0) = y, H(y,1) = \phi(\psi(y))$ for $y \in Y^{k+1}$. By the cellular approximation theorem, we may assume H is cellular. Now we have maps

$$\left[Y^{k+1}|_{Y^k}, Z; g\right]_{\text{rel}.B} \xrightarrow{\nu_\phi} \left[X^{k+1}|_{X^k}, Z; g \circ \phi|_A\right]_{\text{rel}.A} \xrightarrow{\nu_\psi} \left[Y^{k+1}|_{Y^k}, Z; g \circ \phi \circ \psi|_B\right]_{\text{rel}.B}.$$

We claim the composition of the above two maps, $\nu_\psi \circ \nu_\phi$ is equal to

$$\mu_{g \circ H|_{B \times [0,1]}} : \left[Y^{k+1}|_{Y^k}, Z; g\right]_{\text{rel}.B} \longrightarrow \left[Y^{k+1}|_{Y^k}, Z; g \circ \phi \circ \psi|_B\right]_{\text{rel}.B}.$$

In fact, if $v \in C_g\left(Y^{k+1}, Z\right)$, then $v \circ \phi \circ \psi|_{Y^{k+1}} \in C_{g \circ \phi \circ \psi|_B}\left(Y^{k+1}, Z\right)$, in addition, we have a map

$$\overline{w}(y,t) = \begin{cases} v(y), & y \in Y^{k+1}, t = 0; \\ v(H(y,t)), & y \in Y^k, 0 \leq t \leq 1. \end{cases}$$

By the homotopy extension property, we may find a continuous extension of $\overline{w}$, namely $w \in C\left(Y^{k+1} \times [0,1], Z\right)$. By definition, we have

$$\begin{aligned} \mu_{g \circ H|_{B \times [0,1]}}\left([v]_{Y^k,\text{rel}.B}\right) &= [w(\cdot\,,1)]_{Y^k,\text{rel}.B} = \left[v \circ \phi \circ \psi|_{Y^{k+1}}\right]_{Y^k,\text{rel}.B} \\ &= \nu_\psi\left(\nu_\phi\left([v]_{Y^k,\text{rel}.B}\right)\right). \end{aligned}$$

The claim follows.

Using the fact

$$\mu_{g\circ H|_{B\times[0,1]}}:\left[Y^{k+1}\big|_{Y^k},Z;g\right]_{\mathrm{rel}.B}\longrightarrow\left[Y^{k+1}\big|_{Y^k},Z;\,g\circ\phi\circ\psi|_B\right]_{\mathrm{rel}.B}$$

is a bijection, we see

$$\nu_\phi:\left[Y^{k+1}\big|_{Y^k},Z;g\right]_{\mathrm{rel}.B}\longrightarrow\left[X^{k+1}\big|_{X^k},Z;\,g\circ\phi|_A\right]_{\mathrm{rel}.A}$$

must be injective and

$$\nu_\psi:\left[X^{k+1}\big|_{X^k},Z;\,g\circ\phi|_A\right]_{\mathrm{rel}.A}\xrightarrow{\nu_\psi}\left[Y^{k+1}\big|_{Y^k},Z;\,g\circ\phi\circ\psi|_B\right]_{\mathrm{rel}.B}$$

must be surjective.

Applying above assertions for g replaced by $g\circ\phi|_A$ we see

$$\nu_\psi:\left[X^{k+1}\big|_{X^k},Z;\,g\circ\phi|_A\right]_{\mathrm{rel}.A}\longrightarrow\left[Y^{k+1}\big|_{Y^k},Z;\,g\circ\phi\circ\psi|_B\right]_{\mathrm{rel}.B}$$

is also injective, hence it must be a bijection. This together with the fact

$$\mu_{g\circ H|_{B\times[0,1]}}:\left[Y^{k+1}\big|_{Y^k},Z;g\right]_{\mathrm{rel}.B}\longrightarrow\left[Y^{k+1}\big|_{Y^k},Z;\,g\circ\phi\circ\psi|_B\right]_{\mathrm{rel}.B}$$

is a bijection imply that

$$\nu_\phi:\left[Y^{k+1}\big|_{Y^k},Z;g\right]_{\mathrm{rel}.B}\longrightarrow\left[X^{k+1}\big|_{X^k},Z;\,g\circ\phi|_A\right]_{\mathrm{rel}.A}$$

is a bijection too. □

In the future, we will also need the following fact.

Proposition 1. *Let $Y\subset\mathbb{R}^n$ be a compact subset which is a Lipschitz neighborhood retractor i.e. there exists an open set $Y\subset V\subset\mathbb{R}^n$ and a Lipschitz map $r:V\to Y$ such that $r(y)=y$ for $y\in Y$. Let X be a compact metric space and A be a closed subset. Assume $f\in Lip(A,Y)$ such that there exists a $v\in C(X,Y)$ with $v|_A=f$, then for any $\varepsilon>0$, there exists a $u\in Lip(X,Y)$ such that $u|_A=f$ and $|u(x)-v(x)|<\varepsilon$ for $x\in X$. In particular, when ε is small enough, we have $u\sim v$ rel. A.*

Proof. Choose an $\varepsilon_0>0$ such that $\{z\in\mathbb{R}^n:\operatorname{dist}(z,Y)\le\varepsilon_0\}\subset V$. Let $\varepsilon_1=\frac{\varepsilon}{8\left([r]_{\mathrm{Lip}(V)}+1\right)}$. Fix a $w_1\in\mathrm{Lip}(X,\mathbb{R}^n)$ such that $w_1|_A=f$ and a $w_2\in\mathrm{Lip}(X,\mathbb{R}^n)$ such that $|w_2(x)-v(x)|\le\varepsilon_1$ for $x\in X$ (see lemma 2.3 of [7]). Let

$$\delta=\frac{\varepsilon_1}{[w_1]_{\mathrm{Lip}(X)}+[w_2]_{\mathrm{Lip}(X)}+1},\quad U=\{x\in X:\operatorname{dist}(x,A)<\delta\},$$

then for $x\in U$, there exists an $a\in A$ with $d(x,a)<\delta$, this implies

$$|w_1(x)-w_2(x)|\le|w_1(x)-w_1(a)|+|w_1(a)-w_2(a)|+|w_2(a)-w_2(x)|\le3\varepsilon_1.$$

Choose $\eta\in\mathrm{Lip}(X,\mathbb{R})$ such that $0\le\eta\le1$, $\eta|_A=1$ and $\eta|_{X\setminus U}=0$, then for $x\in X$, let

$$w(x)=\eta(x)w_1(x)+(1-\eta(x))w_2(x)=w_2(x)+\eta(x)(w_1(x)-w_2(x)).$$

We see that $|w(x)-w_2(x)|\le3\varepsilon_1$. This implies $|w(x)-v(x)|\le4\varepsilon_1\le\varepsilon_0$ when ε is small enough, hence $w(x)\in V$. Let $u(x)=r(w(x))$ for $x\in X$, then $u\in\mathrm{Lip}(X,Y)$ and $u|_A=f$. In addition, for $x\in X$, $|u(x)-v(x)|=|r(w(x))-r(v(x))|\le[r]_{\mathrm{Lip}(V)}|w(x)-v(x)|\le4[r]_{\mathrm{Lip}(V)}\varepsilon_1\le\frac{\varepsilon}{2}<\varepsilon$. Take $H(x,t)=r((1-t)u(x)+tv(x))$ for $x\in X$, $0\le t\le1$, we see $u\sim v$ rel. A. □

3. **Construction of maps with suitable transversality.** Let M^n be a smooth compact Riemannian manifold without boundary and assume $M^n \subset \mathbb{R}^l$. In [7], when applying the results on slicing of Sobolev functions on M, the nearest point projection plays an important role. In fact, for $\varepsilon_0 = \varepsilon_0(M) > 0$ small enough,

$$V_{2\varepsilon_0}(M) = \left\{ y \in \mathbb{R}^l : \operatorname{dist}(y, M) < 2\varepsilon_0 \right\}$$

is a tubular neighborhood of M. Let $\pi_M : V_{2\varepsilon_0}(M) \to M$ be the nearest point projection. Define

$$\Psi : M \times B^l_{\varepsilon_0} \to M : (y, \xi) \mapsto \pi_M(y + \xi).$$

If Δ is a rectilinear cell and $h : \Delta \to M$ is a Lipschitz map, then the map $H : \Delta \times B^l_{\varepsilon_0} \to M$ given by $H(x, \xi) = \Psi(h(x), \xi)$ satisfies (H_1), (H_2) and (H_3) in section 3 of [7] (see the beginning of section 4 of [7]). This fact plays a crucial role in discussing the homotopy of Sobolev maps between manifolds (see section 4 of [7]). However, the method of using the nearest point projection is not very convenient for the Dirichlet problem. In this section, we describe some other methods of constructing maps satisfying (H_1), (H_2) and (H_3) in section 3 of [7] and will apply them to the Dirichlet problem later.

3.1. **Manifolds without boundary.** Let M^n be a smooth compact Riemannian manifold without boundary. We claim that we may find smooth vector fields $X_1, \cdots, X_l$ on M such that

$$\operatorname{span}\{X_1(p), \cdots, X_l(p)\} = M_p \quad \text{for every } p \in M. \tag{4}$$

Here M_p means the tangent space of M at p. In fact, for any $p \in M$, we may find n smooth vector fields, $X_{p,1}, \cdots, X_{p,n}$ such that $X_{p,1}(p), \cdots, X_{p,n}(p)$ are linearly independent in M_p. By continuity, we may find an open neighborhood $U(p)$ such that for any $q \in U(p)$, $X_{p,1}(q), \cdots, X_{p,n}(q)$ are linearly independent in M_q. By the compactness of M, we may find $p_1, \cdots, p_m \in M$ such that $\cup_{i=1}^m U(p_i) = M$. Then $\{X_{p_i,j} : 1 \le i \le m, 1 \le j \le n\}$ is a collection of vector field satisfying (4).

If X is a smooth vector fields on M, let $\phi_{X,t} : M \to M$, $t \in \mathbb{R}$ be the flow generated by X, that is $\frac{d\phi_{X,t}(p)}{dt} = X(\phi_{X,t}(p))$ and $\phi_{X,0}(p) = 0$. Then we may define

$$\Phi : M \times \mathbb{R}^l \to M : (p, \xi) \mapsto \phi_{X_1,\xi^1} \circ \phi_{X_2,\xi^2} \circ \cdots \circ \phi_{X_l,\xi^l}(p). \tag{5}$$

It is clear that Φ is a smooth map and for any $p \in M$,

$$d\Phi_{(p,0)}\left(0, \left.\frac{\partial}{\partial \xi^i}\right|_0\right) = X_i(p) \quad \text{for } 1 \le i \le l,$$

this implies that $d\Phi_{(p,0)}\left(\{0\} \times \mathbb{R}^l_0\right) = M_p$ and there exists some constant $c = c(M) > 0$ such that the Jacobian $J_{\Phi^p}(0) \ge c(M)$ for any $p \in M$. Here $\Phi^p(\xi) = \Phi(p, \xi)$ for $p \in M$, $\xi \in \mathbb{R}^l$. Hence we may find a $\varepsilon_0 = \varepsilon_0(M) > 0$ such that for every $p \in M$, $J_{\Phi^p}(\xi) \ge \frac{c(M)}{2}$ for $|\xi| \le \varepsilon_0$.

Lemma 4. *Assume $p_0 \in M$, $\xi_0 \in \mathbb{R}^l$, $p_1 = \Phi(p_0, \xi_0)$ and*

$$d\Phi_{(p_0,\xi_0)}\left(\{0\} \times \mathbb{R}^l_{\xi_0}\right) = M_{p_1},$$

then there exists open neighborhoods $U(p_0, \xi_0)$, $U(p_0)$, $V(p_1)$ and a bi-Lipschitz map

$$\phi : U(p_0, \xi_0) \to U(p_0) \times V(p_1) \times B_1^{l-n}$$

such that

(1) $\phi(p,\xi) = (p, \Phi(p,\xi), g(\xi))$ *for* $(p,\xi) \in U(p_0,\xi_0)$.
(2) $\phi^{-1}(p,q,\zeta) = (p, \psi(p,q,\zeta))$ *for* $(p,q,\zeta) \in U(p_0) \times V(p_1) \times B_1^{l-n}$.
(3) $\Phi\left(\phi^{-1}(p,q,\zeta)\right) = q$ *for* $(p,q,\zeta) \in U(p_0) \times V(p_1) \times B_1^{l-n}$.

Proof. Choose a base of $\mathbb{R}^l_{\xi_0}$, namely $e_1, \cdots, e_l$. Let $e^1, \cdots, e^l$ be the corresponding dual base. Without losing of generality, we may assume $d\Phi_{(p_0,\xi_0)}(0,e_1), \cdots, d\Phi_{(p_0,\xi_0)}(0,e_n)$ generates M_{p_1}. Choose $f^1, \cdots, f^{l-n} \in C^\infty\left(\mathbb{R}^l, \mathbb{R}\right)$ such that $f^i(\xi_0) = 0$ and $df^i_{\xi_0} = e^{n+i}$ for $i = 1, 2, \cdots, l-n$. Define $\varphi : M \times \mathbb{R}^l \to M \times M \times \mathbb{R}^{l-n}$ by

$$\varphi(p,\xi) = \left(p, \Phi(p,\xi), f^1(\xi), \cdots, f^l(\xi)\right).$$

It is easy to see $d\varphi_{(p_0,\xi_0)} : M_{p_0} \times \mathbb{R}^l_{\xi_0} \to M_{p_0} \times M_{p_1} \times \mathbb{R}^l_0$ is a linear isomorphism, hence we may find an open neighborhood $U_1(p_0,\xi_0)$ such that $\varphi(U_1(p_0,\xi_0))$ is open and $\varphi|_{U_1(p_0,\xi_0)}$ is a diffeomorphism from $U_1(p_0,\xi_0)$ to $\varphi(U_1(p_0,\xi_0))$. By shrinking $U_1(p_0,\xi_0)$ a little bit, we may assume $\varphi|_{U_1(p_0,\xi_0)}$ is a bi-Lipschitz map. Choose open neighborhoods $U(p_0)$, $V(p_1)$ and $\varepsilon > 0$ such that $U(p_0) \times V(p_1) \times B^{l-n}_\varepsilon \subset \varphi(U_1(p_0,\xi_0))$. Let $U(p_0,\xi_0) = \varphi|^{-1}_{U_1(p_0,\xi_0)}\left(U(p_0) \times V(p_1) \times B^{l-n}_\varepsilon\right)$ and

$$\phi(p,\xi) = \left(p, \Phi(p,\xi), \frac{1}{\varepsilon} f^1(\xi), \cdots, \frac{1}{\varepsilon} f^{l-n}(\xi)\right),$$

we get the lemma. □

Lemma 5. *Assume Δ is a rectilinear cell of dimension d and $h : \Delta \to M$ is a Lipschitz map. Define $H(x,\xi) = \Phi(h(x),\xi)$ for $x \in \Delta$, $\xi \in B^l_{\varepsilon_0}$, then H satisfies (H_1), (H_2) and (H_3) in section 3 of* [7] *, that is,*

(1) *$H \in Lip\left(\Delta \times B^l_{\varepsilon_0}, M\right)$ and $[H_\xi]_{Lip(\Delta)} \le c(M)[h]_{Lip(\Delta)}$ for any $\xi \in B^l_{\varepsilon_0}$. Here $H_\xi(x) = H(x,\xi)$ for $x \in \Delta$.*
(2) *There exists a positive number $c(M)$ such that the Jacobian $J_H(x,\xi) \ge c(M)$, $\mathcal{H}^{d+l}$ a.e. $(x,\xi) \in \Delta \times B^l_{\varepsilon_0}$.*
(3) *There exists a positive number $c(M)$ such that*

$$\mathcal{H}^{d+l-n}\left(H^{-1}(p)\right) \le c(M)\,\mathcal{H}^d(\Delta)\left([h]_{Lip(\Delta)} + 1\right)^{d+l-n}$$

for $p \in M$.

Proof. (1) and (2) are clearly satisfied. To prove (3), fix a $p_0 \in M$, for any $(p,\xi) \in M \times \overline{B}^l_{\varepsilon_0}$, since $J_{\Phi^p}(\xi) \ge c(M) > 0$, by Lemma 4 we may find open neighborhoods $U(p,\xi)$, $U(p)$, $V(\Phi(p,\xi))$ and a bi-Lipschitz map

$$\phi_{p,\xi} : U(p,\xi) \to U(p) \times V(\Phi(p,\xi)) \times B^{l-n}_1$$

such that

$$\begin{aligned} \phi_{p,\xi}(q,\eta) &= (q, \Phi(q,\eta), g_{p,\xi}(\eta)) \quad \text{for } (q,\eta) \in U(p,\xi), \\ \phi^{-1}_{p,\xi}(q_1,q_2,\zeta) &= (q_1, \psi_{p,\xi}(q_1,q_2,\zeta)) \quad \text{for } (q_1,q_2,\zeta) \in U(p) \times V(\Phi(p,\xi)) \times B^{l-n}_1. \end{aligned}$$

By compactness, we may find $(p_i,\xi_i) \in M \times \overline{B}^l_{\varepsilon_0}$ for $i = 1, 2, \cdots, k$ such that

$$M \times \overline{B}^l_{\varepsilon_0} \subset \bigcup_{i=1}^k U(p_i,\xi_i).$$

For convenience, we write $\phi_i = \phi_{p_i,\xi_i}$, $g_i = g_{p_i,\xi_i}$, $\psi_i = \psi_{p_i,\xi_i}$. We claim

$$H^{-1}(p_0) \subset \bigcup_{i=1}^{k} \left\{ (x, \psi_i(h(x), p_0, \zeta)) : x \in \Delta, \zeta \in B_1^{l-n}, h(x) \in U(p_i), p_0 \in V(\Phi(p_i, \xi_i)) \right\}.$$

In fact, if $(x, \xi) \in H^{-1}(p_0)$, then for some i, $(h(x), \xi) \in U(p_i, \xi_i)$. We have $\phi_i(h(x), \xi) = (h(x), p_0, g_i(\xi))$ and $\psi_i(h(x), p_0, g_i(\xi)) = \xi$, hence

$$(x, \xi) = (x, \psi_i(h(x), p_0, g_i(\xi))).$$

The claim follows. (3) clearly follows from the claim. □

3.2. **Manifolds with boundaries.** *In this subsection, we always assume M^n is a smooth compact Riemannian manifold with boundary.* Let V be a product neighborhood of ∂M and $\psi : V \to \partial M \times [0,1)$ be a diffeomorphism such that $\psi(p) = (p, 0)$ for any $p \in \partial M$. We may write $\psi(p) = (r(p), \tau(p))$, where $r : V \to \partial M$ is a smooth retraction and $\tau : V \to [0,1)$ is a smooth function. Recall the double of M, $\mathrm{D}(M)$, is defined as

$$\mathrm{D}(M) = M \times \{0,1\} / \{(p,0) \sim (p,1) : p \in \partial M\}.$$

Let $V_{\mathrm{D}} = \{[p,0], [p,1] : p \in V\}$. We may define maps $i_0 : M \to \mathrm{D}(M)$ by $i_0(p) = [p,0]$ for $p \in M$; $i_1 : M \to \mathrm{D}(M)$ by $i_1(p) = [p,1]$ for $p \in M$; $\psi : V_{\mathrm{D}} \to \partial M \times (-1,1)$ by $\psi([p,0]) = (r(p), \tau(p))$, $\psi([p,1]) = (r(p), -\tau(p))$ for $p \in V$. We still denote the first component of this ψ as r and second component as τ. The differentiable structure on $\mathrm{D}(M)$ is given such that i_0, i_1 are smooth embeddings and ψ is a diffeomorphism. We identify M as a subset of $\mathrm{D}(M)$ by i_0.

Fix a $\eta \in C^\infty(\mathbb{R}, \mathbb{R})$ such that $0 \le \eta \le 1$, $\eta(t) = 1$ for $|t| \le \frac{1}{2}$ and $\eta(t) = 0$ for $|t| \ge \frac{3}{4}$. Let

$$X\left(\psi^{-1}(p,t)\right) = \eta(t) d\left(\psi^{-1}\right)_{(p,t)} \left(0, \frac{d}{dt}\right) \quad \text{for } p \in V, |t| < 1,$$

and $X|_{\mathrm{D}(M) \setminus V_{\mathrm{D}}} = 0$, then X is a smooth vector fields on $\mathrm{D}(M)$.

We claim that there exists $X_1, \cdots, X_l$, smooth vector fields on $\mathrm{D}(M)$, such that

$$\begin{cases} X_1 = X; \\ \operatorname{span}\{X_1(p), \cdots, X_l(p)\} = \mathrm{D}(M)_p \quad \text{for any } p \in \mathrm{D}(M); \\ X_2(p), \cdots, X_l(p) \in (\partial M)_p \quad \text{for any } p \in \partial M. \end{cases} \tag{6}$$

Indeed, given any $p \in \mathrm{D}(M)$. If $p \in \mathrm{D}(M) \setminus \partial M$, then we may find smooth vector fields $X_{p,1}, \cdots, X_{p,n}$ such that $X_{p,1}(p), \cdots, X_{p,n}(p)$ is linearly independent and $X_{p,i}|_{\partial M} = 0$ for $1 \le i \le n$. If $p \in \partial M$, then we may find smooth vector fields $X_{p,1}, \cdots, X_{p,n}$ such that $X_{p,1} = X$, $X_{p,i}(q) \in (\partial M)_q$ for $2 \le i \le n$ and $q \in \partial M$, and $X_{p,1}(p), \cdots, X_{p,n}(p)$ is linearly independent. In both cases, we may find an open neighborhood $U(p)$ such that for any $q \in U(p)$, $X_{p,1}(q), \cdots, X_{p,n}(q)$ are linearly independent. Since $\cup_{p \in M} U(p) = \mathrm{D}(M)$, we may find $p_1, \cdots, p_m \in \mathrm{D}(M)$ such that $\cup_{i=1}^m U(p_i) = \mathrm{D}(M)$. Now $\{X_{p_i,j} : 1 \le i \le m, 1 \le j \le n\}$ is the collection of vector fields we want. Note that X must lie in this collection by our choice.

Define

$$\Phi : \mathrm{D}(M) \times \mathbb{R}^l \to \mathrm{D}(M) : (p, \xi) \mapsto \phi_{X_1,\xi^1} \circ \phi_{X_2,\xi^2} \circ \cdots \circ \phi_{X_l,\xi^l}(p). \tag{7}$$

It is clear that for any $p \in \mathrm{D}(M)$, $d\Phi_{(p,0)}\left(\{0\} \times \mathbb{R}_0^l\right) = \mathrm{D}(M)_p$ and there exists some constant $c = c(M) > 0$ such that the Jacobian $J_{\Phi^p}(0) \ge c(M)$ for any $p \in M$.

We may find a $\frac{1}{4} > \varepsilon_0 = \varepsilon_0(M) > 0$ such that for every $p \in M$, $J_{\Phi^p}(\xi) \geq \frac{c(M)}{2}$ for $\xi \in \mathbb{R}^l_+$, $|\xi| \leq \varepsilon_0$. In the same line as Lemma 5, we have

Lemma 6. *Assume Δ is a rectilinear cell of dimension d and $h : \Delta \to M$ is a Lipschitz map. Define $H(x,\xi) = \Phi(h(x),\xi)$ for $x \in \Delta$, $\xi \in B^l_{\varepsilon_0}$, then H satisfies (H_1), (H_2) and (H_3) in section 3 of* [7] *, that is,*

1. *$H \in Lip\left(\Delta \times B^l_{\varepsilon_0}, D(M)\right)$ and $[H_\xi]_{Lip(\Delta)} \leq c(M)[h]_{Lip(\Delta)}$ for any $\xi \in B^l_{\varepsilon_0}$. Here $H_\xi(x) = H(x,\xi)$ for $x \in \Delta$.*
2. *There exists a positive number $c(M)$ such that the Jacobian $J_H(x,\xi) \geq c(M)$, $\mathcal{H}^{d+l}$ a.e. $(x,\xi) \in \Delta \times B^l_{\varepsilon_0}$.*
3. *There exists a positive number $c(M)$ such that*
$$\mathcal{H}^{d+l-n}\left(H^{-1}(p)\right) \leq c(M)\mathcal{H}^d(\Delta)\left([h]_{Lip(\Delta)} + 1\right)^{d+l-n}$$
for $p \in D(M)$.

For convenience, denote $\mathbb{R}^l_+ = [0,\infty) \times \mathbb{R}^{l-1}$. Let
$$M_{\varepsilon_0} = M \cup \left\{\psi^{-1}(p,t) : p \in \partial M, -\varepsilon_0 \leq t \leq 0\right\},$$
then we know
$$\Phi\left(M \times \mathbb{R}^l_+\right) \subset M, \quad \Phi\left(M \times B^l_{\varepsilon_0}\right) \subset M_{\varepsilon_0}.$$

Besides the map Φ defined in (7), another map is also of interest. To describe it, we claim that there exists smooth vector fields $Y_1, \cdots, Y_l$ on M such that
$$\operatorname{span}\{Y_1(p), \cdots, Y_l(p)\} = \begin{cases} M_p, & \text{when } p \in M \backslash \partial M; \\ (\partial M)_p, & \text{when } p \in \partial M. \end{cases} \tag{8}$$
Indeed, this may be proven directly or we may choose $\rho \in C^\infty(\mathrm{D}(M), \mathbb{R})$ such that $\{\rho = 0\} = \partial M$, then let
$$Y_1 = \rho X_1|_M, Y_2 = X_2|_M, \cdots, Y_l = X_l|_M,$$
here $X_1, \cdots, X_l$ satisfies (6).

Define
$$\Psi : M \times \mathbb{R}^l \to M : (p,\xi) \mapsto \phi_{Y_1,\xi^1} \circ \phi_{Y_2,\xi^2} \circ \cdots \circ \phi_{Y_l,\xi^l}(p), \tag{9}$$
then
$$d\Psi_{(p,0)}\left(\{0\} \times \mathbb{R}^l_0\right) = \begin{cases} M_p, & \text{when } p \in M \backslash \partial M; \\ (\partial M)_p, & \text{when } p \in \partial M. \end{cases} \tag{10}$$

4. **The case domain manifold has a nonempty boundary.** In this section, we always assume M^n is a smooth compact Riemannian manifold with a nonempty boundary, N is a smooth compact Riemannian manifold without boundary and it is embedded into $\mathbb{R}^{\bar{l}}$. For $1 \leq p < \infty$, let
$$W^{1,p}(M,N) = \left\{u \in W^{1,p}\left(M, \mathbb{R}^{\bar{l}}\right) : u(x) \in N \text{ a.e. } x \in M\right\}.$$
The aim of this section is to classify the path connected components of $W^{1,p}(M,N)$ and study when smooth maps will be dense in $W^{1,p}(M,N)$. These questions have been answered in [7] for the case domain manifold has no boundary. We will follow closely arguments in [7], present main statements and discuss some proofs when necessary.

4.1. **Homotopy of Sobolev mappings.** Assume $1 \le p \le n$, $u \in W^{1,p}(M,N)$, K is a finite rectilinear cell complex, the parameter space P is a m-dimensional Riemannian manifold and $H : |K| \times P \to M$ is a map such that $H|_{\Delta \times P}$ satisfies (H_1), (H_2) and (H_2) in section 3 of [7] for any $\Delta \in K$. Assume k is a nonnegative integer and $k \le p$, we want to define a measurable map

$$\chi = \chi_{k,H,u} : P \to \left[\left|K^{[p]}\right|\Big|_{|K^k|}, N \right] \subset \left[\left|K^k\right|, N \right].$$

We note that the map defined here is the same as the one defined in section 4 of [7] but we emphasize its image lies in $\left[\left|K^{[p]}\right|\Big|_{|K^k|}, N \right]$, a subset of $\left[\left|K^k\right|, N \right]$. This modification makes χ conceptually clearer.

To define such a χ, we note that by lemma 3.3 and corollary 3.1 of [7], we may find a Borel set $E \subset P$ such that $\mathcal{H}^m(E) = 0$ and $u \circ H_\xi \in \mathcal{W}^{1,p}(K,N)$ for any $\xi \in P \backslash E$.

If p is not an integer, it follows from Sobolev embedding theorem that $u \circ H_\xi|_{\left|K^{[p]}\right|} \in C\left(\left|K^{[p]}\right|, N\right)$, then we let

$$\chi(\xi) = \left[u \circ H_\xi|_{\left|K^{[p]}\right|} \right]_{|K^k|}.$$

We note that under the identification of $\left[\left|K^{[p]}\right|\Big|_{|K^k|}, N \right]$ as a subset of $\left[\left|K^k\right|, N \right]$, $\chi(\xi) = \left[u \circ H_\xi|_{|K^k|} \right]$.

If p is an integer and $0 \le k < p$, then by lemma 4.4 of [7], for $\xi \in P \backslash E$, we may find a $g \in C(|K^p|, N) \cap \mathcal{W}^{1,p}(K^p, N)$ such that $g|_{|K^{p-1}|} = u \circ H_\xi|_{|K^{p-1}|}$, then we let

$$\chi(\xi) = [g]_{|K^k|}.$$

Note that under the identification of $\left[|K^p|\big|_{|K^k|}, N \right]$ as a subset of $\left[\left|K^k\right|, N \right]$, $\chi(\xi) = \left[u \circ H_\xi|_{|K^k|} \right]$.

If p is an integer and $k = p$, then we let Θ be the map defined in lemma 4.5 of [7] and define

$$\chi(\xi) = \Theta\left(u \circ H_\xi|_{|K^p|} \right) \quad \text{for } \xi \in P \backslash E,$$

one should compare this with lemma 4.5 and lemma 4.6 in [7].

Same proofs as in section 4 of [7] show that χ is measurable. Moreover proposition 4.1 and lemma 4.7 in [7] remain true if we assume the domain manifold has a nonempty boundary. We will use those notations in Section 3.2 below.

Given $1 \le p \le n$, $u \in W^{1,p}(M,N)$. Assume K is a finite rectilinear cell complex, $h : |K| \to M$ is a Lipschitz map. Define $H(x,\xi) = \Phi(h(x), \xi) = H_\xi(x) = h_\xi(x)$ for $x \in |K|$, $\xi \in B^l_{\varepsilon_0} \cap \mathbb{R}^l_+$. It follows from Lemma 6 and (the boundary version of) lemma 4.7 in [7] that $\chi_{[p]-1,H,u} \equiv$ const a.e. on $B^l_{\varepsilon_0} \cap \mathbb{R}^l_+$. We denote this constant as $u_{\#,p}(h) \in \left[\left|K^{[p]}\right|\Big|_{\left|K^{[p]-1}\right|}, N \right] \subset \left[\left|K^{[p]-1}\right|, N \right]$.

Using this definition of $u_{\#,p}(h)$, one sees easily the boundary version of lemma 4.8 and lemma 4.9 in [7] remain true. In the same line as definition 4.1 and theorem 4.1 in [7], we have

Definition 3. Assume $1 \le p \le n$, $u, v \in W^{1,p}(M,N)$. If for every bi-Lipschitz rectilinear cell decomposition $h : K \to M$, we have $u_{\#,p}(h) = v_{\#,p}(h)$, then we say u is $([p]-1)$-homotopic to v.

Theorem 1. *If* $1 \leq p \leq n$, $u, v \in W^{1,p}(M,N)$ *and* $A > 0$, *then there exists a positive number* $\varepsilon = \varepsilon(p, A, M, N) > 0$ *such that* $|du|_{L^p(M)}, |dv|_{L^p(M)} \leq A$ *and* $|u - v|_{L^p(M)} \leq \varepsilon$ *implies* u *is* $([p]-1)$*-homotopic to* v.

4.2. **Path connected components in $W^{1,p}(M,N)$.** Recall that for two maps $u, v \in W^{1,p}(M,N)$, we write $u \sim_p v$ to mean there exists a continuous path in $W^{1,p}(M,N)$ connecting u and v. If $p \geq n$, then it is known ([3, 13]) that we have a natural bijection $[M,N] \longrightarrow W^{1,p}(M,N) / \sim_p$. For $1 \leq p < n$, the boundary version of theorem 5.1 in [7] is also valid. That is

Theorem 2. *Assume* $1 \leq p < n$, *and* $u, v \in W^{1,p}(M,N)$. *Then* $u \sim_p v$ *if and only if* u *is* $([p]-1)$*-homotopic to* v.

Proof. The necessary part follows from Theorem 1. For the sufficient part, assume we have $u, v \in W^{1,p}(M,N)$ such that u is $([p]-1)$-homotopic to v, we want to show $u \sim_p v$. Indeed, take a bi-Lipschitz triangulation $h : K \to M$, we may find a $\xi \in B^l_{\varepsilon_0} \cap \mathbb{R}^l_+$ such that $u \circ h_\xi, v \circ h_\xi \in \mathcal{W}^{1,p}(K,N)$ and $u \circ h_\xi|_{|K^{[p]-1}|} \sim v \circ h_\xi|_{|K^{[p]-1}|}$. Here $h_\xi = \Phi_\xi \circ h$. Let $\bar{u} = u \circ \Phi_\xi$, $\bar{v} = u \circ \Phi_\xi$, then $\bar{u} \sim_p u$, $\bar{v} \sim_p v$. In fact, the map $t \mapsto u \circ \Phi_{t\xi}$ gives the needed path. Note that $\bar{u} \circ h, \bar{v} \circ h \in \mathcal{W}^{1,p}(K,N)$ and $\bar{u} \circ h|_{|K^{[p]-1}|} \sim \bar{v} \circ h|_{|K^{[p]-1}|}$, it follows from the same arguments as in the proof of theorem 5.1 of [7] that $\bar{u} \sim_p \bar{v}$. Hence $u \sim_p v$. □

Using Theorem 2, we may easily classify the path connected components of $W^{1,p}(M,N)$. Indeed, assume $1 \leq p < n$, take a bi-Lipschitz rectilinear cell decomposition of M, namely $h : K \to M$, and let $M^i = h\left(|K|^i\right)$ for $0 \leq i \leq n$, then we have two bijections

$$\left[M^{[p]}\Big|_{M^{[p]-1}}, N\right] \longleftarrow \operatorname{Lip}\left(M^{[p]}, N\right) / \sim_{M^{[p]-1}, \mathrm{Lip}} \longrightarrow W^{1,p}(M,N) / \sim_p .$$

The left pointing arrow is the usual one and the right pointing arrow can be described as follows: take any $u \in \operatorname{Lip}\left(M^{[p]}, N\right)$, let $f = u \circ h|_{|K^{[p]}|}$, fix an interior point for each of the cell in K, and let $\overline{f} : |K| \to N$ be the map deriving from f using homogeneous degree zero extension on each of the cell with dimension bigger than or equal to $[p]+1$. Let $\overline{u} = \overline{f} \circ h^{-1}$, then the right-going arrow sends the equivalence class of u to the equivalence class of $\overline{u}$. In particular, $W^{1,p}(M,N)$ is path connected if and only if $\left[M^{[p]}\big|_{M^{[p]-1}}, N\right]$ has only one element. This is clearly the case when either M or N is $([p]-1)$-connected. The proof of these statements are the same as those in section 5 of [7].

On the other hand, in the same line as proposition 5.2 and corollary 5.4 of [7], we know if $1 \leq p < n$, $u \in W^{1,p}(M,N)$ and $h : K \to M$ is a bi-Lipschitz rectilinear cell decomposition, then u may be connected to a smooth map if and only if $u_{\#,p}(h)$ has a continuous extension to M with respect to N; in particular, every map in $W^{1,p}(M,N)$ can be connected to a smooth map if and only if M satisfies the $([p]-1)$-extension property with respect to N. As an example, it is easy to see that for a M^3 with nonempty boundary, it satisfies 1-extension property with respect to any N, hence for any $1 \leq p < 3$, every map in $W^{1,p}\left(M^3, N\right)$ can be connected to a smooth map. This recovers theorem 0.4 of [2].

4.3. **Strong and weak density of smooth maps in $W^{1,p}(M,N)$.** Recall that

$$\begin{aligned} H^{1,p}_{\mathrm{S}}(M,N) &= \left\{u \in W^{1,p}(M,N) : \text{there exists a sequence } u_i \in C^\infty(M,N)\right. \\ &\qquad \left.\text{such that } u_i \to u \text{ in } W^{1,p}\left(M, \mathbb{R}^l\right)\right\}, \end{aligned}$$

$$H_{\mathrm{W}}^{1,p}(M,N) = \left\{u \in W^{1,p}(M,N) : \text{there exists a sequence } u_i \in C^{\infty}(M,N) \text{ such that } u_i \rightharpoonup u \text{ in } W^{1,p}\left(M,\mathbb{R}^l\right)\right\},$$

and

$$R^{p,\infty}(M,N) = \left\{u \in W^{1,p}(M,N) : \text{there exists a smooth rectilinear cell decomposition of } M,\ h: K \to M, \text{ and a dual } n-[p]-1 \text{ skeleton } L^{n-[p]-1} \text{ s.t. } u \text{ is } C^{\infty} \text{ on } M \backslash h\left(L^{n-[p]-1}\right)\right\}.$$

It is known ([13]) that when $p \geq n$, we have $H_{\mathrm{S}}^{1,p}(M,N) = H_{\mathrm{W}}^{1,p}(M,N) = W^{1,p}(M,N)$. For $1 \leq p < n$, one has

Theorem 3. *Assume $1 \leq p < n$, then $R^{p,\infty}(M,N)$ is dense in $W^{1,p}(M,N)$.*

Proof. Let $u \in W^{1,p}(M,N)$, then for $u \circ \Phi_{\xi} \to u$ as $\xi \to 0$ in $\mathbb{R}^l$. Fix a smooth cubeulation of M, $h: K \to M$ such that each of the cell in K is a normal cube. It follows from Lemma 6 and (the boundary version of) lemma 4.7 in [7] that for a.e. $\xi \in B_{\varepsilon_0}^l \cap \mathbb{R}_+^l$, we have $u \circ \Phi_{\xi} \in \mathcal{W}^{1,p}(K,N)$. For such ξ, the proof of theorem 6.1 of [7] shows $u \circ \Phi_{\xi} \in \overline{R^{p,\infty}(M,N)}$. Hence $u \in \overline{R^{p,\infty}(M,N)}$. □

The same arguments as in the proof of theorem 6.2 in [7] or theorem 5.4 in [8] give us

Theorem 4. *Assume $1 \leq p < n$, $h: K \to M$ is a bi-Lipschitz rectilinear cell decomposition, $M^i = h\left(\left|K^i\right|\right)$ for $i \geq 0$, $L^{n-[p]-1}$ is one of the dual $(n-[p]-1)$-skeletons, and $u \in W^{1,p}(M,N)$ such that u is continuous on $M \backslash h\left(L^{n-[p]-1}\right)$. Then $u \in H_S^{1,p}(M,N)$ if and only if $u|_{M^{[p]}}$ has a continuous extension to M. In addition, if for some $\alpha \in [M,N]$, we have $u|_{M^{[p]}} \in \alpha|_{M^{[p]}}$, then we may find a sequence $u_i \in C^{\infty}(M,N)$ such that $u_i \to u$ in $W^{1,p}(M,N)$.*

In the same line as theorem 6.3 of [7], it follows from Theorem 3 and Theorem 4 that

Theorem 5. *Assume $1 \leq p < n$, then $H_S^{1,p}(M,N) = W^{1,p}(M,N)$ if and only if $\pi_{[p]}(N) = 0$ and M satisfies the $([p]-1)$-extension property with respect to N.*

If $1 < p < n$ and p is not an integer, or $p = 1$, then $H_{\mathrm{W}}^{1,p}(M,N) = H_{\mathrm{S}}^{1,p}(M,N)$. Indeed, given any $u \in H_{\mathrm{W}}^{1,p}(M,N)$, let $\bar{u}: \mathrm{D}(M) \to N$ be defined by $\bar{u}([p,0]) = \bar{u}([p,1]) = u(p)$, then $\bar{u} \in H_{\mathrm{W}}^{1,p}(\mathrm{D}(M),N)$. It follows from [1, 7, 4] that $\bar{u} \in H_{\mathrm{S}}^{1,p}(\mathrm{D}(M),N)$, and this implies $u \in H_{\mathrm{S}}^{1,p}(M,N)$.

On the other hand, it follows easily from Theorem 1 and Theorem 2 that if $1 \leq p < n$, $u \in H_{\mathrm{W}}^{1,p}(M,N)$ and $h: K \to M$ is a bi-Lipschitz rectilinear cell decomposition, then $u_{\#,p}(h)$ has a continuous extension to M with respect to N, in addition, u may be connected to a smooth map by a continuous path in $W^{1,p}(M,N)$.

If $u \in W^{1,1}(M,N)$, then $\bar{u}: \mathrm{D}(M) \to N$, defined in the paragraph after Theorem 5, lies in $W^{1,1}(\mathrm{D}(M),N)$. It follows from [11, 4] that we may find a sequence $\bar{u}_i \in C^{\infty}(\mathrm{D}(M),N)$ such that $|\bar{u}_i - \bar{u}|_{L^1(\mathrm{D}(M))} \to 0$, $d\bar{u}_i \to d\bar{u}$ a.e. on $\mathrm{D}(M)$ and $|d\bar{u}_i|_{L^p(\mathrm{D}(M))} \leq c(N)\,|d\bar{u}|_{L^p(\mathrm{D}(M))}$. Let $u_i = \bar{u}_i|_M$, then $|u_i - u|_{L^1(\mathrm{D}(M))} \to 0$, $du_i \to du$ a.e. on M and $|du_i|_{L^p(M)} \leq c(N)\,|du|_{L^p(M)}$.

If $n \geq 3$, $h : K \to M$ is a bi-Lipschitz rectilinear cell decomposition, then we claim

$$\begin{aligned} & H^{1,2}_{\mathrm{W}}(M,N) \\ = & \left\{u \in W^{1,2}(M,N) : u_{\#,2}(h) \text{ has a continuous extension to } M \text{ w.r.t. } N\right\} \\ = & \left\{u \in W^{1,2}(M,N) : u \text{ may be connected to smooth maps}\right\}. \end{aligned}$$

Moreover, if $u \in W^{1,2}(M,N)$ and $\alpha \in [M,N]$ satisfies $\alpha \circ h|_{|K^1|} = u_{\#,2}(h)$, then we may find a sequence $u_i \in C^\infty(M,N)$ such that $u_i \rightharpoonup u$ in $W^{1,2}(M,N)$, $du_i \to du$ a.e. and $[u_i] = \alpha$. Indeed, take a $v \in \mathrm{Lip}(M,N)$ such that $[v] = \alpha$. Since $u_{\#,2}(h) = \alpha \circ h|_{|K^1|} = v_{\#,2}(h)$, we know u may be connected to v by a continuous path in $W^{1,2}(M,N)$. Let $\overline{u}, \overline{v} : \mathrm{D}(M) \to N$ be the double of maps u and v as in the previous paragraph. Then we may connect u and v continuously in $W^{1,2}(\mathrm{D}(M), N)$. It follows from [5], which is based on an important local result in [12], that we may find a sequence $\overline{u}_i \in C^\infty(\mathrm{D}(M), N)$ such that $\overline{u}_i \rightharpoonup \overline{u}$ in $W^{1,2}(M,N)$, $d\overline{u}_i \to d\overline{u}$ a.e. and $[\overline{u}_i] = [\overline{v}]$. Let $u_i = \overline{u}_i|_M$, then clearly u_i satisfies all the requirements.

For $3 \leq p < n$, $p \in \mathbb{N}$, it remains a hard problem to characterize $H^{1,p}_{\mathrm{W}}(M,N)$. Under the additional assumption that N satisfies the p-vanishing condition (see definition 8.4 of [8]), we know

$$\begin{aligned} & H^{1,p}_{\mathrm{W}}(M,N) \\ = & \left\{u \in W^{1,p}(M,N) : u_{\#,2}(h) \text{ has a continuous extension to } M \text{ w.r.t. } N\right\} \\ = & \left\{u \in W^{1,p}(M,N) : u \text{ may be connected to smooth maps}\right\}, \end{aligned}$$

here $h : K \to M$ is a bi-Lipschitz rectilinear cell decomposition. Moreover, if $u \in W^{1,p}(M,N)$ and $\alpha \in [M,N]$ satisfies $\alpha \circ h|_{|K^{p-1}|} = u_{\#,p}(h)$, then we may find a sequence $u_i \in C^\infty(M,N)$ such that $u_i \rightharpoonup u$ in $W^{1,p}(M,N)$, $du_i \to du$ a.e., $|du_i| \leq c(p,M,N)\, |du|_{L^p(M)}$ and $[u_i] = \alpha$. This may be proved using $\bar{u} : \mathrm{D}(M) \to N$ and theorem 8.5 in [8], just the same way as in the previous paragraph. We remark that the method in [5] reduces the problem for general target N to a local question. There is also a very interesting recent work for $W^{1,3}(B^4, S^2)$ in [9].

5. **Dirichlet boundary conditions.** In this section, we always assume M^n is a smooth compact Riemannian manifold with a nonempty boundary, N is a smooth compact Riemannian manifold without boundary and it is embedded into $\mathbb{R}^{\bar{l}}$. Let $\overline{\varepsilon}_0 = \overline{\varepsilon}_0(N) > 0$ be a small positive number such that

$$V_{2\overline{\varepsilon}_0}(N) = \left\{y \in \mathbb{R}^{\bar{l}} : \mathrm{dist}(y,N) < 2\overline{\varepsilon}_0\right\}$$

is a tubular neighborhood of N. Denote $\pi_N : V_{2\overline{\varepsilon}_0}(N) \to N$ as the nearest point projection. Assume $1 \leq p < \infty$, $\varphi \in C(\partial M, N) \cap W^{1,p}(\partial M, N)$, we let

$$W^{1,p}_{\varphi}(M,N) = \left\{u \in W^{1,p}(M,N) : u|_{\partial M} = \varphi\right\}.$$

The aim of this section is to classify the path connected components of $W^{1,p}_{\varphi}(M,N)$ and study when continuous $W^{1,p}$ maps are dense in $W^{1,p}_{\varphi}(M,N)$. We will use those notations from Section 3.2.

5.1. **Homotopy of Sobolev mappings.** In this subsection, we will study the topological information carried by a Sobolev map. These issues were studied systematically in [15]. We will use somewhat different methods to obtain the main conclusions in section 4 of [15] as well as some generalizations. We shall closely follow section 4 of [7].

Let X be a compact metric space, $A \subset X$ be a closed subset and $f \in C(A, N)$, then $[X, N; f]_{\mathrm{rel}.A}$ is countable. Indeed, since $C_f(X, N) \subset C\left(X, \mathbb{R}^{\bar{l}}\right)$, which has a countable topological base, we may find a countable dense subset $\mathcal{N}$ of $C_f(X, N)$. If $u \in C_f(X, N)$, we may find a $v \in \mathcal{N}$ such that $|u(x) - v(x)| < \bar{\varepsilon}_0$ for $x \in X$. Let $H(x, t) = \pi_N((1-t)u(x) + tv(x))$ for $x \in X, 0 \le t \le 1$, then we see $u \sim_{\mathrm{rel}.A} v$. The countability of $[X, N; f]_{\mathrm{rel}.A}$ follows. This countability is needed later to discuss the measurability of maps into $[X, N; f]_{\mathrm{rel}.A}$.

Assume $1 \le p \le n$, $u \in W^{1,p}(M, N)$, $C \subset M$ is a subset such that $\left.u\right|_C \in C(C, N)$, K is a finite rectilinear cell complex, $K_0 \subset K$ is a subcomplex. It follows that (K, K_0) is a relative complex with k-skeleton $(K, K_0)^k = K^k \cup K_0$. Assume P^m is a simply connected smooth Riemannian manifold of dimension m, $H : |K| \times P \to M$ is a map such that $\left.H\right|_{\Delta \times P}$ satisfies (H_1), (H_2) and (H_3) in section 3 of [7] for any $\Delta \in K$ and $H(|K_0| \times P) \subset C$.

Let k_1 and k_2 be nonnegative integers such that $k_1 \ge k_2$. Given $\xi_1, \xi_2 \in P$, we want to define a map μ_{ξ_2,ξ_1} from

$$\left[\left|K^{k_1} \cup K_0\right|\big|_{\left|K^{k_2} \cup K_0\right|}, N; u \circ H_{\xi_1}|_{|K_0|}\right]_{\mathrm{rel}.|K_0|}$$

to

$$\left[\left|K^{k_1} \cup K_0\right|\big|_{\left|K^{k_2} \cup K_0\right|}, N; u \circ H_{\xi_2}|_{|K_0|}\right]_{\mathrm{rel}.|K_0|}.$$

Choose a continuous curve $\xi(\cdot)$ in P such that $\xi(0) = \xi_1$, $\xi(1) = \xi_2$. Let $\Sigma : |K_0| \times [0,1] \to N$ be given by $\Sigma(x, t) = u(H(x, \xi(t)))$ for $x \in |K_0|$, $0 \le t \le 1$. It follows from simply connected connectedness of P and the discussions in Section 2 that the map μ_Σ from

$$\left[\left|K^{k_1} \cup K_0\right|\big|_{\left|K^{k_2} \cup K_0\right|}, N; u \circ H_{\xi_1}|_{|K_0|}\right]_{\mathrm{rel}.|K_0|}$$

to

$$\left[\left|K^{k_1} \cup K_0\right|\big|_{\left|K^{k_2} \cup K_0\right|}, N; u \circ H_{\xi_2}|_{|K_0|}\right]_{\mathrm{rel}.|K_0|}.$$

does not depend on the specific choice of the path $\xi(\cdot)$. Hence we define this map as μ_{ξ_2,ξ_1}. It is easy to see that for $\xi_1, \xi_2, \xi_3 \in P$, $\mu_{\xi_3,\xi_2} \circ \mu_{\xi_2,\xi_1} = \mu_{\xi_3,\xi_1}$.

Fix a point $\xi_0 \in P$, we want to define a measurable map

$$\chi = \chi_{k,H,|K_0|,u} : P \to \left[\left|K^{[p]} \cup K_0\right|\Big|_{\left|K^k \cup K_0\right|}, N; u \circ H_{\xi_0}|_{|K_0|}\right]_{\mathrm{rel}.|K_0|}.$$

Indeed, it follows from lemma 3.3 and corollary 3.1 of [7] that we may find a Borel set $E \subset P$ such that $\mathcal{H}^m(E) = 0$ and $u \circ H_\xi \in \mathcal{W}^{1,p}(K, N)$ for $\xi \in P \backslash E$.

If p is not an integer, by Sobolev embedding theorem we see $u \circ H_\xi|_{\left|K^k \cup K_0\right|} \in C\left(\left|K^k \cup K_0\right|\right)$, then we let

$$\chi(\xi) = \mu_{\xi_0,\xi}\left(\left[u \circ H_\xi|_{\left|K^{[p]} \cup K_0\right|}\right]_{\left|K^k \cup K_0\right|, \mathrm{rel}.|K_0|}\right).$$

We remark that, under the identification made in Section 2,

$$\chi(\xi)=\mu_{\xi_0,\xi}\left(\left[u\circ H_\xi|_{|K^k\cup K_0|}\right]_{\text{rel.}|K_0|}\right).$$

To prove the measurability of χ, we observe that it follows from corollary 3.1 in [7] that if we set $\widetilde{u}(\xi)=u\circ H_\xi$, then $\widetilde{u}\in L^p\left(P,\mathcal{W}^{1,p}(K,N)\right)$. It follows from Lusin's theorem that for any $\varepsilon>0$, we may find a closed subset $F\subset P$ such that $\mathcal{H}^m(P\backslash F)<\varepsilon$ and $\widetilde{u}|_F\in C\left(F,\mathcal{W}^{1,p}(K,N)\right)$. Given $\xi_1\in F$, we may find a neighborhood $U(\xi_1)$ in P, such that for any $\xi_2\in U(\xi_1)$, we may find a path $\xi(\cdot)$ in P from ξ_1 to ξ_2, such that for any $0\le t\le 1$ and any $x\in|K_0|$, we have $|u(H(x,\xi(t)))-u(H(x,\xi_1))|\le\frac{\overline{\varepsilon}_0}{4}$. On the other hand, by Sobolev embedding theorem, we know if $U(\xi_1)$ is small enough, then $|u(H(x,\xi_2))-u(H(x,\xi_1))|\le\frac{\overline{\varepsilon}_0}{4}$ for any $x\in\left|K^k\right|$. We claim

$$\mu_{\xi_2,\xi_1}\left(\left[u\circ H_{\xi_1}|_{|K^k\cup K_0|}\right]_{\text{rel.}|K_0|}\right)=\left[u\circ H_{\xi_2}|_{|K^k\cup K_0|}\right]_{\text{rel.}|K_0|}.$$

In fact, let $\Sigma(x,t)=u(H(x,\xi(t)))$ for $x\in|K_0|$, $0\le t\le 1$, and

$$\Sigma_1(x,t)=\pi_N((1-t)\,u(H(x,\xi_1))+tu(H(x,\xi_2)))$$

for $x\in\left|K^k\cup K_0\right|$, then we know $|\Sigma(x,t)-\Sigma_1(x,t)|\le\overline{\varepsilon}_0$ for $x\in|K_0|$, $0\le t\le 1$, this implies $\Sigma\sim_{\text{rel.}|K_0|\times\{0,1\}}\Sigma_1|_{|K_0|\times[0,1]}$. By the discussions in Section 2, we have

$$\begin{aligned}\mu_{\xi_2,\xi_1}\left(\left[u\circ H_{\xi_1}|_{|K^k\cup K_0|}\right]_{\text{rel.}|K_0|}\right)&=\mu_\Sigma\left(\left[u\circ H_{\xi_1}|_{|K^k\cup K_0|}\right]_{\text{rel.}|K_0|}\right)\\&=\mu_{\Sigma_1|_{|K_0|\times[0,1]}}\left(\left[u\circ H_{\xi_1}|_{|K^k\cup K_0|}\right]_{\text{rel.}|K_0|}\right)\\&=\left[u\circ H_{\xi_2}|_{|K^k\cup K_0|}\right]_{\text{rel.}|K_0|}.\end{aligned}$$

Hence

$$\begin{aligned}\mu_{\xi_0,\xi_1}\left(\left[u\circ H_{\xi_1}|_{|K^k\cup K_0|}\right]_{\text{rel.}|K_0|}\right)&=\mu_{\xi_0,\xi_2}\left(\mu_{\xi_2,\xi_1}\left(\left[u\circ H_{\xi_1}|_{|K^k\cup K_0|}\right]_{\text{rel.}|K_0|}\right)\right)\\&=\mu_{\xi_0,\xi_2}\left(\left[u\circ H_{\xi_2}|_{|K^k\cup K_0|}\right]_{\text{rel.}|K_0|}\right).\end{aligned}$$

This shows χ is locally constant on F. The measurability follows.

If p is an integer and $0\le k<p$, then by lemma 4.4 of [7], for $\xi\in P\backslash E$, we may find a $g\in C(|K^p\cup K_0|,N)\cap\mathcal{W}^{1,p}(K^p\cup K_0,N)$ such that $g|_{|K^{p-1}\cup K_0|}=u\circ H_\xi|_{|K^{p-1}\cup K_0|}$. Define

$$\chi(\xi)=\mu_{\xi_0,\xi}\left([g]_{|K^k\cup K_0|,\text{rel.}|K_0|}\right).$$

Under the identification made in Section 2, we have

$$\chi(\xi)=\mu_{\xi_0,\xi}\left(\left[u\circ H_\xi|_{|K^k\cup K_0|}\right]_{\text{rel.}|K_0|}\right).$$

For the case p is an integer and $k=p$, we need some preparations. Let K and K_0 be the same as above and $f\in C(|K_0|,N)\cap\mathcal{W}^{1,k}(K_0,N)$. We want to define a map

$$\Theta:\left\{v\in\mathcal{W}^{1,k}\left(K^k\cup K_0,N\right):v|_{K_0}=f\right\}\to\left[\left|K^k\cup K_0\right|,N;f\right]_{\text{rel.}|K_0|}.$$

Indeed, it follows from lemma 4.4 of [7] that we may find an $\varepsilon = \varepsilon(k, K, K_0, N, u) > 0$ and a $w \in C\left(\left|K^k \cup K_0\right|, N\right) \cap \mathcal{W}^{1,k}\left(K^k \cup K_0, N\right)$ such that $w|_{\left|K^{k-1} \cup K_0\right|} = v|_{\left|K^{k-1} \cup K_0\right|}$ and $|v - w|_{\mathcal{W}^{1,k}(K^k \cup K_0)} \leq \varepsilon$. Then we define $\Theta(v) = [w]_{\text{rel.}|K_0|}$. By lemma 4.4 of [7], $\Theta(v)$ does not depend on the choice when ε is small enough. Hence Θ is well defined. In addition, it follows from corollary 4.1 in [7] that Θ is a locally constant map. For $\xi \in P \backslash E$, define

$$\chi(\xi) = \mu_{\xi_0, \xi}\left(\Theta\left(u \circ H_\xi|_{|K^p| \cup |K_0|}\right)\right).$$

Then similar arguments as before show that χ is a measurable map.

A similar argument as the one for proposition 4.1 in [7] gives us

Proposition 2. *Assume* $1 \leq p \leq n$, *k is a nonnegative integer,* $u, u_i \in W^{1,p}(M, N)$, $C \subset M$ *is a subset such that* $u_i|_C = u|_C \in C(C, N)$, *$K$ is a finite rectilinear cell complex,* $K_0 \subset K$ *is a subcomplex. Also assume* P^m *is a simply connected smooth Riemannian manifold of dimension m,* $\xi_0 \in P$ *is a given point,* $H : |K| \times P \to M$ *is a map such that* $H|_{\Delta \times P}$ *satisfies* (H_1), (H_2) *and* (H_3) *in section 3 of* [7] *for any* $\Delta \in K$, *and* $H(|K_0| \times P) \subset C$. *If either*

- $0 \leq k \leq p$ *and* $u_i \to u$ *in* $W^{1,p}(M, N)$;

or

- $0 \leq k < p$ *and* $u_i \rightharpoonup u$ *in* $W^{1,p}(M, N)$,

then after passing to a subsequence we have $\chi_{k,H,|K_0|,u_{i'}} \to \chi_{k,H,|K_0|,u}$ *a.e. on P.*

Next we want to introduce the $([p]-1)$-homotopy class of a Sobolev map. Assume we have the same $p, u, C, K, K_0, P, \xi_0$ and H as in the assumption of Proposition 2. If we know $k \leq [p] - 1$, then we claim

$$\chi_{k,H,|K_0|,u} \equiv \text{const} \quad \mathcal{H}^m \text{ a.e. on } P.$$

The proof of this claim follows from the arguments for lemma 4.7 in [7] with minor modifications.

Let $1 \leq p \leq n$, $u \in W^{1,p}_\varphi(M, N)$, K be a finite rectilinear cell complex, $K_0 \subset K$ be a subcomplex, and $h : |K| \to M$ be a Lipschitz map such that $h(|K_0|) \subset \partial M$. Denote $\mathbb{R}^l_- = (-\infty, 0] \times \mathbb{R}^{l-1}$. Let $H : |K| \times \left(B^l_{\varepsilon_0} \cap \mathbb{R}^l_-\right) \to M_{\varepsilon_0}$ (see Section 3.2 for definitions) be the map given by $H(x, \xi) = \Phi(h(x), \xi) = \Phi_\xi(h(x)) = h_\xi(x)$. Let $C = \left\{\psi^{-1}(q, t) : q \in \partial M, -\varepsilon_0 \leq t \leq 0\right\}$, then $H\left(|K_0| \times \left(B^l_{\varepsilon_0} \cap \mathbb{R}^l_-\right)\right) \subset C$. We define $\overline{u} : M_{\varepsilon_0} \to N$ by $\overline{u}|_M = u$ and $\overline{u}\left(\psi^{-1}(q, t)\right) = \varphi(q)$ for $q \in \partial M$ and $-\varepsilon_0 \leq t \leq 0$. Then $\overline{u} \in W^{1,p}(M_{\varepsilon_0}, N)$ and $\overline{u}|_C$ is continuous. Let $\xi_0 = 0 \in B^l_{\varepsilon_0} \cap \mathbb{R}^l_-$, then it follows from the previous paragraph that $\chi_{[p]-1,H,|K_0|,\overline{u}} \equiv \text{const } \mathcal{H}^l$ a.e. on $B^l_{\varepsilon_0} \cap \mathbb{R}^l_-$. We denote this constant as $u_{\#,p,|K_0|}(h)$. Note that $u_{\#,p,|K_0|}(h) \in \left[\left|K^{[p]} \cup K_0\right|\big|_{\left|K^{[p]-1} \cup K_0\right|}, N; \varphi \circ h|_{|K_0|}\right]_{\left|K^{[p]-1} \cup K_0\right|, \text{rel.}|K_0|}$, which may be identified as a subset of $\left[\left|K^{[p]-1} \cup K_0\right|, N; \varphi \circ h|_{|K_0|}\right]_{\text{rel.}|K_0|}$.

This homotopy class behaves well topologically. Indeed, if $1 \leq p \leq n$, $u \in W^{1,p}_\varphi(M, N)$, K is a finite rectilinear cell complex, $K_0 \subset K$ is a subcomplex. $h_0, h_1 : |K| \to M$ are two Lipschitz maps such that $h_0(|K_0|), h_1(|K_0|) \subset \partial M$ and we may find a homotopy $g : |K| \times [0,1] \to M$ from h_0 to h_1 with $g(|K_0| \times [0,1]) \subset \partial M$. Let $\Sigma(x, t) = u(g(x, t))$ for $x \in |K_0|$, $0 \leq t \leq 1$, then we have

$$\mu_\Sigma\left(u_{\#,p,|K_0|}(h_0)\right) = u_{\#,p,|K_0|}(h_1). \tag{11}$$

In fact, first we observe that we may assume g is a Lipschitz map. To see this, we note that it follows from Proposition 1 that there exists a map $g_1 \in \mathrm{Lip}\left(|K_0| \times [0,1], \partial M\right)$ such that $g_1|_{|K_0| \times \{0,1\}} = g|_{|K_0| \times \{0,1\}}$ and $g_1 \sim_{\mathrm{rel.}|K_0| \times \{0,1\}} g|_{|K_0| \times [0,1]}$. Let $g_1(x,0) = h_0(x)$, $g_1(x,1) = h_1(x)$ for $x \in K$, then clearly $g_1 \sim_{\mathrm{rel.}|K| \times \{0,1\}} g|_{(|K| \times \{0,1\}) \cup (|K_0| \times [0,1])}$. By homotopy extension property, g_1 has a continuous extension to $|K| \times [0,1]$, still denoted as g_1. Applying Proposition 1 again, we may find $g_2 \in \mathrm{Lip}\left(|K| \times [0,1], M\right)$ such that $g_2|_{(|K| \times \{0,1\}) \cup (|K_0| \times [0,1])} = g_1|_{(|K| \times \{0,1\}) \cup (|K_0| \times [0,1])}$ and $g_2 \sim g_1$ rel. $(|K| \times \{0,1\}) \cup (|K_0| \times [0,1])$. Clearly we may replace g by g_2. Next, define a new rectilinear cell complex $\widetilde{K}$ by

$$\widetilde{K} = \left\{\Delta \times \{0\}, \Delta \times \{1\}, \Delta \times [0,1] : \Delta \in K\right\}.$$

Then $\left|\widetilde{K}\right| = |K| \times [0,1]$. We may find a Borel set $E \subset B_{\varepsilon_0}^l \cap \mathbb{R}_-^l$ with $\mathcal{H}^l(E) = 0$ and for any $\xi \in \left(B_{\varepsilon_0}^l \cap \mathbb{R}_-^l\right) \backslash E$, $\overline{u} \circ g_\xi \in \mathcal{W}^{1,p}\left(\widetilde{K}, N\right)$. Observing that $\overline{u} \circ g_\xi(x,0) = \overline{u} \circ (h_0)_\xi(x)$ and $\overline{u} \circ g_\xi(x,1) = \overline{u} \circ (h_1)_\xi(x)$ for $x \in K$, using lemma 4.4 of [7] when $p \in \mathbb{N}$, we see

$$\mu_{\Sigma_1}\left(\left[\overline{u} \circ (h_0)_\xi\Big|_{\left|K^{[p]-1} \cup K_0\right|}\right]_{\mathrm{rel.}|K_0|}\right) = \left[\overline{u} \circ (h_1)_\xi\Big|_{\left|K^{[p]-1} \cup K_0\right|}\right]_{\mathrm{rel.}|K_0|},$$

here $\Sigma_1(x,t) = \overline{u}\left(g_\xi(x,t)\right)$ for $x \in |K_0|$, $0 \le t \le 1$. We claim

$$\begin{aligned} & \mu_{0,\xi}^{h_1}\left(\mu_{\Sigma_1}\left(\left[\overline{u} \circ (h_0)_\xi\Big|_{\left|K^{[p]-1} \cup K_0\right|}\right]_{\mathrm{rel.}|K_0|}\right)\right) \\ = \; & \mu_\Sigma\left(\mu_{0,\xi}^{h_0}\left(\left[\overline{u} \circ (h_0)_\xi\Big|_{\left|K^{[p]-1} \cup K_0\right|}\right]_{\mathrm{rel.}|K_0|}\right)\right). \end{aligned}$$

In fact, let

$$\begin{aligned} \Lambda_0(x,t) &= \begin{cases} \overline{u}\left(\Phi\left(h_0(x), (1-2t)\xi\right)\right), & x \in |K_0|, 0 \le t \le \frac{1}{2}; \\ \overline{u}\left(\Phi\left(g(x,2t-1), 0\right)\right), & x \in |K_0|, \frac{1}{2} \le t \le 1, \end{cases} \\ \Lambda_1(x,t) &= \begin{cases} \overline{u}\left(\Phi\left(g(x,2t), \xi\right)\right), & x \in |K_0|, 0 \le t \le \frac{1}{2}; \\ \overline{u}\left(\Phi\left(h_1(x), (2-2t)\xi\right)\right) & x \in |K_0|, \frac{1}{2} \le t \le 1, \end{cases} \end{aligned}$$

we see easily that $\Lambda_0 \sim_{\mathrm{rel.}|K_0| \times \{0,1\}} \Lambda_1$. The claim follows from discussions in Section 2. Hence we have

$$\begin{aligned} & \mu_\Sigma\left(\mu_{0,\xi}^{h_0}\left(\left[\overline{u} \circ (h_0)_\xi\Big|_{\left|K^{[p]-1} \cup K_0\right|}\right]_{\mathrm{rel.}|K_0|}\right)\right) \\ = \; & \mu_{0,\xi}^{h_1}\left(\left[\overline{u} \circ (h_1)_\xi\Big|_{\left|K^{[p]-1} \cup K_0\right|}\right]_{\mathrm{rel.}|K_0|}\right), \end{aligned}$$

this clearly implies $\mu_\Sigma\left(u_{\#,p,|K_0|}(h_0)\right) = u_{\#,p,|K_0|}(h_1)$.

Besides the above stated homotopy invariance, we have the following statement. Assume $1 \le p \le n$, $u, v \in W_\varphi^{1,p}(M,N)$, K is a finite rectilinear cell complex and $K_0 \subset K$ is a subcomplex, and $h : (|K|, |K_0|) \to (M, \partial M)$ is a Lipschitz map. If $h : |K| \to M$ is a homeomorphism, $h(|K_0|) = \partial M$ and $u_{\#,p,|K_0|}(h) = v_{\#,p,|K_0|}(h)$, then for any finite rectilinear cell complex L, a subcomplex $L_0 \subset L$ and a Lipschitz map $g : (|L|, |L_0|) \to (M, \partial M)$, we have $u_{\#,p,|L_0|}(g) = v_{\#,p,|L_0|}(g)$.

To prove this statement, we may assume $L = L^{[p]-1} \cup L_0$. Observe that by the cellular approximation theorem, we may find a $g_0 \in C(|L|, M)$ such that

$g \sim_{\mathrm{rel.}|L_0|} g_0$ and $g_0(|L|) \subset h\left(\left|K^{[p]-1} \cup K_0\right|\right)$. This shows $h^{-1} \circ g_0$ is a continuous map from $(|L|, |L_0|)$ to $\left(\left|K^{[p]-1} \cup K_0\right|, |K_0|\right)$. We may find a Lipschitz map $f : (|L|, |L_0|) \to \left(\left|K^{[p]-1} \cup K_0\right|, |K_0|\right)$ such that $f \sim h^{-1} \circ g_0$ as pairs from $(|L|, |L_0|)$ to $\left(\left|K^{[p]-1} \cup K_0\right|, |K_0|\right)$. This implies $h \circ f \sim g_0 \sim g$ as pairs from $(|L|, |L_0|)$ to $(M, \partial M)$. It is clearly true that $u_{\#,p,|L_0|}(h \circ f) = v_{\#,p,|L_0|}(h \circ f)$. This and (11) implies $u_{\#,p,|L_0|}(g) = v_{\#,p,|L_0|}(g)$.

Definition 4. Assume $1 \leq p \leq n$, $u, v \in W^{1,p}_{\varphi}(M, N)$. If for any bi-Lipschitz rectilinear cell decomposition $h : K \to M$ with $K_0 \subset K$, the subcomplex such that $h(|K_0|) = \partial M$, we have $u_{\#,p,|K_0|}(h) = v_{\#,p,|K_0|}(h)$, then we say u is $([p]-1)$-homotopic to v rel. ∂M.

Similar arguments for theorem 4.1 in [7] give us the following

Theorem 6. *If $1 \leq p \leq n$, $u, v \in W^{1,p}_{\varphi}(M, N)$ and $A > 0$ with $|d\varphi|_{L^p(\partial M)} \leq A$, then there exists a positive number $\varepsilon = \varepsilon(p, A, M, N)$ such that $|du|_{L^p(M)}, |dv|_{L^p(M)} \leq A$ and $|u - v|_{L^p(M)} \leq \varepsilon$ imply that u is $([p]-1)$-homotopic to v rel. ∂M.*

5.2. **Path connected components in $W^{1,p}_{\varphi}(M, N)$.** Recall that for any two maps $u, v \in W^{1,p}_{\varphi}(M, N)$, we write $u \sim_{p,\mathrm{rel}.\partial M} v$ to mean we may find a continuous path in $W^{1,p}_{\varphi}(M, N)$ connecting u and v. When $p \geq n$, it is known ([3, 13]) that there is a natural bijection $[M, N; \varphi]_{\mathrm{rel}.\partial M} \to W^{1,p}_{\varphi}(M, N) / \sim_{p,\mathrm{rel}.\partial M}$. For $1 \leq p < n$, in the same line as theorem 5.1 in [7], we have

Theorem 7. *Assume $1 \leq p < n$, and $u, v \in W^{1,p}_{\varphi}(M, N)$. Then $u \sim_{p,rel.\partial M} v$ if and only if u is $([p]-1)$-homotopic to v.*

To prove this theorem, we need to do some preparations.

Lemma 7. *Assume $1 \leq p < \infty$, K is a finite rectilinear cell complex, $K_0 \subset K$ is a subcomplex and $f \in C(|K_0|, \mathbb{R}) \cap \mathcal{W}^{1,p}(K_0, \mathbb{R})$, then there exists a $u \in C(|K|, \mathbb{R}) \cap \mathcal{W}^{1,p}(K, \mathbb{R})$ such that $u|_{|K_0|} = f$.*

Proof. For each $\Delta \in K$, take an interior point $y_\Delta \in \mathrm{int}\Delta$. For $x \in \Delta$, let $|x|_\Delta$ be the Minkowski norm of x with respect to y_Δ i.e.

$$|x|_\Delta = \inf\{t \in \mathbb{R} : t > 0, x - y_\Delta \in t(\Delta - y_\Delta)\}.$$

We will define u on $|K|$ by induction. For $x \in |K_0|$, we let $u(x) = f(x)$. For $x \in \left|K^0\right| \setminus |K_0|$, we let $u(x) = 0$. Then clearly $u \in C\left(\left|K^0 \cup K_0\right|, \mathbb{R}\right) \cap \mathcal{W}^{1,p}\left(K^0 \cup K_0, \mathbb{R}\right)$. Assume for some $i \geq 0$, we have a $u \in C\left(\left|K^i \cup K_0\right|, \mathbb{R}\right) \cap \mathcal{W}^{1,p}\left(K^i \cup K_0, \mathbb{R}\right)$ such that $u|_{|K_0|} = f$, then for any $\Delta \in K^{i+1} \setminus \left(K^i \cup K_0\right)$, we let $u(x) = |x|_\Delta u\left(y_\Delta + \frac{x - y_\Delta}{|x|_\Delta}\right)$ for $x \in \Delta$, then clearly $u \in C\left(\left|K^{i+1} \cup K_0\right|, \mathbb{R}\right) \cap \mathcal{W}^{1,p}\left(K^{i+1} \cup K_0, \mathbb{R}\right)$. After finite many steps, we get $u \in C(|K|, \mathbb{R}) \cap \mathcal{W}^{1,p}(K, \mathbb{R})$ such that $u|_{|K_0|} = f$. ☐

Lemma 8. *Assume $1 \leq p < \infty$, K is a finite rectilinear cell complex, $K_0 \subset K$ is a subcomplex and $f \in C(|K_0|, N) \cap \mathcal{W}^{1,p}(K_0, N)$ such that there exists a $v \in C(|K|, N)$ with $v|_{|K_0|} = f$. Then for any $\varepsilon > 0$, there exists a $u \in C(|K|, N) \cap \mathcal{W}^{1,p}(K, N)$ such that $u|_{|K_0|} = f$ and $|u(x) - v(x)| < \varepsilon$ for $x \in |K|$. In particular, this implies $u \sim_{rel.|K_0|} v$ when ε is small enough.*

Proof. By Lemma 7, we may find a $v_1 \in C\left(|K|, \mathbb{R}^{\bar{l}}\right) \cap \mathcal{W}^{1,p}\left(K, \mathbb{R}^{\bar{l}}\right)$ such that $v_1|_{|K_0|} = f$. By lemma 2.3 of [7] we may find a $v_2 \in \mathrm{Lip}\left(|K|, \mathbb{R}^{\bar{l}}\right)$ such that

$|v_2(x) - v(x)| < \frac{\varepsilon}{4}$ for $x \in |K|$. By continuity, we may find an open neighborhood of $|K_0|$ (open in $|K|$) such that for any $x \in U$, there exists a $x_0 \in |K_0|$ with $|v_1(x) - v_1(x_0)| < \frac{\varepsilon}{8}$ and $|v(x) - v(x_0)| < \frac{\varepsilon}{8}$. In particular, this shows for any $x \in U$, $|v_1(x) - v(x)| < \frac{\varepsilon}{4}$. Choose a function $\eta \in \mathrm{Lip}(|K|, \mathbb{R})$ such that $0 \le \eta \le 1$, $\eta|_{|K_0|} = 1$, $\eta|_{|K| \setminus U} = 0$. Let $w(x) = \eta(x) v_1(x) + (1 - \eta(x)) v_2(x)$ for $x \in |K|$, then

$$|w(x) - v(x)| \le \eta(x) |v_1(x) - v(x)| + (1 - \eta(x)) |v_2(x) - v(x)| \le \frac{\varepsilon}{4}.$$

When $\varepsilon < \overline{\varepsilon}_0$, we let $u(x) = \pi_N(w(x))$ for $x \in |K|$, then $u \in C(|K|, N) \cap \mathcal{W}^{1,p}(K, N)$ and

$$|u(x) - v(x)| \le |u(x) - w(x)| + |w(x) - v(x)| \le 2|w(x) - v(x)| \le \frac{\varepsilon}{2} < \varepsilon.$$

□

Lemma 9. *Assume* $1 \le p < \infty$, *K is a finite rectilinear cell complex,* $K_0 \subset K$ *is a subcomplex,* $f \in C(|K_0|, N) \cap \mathcal{W}^{1,p}(K_0, N)$ *and* $u, v \in C(|K|, N) \cap \mathcal{W}^{1,p}(K, N)$ *such that* $u|_{|K_0|} = v|_{|K_0|} = f$. *If* $u \sim_{rel.|K_0|} v$, *then we may find a* $H \in C\left([0,1], C(|K|, N) \cap \mathcal{W}^{1,p}(K, N)\right)$ *such that* $H(0) = u$, $H(1) = v$ *and* $H(t)|_{|K_0|} = f$ *for* $0 \le t \le 1$.

Proof. The argument is very similar to those for part (3) of proposition 2.3 in [7]. The only difference is that we need to use Lemma 8 at suitable place instead of part (1) of proposition 2.3 in [7]. □

Proof of Theorem 7. The necessary part follows from Theorem 6. To prove the sufficient part, we will use notations from Section 3.2 and Section 5.1. Assume we have $u, v \in W^{1,p}_{\varphi}(M, N)$ such that u is $([p]-1)$-homotopic to v. Take a bi-Lipschitz triangulation $h : K \to M$ and let $K_0 \subset K$ be the subcomplex such that $h(|K_0|) = \partial M$. Let $\overline{u}, \overline{v} \in W^{1,p}(M_{\varepsilon_0}, N)$ be the corresponding extension maps, then we may find a $\xi \in B^l_{\varepsilon_0} \cap \mathbb{R}^l_-$ such that $\overline{u} \circ h_\xi, \overline{v} \circ h_\xi \in \mathcal{W}^{1,p}(K, N)$ and $\overline{u} \circ h_\xi|_{\left|K^{[p]-1} \cup K_0\right|} \sim_{\mathrm{rel.}|K_0|} \overline{v} \circ h_\xi|_{\left|K^{[p]-1} \cup K_0\right|}$. Let $\xi' = \left(0, \xi^2, \cdots, \xi^l\right)$, then $h_{\xi'} : K \to M$ is still a bi-Lipschitz triangulation with $h_{\xi'}(|K_0|) = \partial M$. Let $\widetilde{u} = \overline{u} \circ \phi_{X,\xi^1}\big|_M$ and $\widetilde{v} = \overline{v} \circ \phi_{X,\xi^1}\big|_M$ (see Section 3.2), then we know $\widetilde{u}, \widetilde{v} \in W^{1,p}_{\varphi}(M, N)$ and $\widetilde{u} \sim_{p,\mathrm{rel}.\partial M} u$, $\widetilde{v} \sim_{p,\mathrm{rel}.\partial M} v$. Indeed, the map $t \mapsto \overline{u} \circ \phi_{X,t\xi^1}\big|_M$ gives the needed path. Observe that $\widetilde{u} \circ h_{\xi'}, \widetilde{v} \circ h_{\xi'} \in \mathcal{W}^{1,p}(K, N)$ and

$$\widetilde{u} \circ h_{\xi'}|_{\left|K^{[p]-1} \cup K_0\right|} \sim_{\mathrm{rel.}|K_0|} \widetilde{v} \circ h_{\xi'}|_{\left|K^{[p]-1} \cup K_0\right|},$$

arguments similar to those in the proof of theorem 5.1 in [7] show $\widetilde{u} \sim_{p,\mathrm{rel}.\partial M} v$ (one needs to use Lemma 8 and Lemma 9 when necessary). This implies $u \sim_{p,\mathrm{rel}.\partial M} v$. □

Next we will identify all the path connected components of $W^{1,p}_{\varphi}(M, N)$ for $1 \le p < n$. First we claim there exists a smooth triangulation $h : K \to M$ and a subcomplex $K_0 \subset K$ such that $h(|K_0|) = \partial M$ and $\varphi \circ h|_{|K_0|} \in \mathcal{W}^{1,p}(K_0, N)$.

Indeed, take an arbitrary smooth triangulation $h : K \to M$ and a subcomplex $K_0 \subset K$ such that $h(|K_0|) = \partial M$. For any $\xi \in \{0\} \times \mathbb{R}^{l-1}$, $\Phi_\xi : M \to M$ is a diffeomorphism such that $\Phi_\xi(\partial M) = \partial M$ (see Section 3.2), hence $h_\xi = \Phi_\xi \circ h$ is still a smooth triangulation of M with $h_\xi(|K_0|) = \partial M$. On the other hand, it follows from (6), Lemma 5 and corollary 3.1 of [7] that for $\mathcal{H}^{l-1}$ a.e. $\xi \in B^l_{\varepsilon_0} \cap \left(\{0\} \times \mathbb{R}^{l-1}\right)$, $\varphi \circ h_\xi|_{|K_0|} \in \mathcal{W}^{1,p}(K_0, N)$. The claim follows.

Take a bi-Lipschitz rectilinear cell decomposition $h : K \to M$ and a subcomplex $K_0 \subset K$ such that $h(|K_0|) = \partial M$ and $\varphi \circ h|_{|K_0|} \in \mathcal{W}^{1,p}(K_0, N)$. Denote

$$\mathcal{X} = \mathcal{W}^{1,p}\left(K^{[p]} \cup K_0, N\right) \cap C_{\varphi\circ h|_{|K_0|}}\left(\left|K^{[p]} \cup K_0\right|, N\right).$$

We may define an equivalence relation on $\mathcal{X}$ by: for $f, g \in \mathcal{X}$, $f \sim_{\mathcal{X}} g$ if and only if there exists a $H \in C\left([0,1], \mathcal{W}^{1,p}\left(K^{[p]-1} \cup K_0, N\right) \cap C\left(\left|K^{[p]-1} \cup K_0\right|, N\right)\right)$ such that $H(0) = f|_{\left|K^{[p]-1}\cup K_0\right|}$, $H(1) = g|_{\left|K^{[p]-1}\cup K_0\right|}$ and $H(t)|_{|K_0|} = \varphi \circ h|_{|K_0|}$ for $0 \le t \le 1$.

We have a natural map

$$\Gamma_1 : \mathcal{X}/\sim_{\mathcal{X}} \longrightarrow \left[M^{[p]} \cup \partial M\Big|_{M^{[p]-1}\cup\partial M}, N; \varphi\right]_{\text{rel.}\partial M}$$

given by $\Gamma_1([f]_{\mathcal{X}}) = \left[f \circ h^{-1}|_{M^{[p]}\cup\partial M}\right]_{M^{[p]-1}\cup\partial M,\text{rel.}\partial M}$ for any $f \in \mathcal{X}$. We claim that Γ_1 is a bijection. Indeed for any $u \in C_\varphi\left(M^{[p]} \cup \partial M, N\right)$, it follows from Lemma 8 that we may find a $g \in C\left(\left|K^{[p]} \cup K_0\right|, N\right) \cap \mathcal{W}^{1,p}\left(K^{[p]} \cup K_0, N\right)$ such that $g|_{|K_0|} = \varphi \circ h|_{|K_0|}$ and $g \sim_{\text{rel.}|K_0|} u \circ h|_{\left|K^{[p]}\cup K_0\right|}$. Hence $\Gamma_1([g]_{\mathcal{X}}) = [u]_{M^{[p]-1}\cup\partial M,\text{rel.}\partial M}$. This shows Γ_1 is onto. On the other hand, if $f_1, f_2 \in \mathcal{X}$ such that $\Gamma_1([f_1]_{\mathcal{X}}) = \Gamma_1([f_2]_{\mathcal{X}})$, then $f_1|_{\left|K^{[p]-1}\cup K_0\right|} \sim_{\text{rel.}|K_0|} f_2|_{\left|K^{[p]-1}\cup K_0\right|}$. It follows from Lemma 9 that $f_1 \sim_{\mathcal{X}} f_2$. Hence $[f_1]_{\mathcal{X}} = [f_2]_{\mathcal{X}}$. This show Γ_1 is an injection.

Pick up an interior point $y_\Delta \in \text{int}\Delta$ for each $\Delta \in K$. For any $f \in \mathcal{X}$, we want to define inductively a $\overline{f} \in \mathcal{W}^{1,p}(K, N)$ such that $\overline{f}|_{\left|K^{[p]}\cup K_0\right|} = f$. First we let $\overline{f} = f$ on $\left|K^{[p]} \cup K_0\right|$. Next for any $\Delta \in K^{[p]+1} \backslash \left(K^{[p]} \cup K_0\right)$, $\overline{f}$ has been defined on $\partial\Delta$, then on Δ we let $\overline{f}$ be the homogeneous degree zero extension of $\overline{f}|_{\partial\Delta}$ with respect to y_Δ. Then we do similar things for $\Delta \in K^{[p]+2} \backslash \left(K^{[p]+1} \cup K_0\right)$. Keeping this procedure, we finish after finite many steps. Let $u = \overline{f} \circ h^{-1}$, then $u \in W^{1,p}_\varphi(M, N)$. It is clear that $u_{\#,p,|K_0|}(h) = [f]_{\left|K^{[p]-1}\cup K_0\right|,\text{rel.}|K_0|}$. We define a map

$$\Gamma_2 : \mathcal{X}/\sim_{\mathcal{X}} \longrightarrow W^{1,p}_\varphi(M, N) / \sim_{p,\text{rel.}\partial M}$$

by $\Gamma_2([f]_{\mathcal{X}}) = [u]_{p,\text{rel.}\partial M}$ for any $f \in \mathcal{X}$. It follows from Theorem 7 that Γ_2 is well defined. We claim Γ_2 is a bijection too. Indeed if $f_1, f_2 \in \mathcal{X}$ such that $\Gamma_2([f_1]_{\mathcal{X}}) = \Gamma_2([f_2]_{\mathcal{X}})$, then by Theorem 7 we know $[f_1]_{\left|K^{[p]-1}\cup K_0\right|,\text{rel.}|K_0|} = [f_2]_{\left|K^{[p]-1}\cup K_0\right|,\text{rel.}|K_0|}$. It follows from Lemma 9 that $f_1 \sim_{\mathcal{X}} f_2$. Hence $[f_1]_{\mathcal{X}} = [f_2]_{\mathcal{X}}$. This shows Γ_2 is an injection. On the other hand, if $v \in W^{1,p}_\varphi(M, N)$, then

$$v_{\#,p,|K_0|}(h) \in \left[\left|K^{[p]} \cup K_0\right|\Big|_{\left|K^{[p]-1}\cup K_0\right|}, N; \varphi \circ h|_{|K_0|}\right]_{\text{rel.}|K_0|}.$$

Take a $f \in \mathcal{W}^{1,p}\left(K^{[p]} \cup K_0, N\right) \cap C_{\varphi\circ h|_{|K_0|}}\left(\left|K^{[p]} \cup K_0\right|, N\right)$ such that

$$[f]_{\left|K^{[p]-1}\cup K_0\right|,\text{rel.}|K_0|} = v_{\#,p,|K_0|}(h),$$

the existence of such a f follows from Lemma 8. Let $u = \overline{f} \circ h^{-1}$, then $u \in W^{1,p}_\varphi(M, N)$ and $u_{\#,p,|K_0|}(h) = [f]_{\left|K^{[p]-1}\cup K_0\right|,\text{rel.}|K_0|} = v_{\#,p,|K_0|}(h)$. This implies $\Gamma_2([f]_{\mathcal{X}}) = [v]_{p,\text{rel.}\partial M}$. Hence Γ_2 is onto.

Sum up the discussions above, we get

Proposition 3. *Assume $1 \le p < n$, $h : K \to M$ is a bi-Lipschitz rectilinear cell decomposition and $K_0 \subset K$ is a subcomplex such that $h(|K_0|) = \partial M$ and*

$\varphi \circ h|_{|K_0|} \in \mathcal{W}^{1,p}(K_0, N)$. *Let* $\mathcal{X}$ *and* $\sim_{\mathcal{X}}$ *be defined as above, then we have two natural bijections*

$$\left[M^{[p]} \cup \partial M \Big|_{M^{[p]-1} \cup \partial M}, N; \varphi \right]_{rel.\partial M} \xleftarrow{\Gamma_1} \mathcal{X}/\sim_{\mathcal{X}} \xrightarrow{\Gamma_2} W^{1,p}_{\varphi}(M, N) / \sim_{p,rel.\partial M},$$

Γ_1 *and* Γ_2 *are described above.*

In particular, it follows from Proposition 3 that $W^{1,p}_{\varphi}(M, N) \neq \emptyset$ if and only if $\left[M^{[p]} \cup \partial M|_{M^{[p]-1} \cup \partial M}, N; \varphi \right]_{\mathrm{rel}.\partial M} \neq \emptyset$, and the latter is equivalent to the statement that φ has a continuous extension to $M^{[p]} \cup \partial M$. This fact was proved in [15] (see theorem 4.1 in [15]).

For the question on when a map in $W^{1,p}_{\varphi}(M, N)$ can be connected to a continuous map in $W^{1,p}_{\varphi}(M, N)$ by a continuous path in $W^{1,p}_{\varphi}(M, N)$, we have

Proposition 4. *Assume* $1 \leq p < n$, $u \in W^{1,p}_{\varphi}(M, N)$, $h : K \to M$ *is a bi-Lipschitz rectilinear cell decomposition and* $K_0 \subset K$ *is a subcomplex such that* $h(|K_0|) = \partial M$. *Then* u *may be connected to a map in* $C(M, N) \cap W^{1,p}_{\varphi}(M, N)$ *by a continuous path in* $W^{1,p}_{\varphi}(M, N)$ *if and only if* $u_{\#,p,|K_0|}(h)$ *is extendible to* M *with respect to* N.

Proof. Assume $u \sim_{p,\mathrm{rel}.\partial M} v$ for some $v \in C(M, N) \cap W^{1,p}_{\varphi}(M, N)$. It follows from Theorem 7 that $u_{\#,p,|K_0|}(h) = v_{\#,p,|K_0|}(h) = \left[v \circ h|_{|K^{[p]} \cup K_0|} \right]_{|K^{[p]-1} \cup K_0|, \mathrm{rel}.|K_0|}$. This clearly implies $u_{\#,p,|K_0|}(h)$ is extendible to M with respect to N. On the other hand, assume $u_{\#,p,|K_0|}(h)$ is extendible to M with respect to N, then we may find a $v \in C_{\varphi}(M, N)$ such that $\left[v \circ h|_{|K^{[p]-1} \cup K_0|} \right]_{\mathrm{rel}.|K_0|} = u_{\#,p,|K_0|}(h)$. It follows from Lemma 8 that we may find a $f \in C(|K|, N) \cap \mathcal{W}^{1,p}(K, N)$ such that $f|_{|K_0|} = \varphi \circ h|_{|K_0|}$ and $f \sim_{\mathrm{rel}.|K_0|} v \circ h$. Let $w = f \circ h^{-1}$, then $w \in C(M, N) \cap W^{1,p}_{\varphi}(M, N)$ and $w_{\#,p,|K_0|}(h) = \left[f|_{|K^{[p]} \cup K_0|} \right]_{|K^{[p]-1} \cup K_0|, \mathrm{rel}.|K_0|} = \left[v \circ h|_{|K^{[p]-1} \cup K_0|} \right]_{\mathrm{rel}.|K_0|} = u_{\#,p,|K_0|}(h)$. This implies $u \sim_{p,\mathrm{rel}.\partial M} w$. $\square$

Corollary 1. *Assume* $1 \leq p < n$. *Then every map in* $W^{1,p}_{\varphi}(M, N)$ *can be connected by a continuous path in* $W^{1,p}_{\varphi}(M, N)$ *to a map in* $C(M, N) \cap W^{1,p}_{\varphi}(M, N)$ *if and only if* $(M, \partial M; \varphi)$ *satisfies the* $([p] - 1)$*-extension property with respect to* N.

Proof. Fix a bi-Lipschitz triangulation $h : K \to M$ and a subcomplex $K_0 \subset K$ such that $h(|K_0|) = \partial M$ and $\varphi \circ h|_{|K_0|} \in \mathcal{W}^{1,p}(K_0, N)$. Let $M^i = h\left(\left|K^i\right|\right)$. As a relative CW complex, we have $(M, \partial M)^i = M^i \cup \partial M$.

If $(M, \partial M; \varphi)$ satisfies the $([p] - 1)$-extension property with respect to N, then for any $u \in W^{1,p}_{\varphi}(M, N)$, $u_{\#,p,|K_0|}(h) \in \left[\left|K^{[p]} \cup K_0\right| |_{|K^{[p]-1} \cup K_0|}, N; \varphi \circ h|_{|K_0|} \right]_{\mathrm{rel}.|K_0|}$ is extendible to M with respect to N, hence u may be connected to a map in $C(M, N) \cap W^{1,p}_{\varphi}(M, N)$.

On the other hand, assume every map in $W^{1,p}_{\varphi}(M, N)$ can be connected to a map in $C(M, N) \cap W^{1,p}_{\varphi}(M, N)$, then for any $v \in C_{\varphi}\left(M^{[p]} \cup \partial M, N\right)$, let $f = v \circ h|_{|K^{[p]} \cup K_0|}$, we have $f \in C\left(\left|K^{[p]} \cup K_0\right|, N\right)$ and $f|_{|K_0|} = \varphi \circ h|_{|K_0|}$. By Lemma 8 we may find a $g \in C\left(\left|K^{[p]} \cup K_0\right|, N\right) \cap \mathcal{W}^{1,p}\left(\left|K^{[p]} \cup K_0\right|, N\right)$ such that $g|_{|K_0|} = \varphi \circ h|_{|K_0|}$ and $g \sim_{\mathrm{rel}.|K_0|} f$. Let $\overline{g} \in \mathcal{W}^{1,p}(K, N)$ be the map we get by doing homogeneous degree zero extension on higher dimensional cells as described before Proposition 3 and $u = \overline{g} \circ h^{-1}$, then $u \in W^{1,p}_{\varphi}(M, N)$ and $u_{\#,p,|K_0|}(h) =$

$[g]|_{\left|K^{[p]-1}\cup K_0\right|,\text{rel.}|K_0|} = [f]|_{\left|K^{[p]-1}\cup K_0\right|,\text{rel.}|K_0|}$. Since u may be connected to a map in $C(M,N)\cap W^{1,p}_{\varphi}(M,N)$, it follows from Proposition 4 that $[f]|_{\left|K^{[p]-1}\cup K_0\right|,\text{rel.}|K_0|}$ is extendible to M with respect to N. In particular, $v|_{M^{[p]-1}\cup\partial M}$ has a continuous extension to M. This shows $(M,\partial M;\varphi)$ satisfies the $([p]-1)$-extension property with respect to N. □

5.3. **Density of continuous $W^{1,p}$ maps in $W^{1,p}_{\varphi}(M,N)$.** The following spaces are of interest in the calculus of variations :

$$\begin{aligned} & H^{1,p}_{\varphi,\mathrm{S}}(M,N) \\ = \ & \left\{u\in W^{1,p}_{\varphi}(M,N) : \text{there exists a sequence } u_i\in W^{1,p}_{\varphi}(M,N)\cap C(M,N) \right. \\ & \left. \text{such that } u_i\to u \text{ in } W^{1,p}\left(M,\mathbb{R}^l\right)\right\}, \end{aligned}$$

and

$$\begin{aligned} & H^{1,p}_{\varphi,\mathrm{W}}(M,N) \\ = \ & \left\{u\in W^{1,p}(M,N) : \text{there exists a sequence } u_i\in W^{1,p}_{\varphi}(M,N)\cap C(M,N) \right. \\ & \left. \text{such that } u_i\rightharpoonup u \text{ in } W^{1,p}\left(M,\mathbb{R}^l\right)\right\}. \end{aligned}$$

It is known ([13]) that when $p\geq n$, we have $H^{1,p}_{\varphi,\mathrm{S}}(M,N) = H^{1,p}_{\varphi,\mathrm{W}}(M,N) = W^{1,p}_{\varphi}(M,N)$. To deal with the case $1\leq p<n$, we need to discuss the deformation with respect to the dual skeletons for a relative rectilinear cell complex (K,K_0). We follow closely section 6 of [7]. Let K be a finite rectilinear cell complex with $\dim K=m$ and $K_0\subset K$ be a subcomplex, then (K,K_0) is a relative complex with $(K,K_0)^i = K^i\cup K_0$. For each $\Delta\in K$, we fix an interior point $y_\Delta\in\operatorname{int}\Delta$. Denote $\mathcal{Y}=(y_\Delta)_{\Delta\in K}$. Given an integer $0\leq k\leq m-1$.

For $x\in\left|(K,K_0)^k\right|$, we let $|x|_{K_0,k}=1$. Then for each $\Delta\in (K,K_0)^{k+1}\setminus(K,K_0)^k = K^{k+1}\setminus(K,K_0)^k$ and $x\in\Delta$, we set

$$|x|_{K_0,k} = |x|_\Delta\left|y_\Delta+\frac{x-y_\Delta}{|x|_\Delta}\right|_{K_0,k},$$

here $|x|_\Delta$ is the Minkowski norm of x with respect to y_Δ. Next we define $|x|_{K_0,k}$ for $x\in\Delta$ with $\Delta\in (K,K_0)^{k+2}\setminus(K,K_0)^{k+1}$ similarly. Keeping this procedure going, after finite many steps, we get a function $|\cdot|_{K_0,k}$ on $|K|$.

For $0\leq\varepsilon\leq 1$, we let $\Gamma^k_{K_0,\varepsilon}=\left\{x\in|K| : |x|_{K_0,k}=\varepsilon\right\}$, then

$$|K| = \bigcup_{0\leq\varepsilon\leq 1}\Gamma^k_{K_0,\varepsilon},\quad \Gamma^k_{K_0,1}=\left|(K,K_0)^k\right|.$$

Denote $L^{m-k-1}=\Gamma^k_{K_0,0}$. We call L^{m-k-1} as the dual $m-k-1$ skeleton of (K,K_0).

Next, we define inductively a map

$$\phi^k_{K_0,1} : \left\{x\in|K| : 0<|x|_{K_0,k}\leq 1\right\}\longrightarrow\Gamma^k_{K_0,1}=\left|(K,K_0)^k\right|.$$

For $x\in\left|(K,K_0)^k\right|$, let $\phi^k_{K_0,1}(x)=x$. For $x\in\left|(K,K_0)^{k+1}\right|\setminus\left|(K,K_0)^k\right|$ with $0<|x|_{K_0,k}\leq 1$, there exists a unique $\Delta\in(K,K_0)^{k+1}\setminus(K,K_0)^k$ such that $x\in\operatorname{int}\Delta$. We let

$$\phi^k_{K_0,1}(x)=\phi^k_{K_0,1}\left(y_\Delta+\frac{x-y_\Delta}{|x|_\Delta}\right).$$

In the next step we deal with points in $\left|(K,K_0)^{k+2}\right|$ similarly. After finitely many steps we get the needed $\phi^k_{K_0,1}$.

Now we will define a map

$$\phi^k_{K_0}:\left\{x\in|K|:0<|x|_{K_0,k}<1\right\}\times(0,1)\to|K|.$$

We write $\phi^k_{K_0}(x,\varepsilon)=\phi^k_{K_0,\varepsilon}(x)$, then $\phi^k_{K_0}(x,1)$ is defined for $0<|x|_{K_0,k}\le 1$. For $x\in\left|(K,K_0)^{k+1}\right|$ with $0<|x|_{K_0,k}<1$, there exists a unique $\Delta\in(K,K_0)^{k+1}$ such that $x\in\operatorname{int}\Delta$, then we set

$$\phi^k_{K_0}(x,\varepsilon)=y_\Delta+\frac{\varepsilon}{|x|_\Delta}(x-y_\Delta)\quad\text{for } 0<\varepsilon<1.$$

If for some $k+2\le i\le m$, $\phi^k_{K_0}(x,\varepsilon)$ has been defined for $x\in\left|(K,K_0)^{i-1}\right|$ with $0<|x|_{K_0,k}<1$ and $0<\varepsilon<1$, then for any $x\in\left|(K,K_0)^{i}\right|\setminus\left|(K,K_0)^{i-1}\right|$, there exists a unique $\Delta\in(K,K_0)^i$ such that $x\in\operatorname{int}\Delta$. We let

$$\begin{aligned}\theta&=1-(1-\varepsilon)\frac{1-|x|_\Delta}{1-|x|_{K_0,k}},\\ \phi^k_{K_0}(x,\varepsilon)&=y_\Delta+\theta\cdot\left(\phi^k_{K_0}\left(y_\Delta+\frac{x-y_\Delta}{|x|_\Delta},\frac{\varepsilon}{\theta}\right)-y_\Delta\right).\end{aligned}$$

After finitely many steps, we get the needed $\phi^k_{K_0}$. As in appendix B of [7], for $\delta,\varepsilon\in(0,1)$, $\phi^k_{K_0,\varepsilon}\big|_{\Gamma^k_{K_0,\delta}}$ is a bijection from $\Gamma^k_{K_0,\delta}$ to $\Gamma^k_{K_0,\varepsilon}$ and its inverse is $\phi^k_{K_0,\delta}\Big|_{\Gamma^k_{K_0,\varepsilon}}$. Similar to (6.9) in section 6 of [7], for $0<\delta\le\varepsilon\le 1$, we may define a map $F^k_{K_0,\delta,\varepsilon}:|K|\to|K|$ by

$$F^k_{K_0,\delta,\varepsilon}(x)=\begin{cases}x, & \text{when } \varepsilon\le|x|_{K_0,k}\le 1;\\ \phi^k_{K_0}(x,\varepsilon), & \text{when } \delta\le|x|_{K_0,k}\le\varepsilon;\\ \phi^k_{K_0}\left(x,\delta^{-1}\varepsilon|x|_{K_0,k}\right), & \text{when } 0<|x|_{K_0,k}\le\delta;\\ x, & \text{when } |x|_{K_0,k}=0.\end{cases}$$

Assume K is a finite rectilinear cell complex with $\dim K=n$ and for any $x\in K$, there exists a $\Delta\in K$ with $\dim\Delta=n$ and $x\in\Delta$. $K_0\subset K$ is a subcomplex with $\dim K_0\le n-1$. For each $x\in K$, we take a point $y_\Delta\in\operatorname{int}\Delta$. Let $\mathcal{Y}=(y_\Delta)_{\Delta\in K}$. Fix an integer $0\le k\le n-1$. Then similar arguments as those in appendix B of [7] show that

(P_1) $\mathcal{H}^n\left(\left\{x\in|K|:|x|_{K_0,k}\le\varepsilon\right\}\right)\le c(K,K_0,\mathcal{Y})\,\varepsilon^{k+1}\quad$ for $0<\varepsilon\le\frac12$;

(P_2) $0<c(K,K_0,\mathcal{Y})^{-1}\le\left|d\,|\cdot|_{K_0,k}\right|\le c(K,K_0,\mathcal{Y})\quad\mathcal{H}^n$ a.e. on $|K|$;

(P_3) $\left|dF^k_{K_0,\delta,\varepsilon}(x)\right|\le c(K,K_0,\mathcal{Y})\,\frac{\varepsilon}{|x|_{K_0,k}}\quad$ for $\delta\le|x|_{K_0,k}\le\varepsilon\le\frac12$;

(P_4) $\left|dF^k_{K_0,\delta,\varepsilon}(x)\right|\le c(K,K_0,\mathcal{Y})\,\frac{\varepsilon}{\delta}\quad$ for $|x|_{K_0,k}\le\delta\le\varepsilon\le\frac12$;

(P_5) For $0<\delta\le\varepsilon\le\frac12$, $J_{\left(\phi^k_{K_0,\delta}\big|_{\Gamma^k_{K_0,\varepsilon}}\right)}\le c(K,K_0,\mathcal{Y})\left(\frac{\delta}{\varepsilon}\right)^k\quad\mathcal{H}^{n-1}$ a.e. on $\Gamma^k_{K_0,\varepsilon}$.

For $1 \le p < n$, we denote

$$R_{\varphi}^{p}(M,N) = \{u : u \in W_{\varphi}^{1,p}(M,N), \text{ there exists a smooth rectilinear cell decomposition } h : K \to M, \text{ a subcomplex } K_0 \subset K \text{ such that } h(|K_0|) = \partial M \text{ and a dual } (n-[p]-1) \text{ skeleton of } (K,K_0), \text{ namely } L^{n-[p]-1} \text{ such that } u|_{M \backslash h(L^{n-[p]-1})} \text{ is continuous}\}.$$

Theorem 8. *Assume $1 \le p < n$, then $R_{\varphi}^{p}(M,N)$ is strongly dense in $W_{\varphi}^{1,p}(M,N)$.*

Proof. Let $\mathcal{K} = \{\text{all faces of } [0,1]^m\}$, where m is a large natural number. If $h : K \to M$ is a smooth cubeulation, $K \subset \mathcal{K}$, $K_0 \subset K$ is a subcomplex with $h(|K_0|) = \partial M$, $u \in W_{\varphi}^{1,p}(M,N)$ and $u \circ h \in \mathcal{W}^{1,p}(K,N)$, then the proof of theorem 6.1 in [7] shows $u \in \overline{R_{\varphi}^{p}(M,N)}$.

Given any $u \in W_{\varphi}^{1,p}(M,N)$. Fix a smooth cubeulation $h : K \to M$, $K \subset \mathcal{K}$, $K_0 \subset K$ is a subcomplex with $h(|K_0|) = \partial M$. We will use notations from Section 3.2. Let $\overline{u} : M_{\varepsilon_0} \to N$ be the map given by $\overline{u}|_M = u$ and $\overline{u}(\psi^{-1}(q,t)) = \varphi(q)$ for $q \in \partial M$ and $-\varepsilon_0 \le t \le 0$. Then $\overline{u} \circ \Phi_{\xi}|_M \to u$ in $W^{1,p}(M,N)$ as $\mathbb{R}^l \ni \xi \to 0$. For $\mathcal{H}^l$ a.e. $\xi \in B_{\varepsilon_0}^l \cap \mathbb{R}_-^l$ we have $\overline{u} \circ h_{\xi} \in \mathcal{W}^{1,p}(K,N)$. Let $\xi' = (0,\xi^2,\cdots,\xi^l)$, $v = \overline{u} \circ \phi_{X,\xi^1}|_M$, then we know $v \in W_{\varphi}^{1,p}(M,N)$ and $v \circ h_{\xi'} \in \mathcal{W}^{1,p}(K,N)$. Note that $h_{\xi'} : K \to M$ is a smooth cubeulation and $h_{\xi'}(|K_0|) = \partial M$, we see $v \in \overline{R_{\varphi}^{p}(M,N)}$. Hence $u \in \overline{R_{\varphi}^{p}(M,N)}$. □

There is another way to prove Theorem 8. Since this method will be used again later in determining $H_{\varphi,\mathrm{W}}^{1,p}(M,N)$, we present it here. We claim that for any $u \in W_{\varphi}^{1,p}(M,N)$, there exists a smooth cubeulation $h : K \to M$ and a subcomplex $K_0 \subset M$ such that $K \subset \mathcal{K} = \{\text{all faces of } [0,1]^m\}$, $h(|K_0|) = \partial M$ and $u \circ h \in \mathcal{W}^{1,p}(K,N)$.

Indeed, fix a smooth cubeulation $h : K \to M$ and a subcomplex $K_0 \subset K$ such that $K \subset \mathcal{K}$ and $h(|K_0|) = \partial M$. Consider the map $\Psi : M \times \mathbb{R}^l \to M$ defined in (9). Denote $H(x,\xi) = \Psi(h(x),\xi)$ for $x \in |K|$, $\xi \in \mathbb{R}^l$. Assume $\Delta \in K_0$, then it is clear that for any $x \in \Delta$, $dH_{(x,0)}(\Delta_x \times \mathbb{R}_0^l) = (\partial M)_{h(x)}$, here Δ_x means the tangent space at x of the smallest affine space containing Δ. Hence for some $\varepsilon_1 > 0$, we know $dH_{(x,\xi)}\left(\Delta_x \times \mathbb{R}_{\xi}^l\right) = (\partial M)_{H(x,\xi)}$ for $x \in \Delta$, $\xi \in B_{2\varepsilon_1}^l$. Assume $\Delta \in K \backslash K_0$ and $x \in \Delta$. If $h(x) \notin \partial M$, then it is clear that $dH_{(x,0)}(\Delta_x \times \mathbb{R}_0^l) = M_{h(x)}$. If $h(x) \in \partial M$, then since $dh_x(\Delta_x)$ is not contained in $(\partial M)_{h(x)}$, we know $dH_{(x,0)}(\Delta_x \times \mathbb{R}_0^l) = M_{h(x)}$ too. Hence when ε_1 is small enough, we know $dH_{(x,\xi)}\left(\Delta_x \times \mathbb{R}_{\xi}^l\right) = M_{H(x,\xi)}$ for $x \in \Delta$, $\xi \in B_{2\varepsilon_1}^l$. It follows from implicit function theorem arguments as those in Section 3 that when ε_1 is small enough, for $\Delta \in K_0$, $H|_{\Delta \times B_{\varepsilon_1}^l} : \Delta \times B_{\varepsilon_1}^l \to \partial M$ satisfies (H_1), (H_2) and (H_3) in section 3 of [7]; for $\Delta \in K \backslash K_0$, $H|_{\Delta \times B_{\varepsilon_1}^l} : \Delta \times B_{\varepsilon_1}^l \to M$ satisfies (H_1), (H_2) and (H_3) in section 3 of [7]. Next we want to show this implies for $\mathcal{H}^l$ a.e. $\xi \in B_{\varepsilon_1}^l$, $u \circ H_{\xi} \in \mathcal{W}^{1,p}(K,N)$. Here $H_{\xi}(x) = H(x,\xi) = \Psi_{\xi}(h(x))$. It is clear that $H_{\xi} : K \to M$ is a smooth cubeulation with $H_{\xi}(|K_0|) = \partial M$. The claim in the previous paragraph follows. To prove the required genericity statement, we follow the arguments in the proof of corollary 3.1 in [7]. Choose a sequence $u_i \in C^{\infty}\left(M,\mathbb{R}^{\bar{l}}\right)$ such that $u_i \to u$ in $W^{1,p}(M)$ and $u_i|_{\partial M} \to \varphi$ in $W^{1,p}(\partial M)$. It follows from lemma 3.4 in [7] that for $\mathcal{H}^l$ a.e. $\xi \in B_{\varepsilon_1}^l$, $u \circ H_{\xi}|_{\Delta} \in W^{1,p}(\Delta)$ and

$u_i \circ H_\xi|_\Delta \to u \circ H_\xi|_\Delta$ in $W^{1,p}(\Delta)$ for any $\Delta \in K$. This implies for $\mathcal{H}^l$ a.e. $\xi \in B^l_{\varepsilon_1}$ and $\Delta \in K$, $\operatorname{tr}\left(u \circ H_\xi|_\Delta\right) = u \circ H_\xi|_{\partial\Delta}$, hence $u \circ H_\xi \in \mathcal{W}^{1,p}(K,N)$.

Next we study when a map in $R^p_\varphi(M,N)$ lies in $H^{1,p}_{\varphi,\mathrm{S}}(M,N)$. We have the following

Theorem 9. *Assume* $1 \le p < n$, $h : K \to M$ *is a bi-Lipschitz rectilinear cell decomposition,* $K_0 \subset K$ *is a subcomplex such that* $h(|K_0|) = \partial M$. *Denote* $M^i = h\left(\left|K^i\right|\right)$. $(M,\partial M)$ *is a relative CW complex with i-skeleton* $(M,\partial M)^i = M^i \cup \partial M$. *Assume* $L^{n-[p]-1}$ *is a dual* $n-[p]-1$ *skeleton of* (K,K_0), $u \in W^{1,p}_\varphi(M,N)$ *such that* u *is continuous on* $M \backslash h\left(L^{n-[p]-1}\right)$. *Then* $u \in H^{1,p}_{\varphi,S}(M,N)$ *if and only if* $u|_{(M,\partial M)^{[p]}}$ *has a continuous extension to* M. *In addition, if for some* $\alpha \in [M,N;\varphi]_{rel.\partial M}$, *we have* $\left[u|_{(M,\partial M)^{[p]}}\right]_{rel.\partial M} = \alpha|_{(M,\partial M)^{[p]}}$, *then we may find a sequence* $u_i \in C(M,N) \cap W^{1,p}_\varphi(M,N)$ *such that* $u_i \to u$ *in* $W^{1,p}(M,N)$ *and* $[u_i]_{rel.\partial M} = \alpha$.

Proof. We follow the arguments in the proof of theorem 5.4 and theorem 5.5 in [8].

We use notations in Section 3.2. Let $\overline{u} : M_{\varepsilon_0} \to N$ be the map given by $\overline{u}|_M = u$ and $\overline{u}\left(\psi^{-1}(q,t)\right) = \varphi(q)$ for $q \in \partial M$ and $-\varepsilon_0 \le t \le 0$. For $\varepsilon_1 > 0$ small enough, define $H(x,\xi) = \Phi(h(x),\xi)$ for $x \in |K|$ and $\xi \in B^l_{\varepsilon_1} \cap \mathbb{R}^l_-$. Take $\xi_0 = 0$. It follows that $\chi_{[p],H,|K_0|,\overline{u}} = \left[u \circ h|_{|K^p \cup K_0|}\right]_{\mathrm{rel.}|K_0|}$. If $u \in H^{1,p}_{\varphi,\mathrm{S}}(M,N)$, then we may find a sequence $u_i \in C(M,N) \cap W^{1,p}_\varphi(M,N)$ such that $u_i \to u$ in $W^{1,p}(M,N)$. Let $\overline{u}_i$ be defined similarly, then $\overline{u}_i \to \overline{u}$ in $W^{1,p}(M_{\varepsilon_0},N)$. Since $\chi_{[p],H,|K_0|,\overline{u}_i} = \left[u_i \circ h|_{|K^p \cup K_0|}\right]_{\mathrm{rel.}|K_0|}$, it follows from Proposition 2 that after passing to a subsequence, for i large enough, we have $\left[u_i \circ h|_{|K^p \cup K_0|}\right]_{\mathrm{rel.}|K_0|} = \left[u \circ h|_{|K^p \cup K_0|}\right]_{\mathrm{rel.}|K_0|}$. This implies $u|_{(M,\partial M)^{[p]}}$ has a continuous extension to M by the homotopy extension property.

To prove the inverse, we use the idea in section 4 of [14], but with the deformations with respect to the dual skeleton of (K,K_0) constructed at the beginning of this subsection. We point out two modifications of arguments in the proof of theorem 5.4 of [8]. For convenience, denote $k = [p]$. Choose a $v \in C_\varphi(M,N)$ such that $[v]_{\mathrm{rel.}\partial M} = \alpha$. Let $g_0 = v \circ h$ and $f = u \circ h$. Then we may find a $g \in C(|K|,N) \cap \mathcal{W}^{1,p}(K,N)$ such that $g = f$ on $\left\{x \in |K| : |x|_{K_0,k} \ge \frac{1}{2}\right\}$ and $g \sim_{\mathrm{rel.}|K_0|} g_0$. Indeed, since $g_0|_{\left\{|x|_{K_0,k} \ge \frac{1}{16}\right\}} \sim_{\mathrm{rel.}|K_0|} f|_{\left\{|x|_{K_0,k} \ge \frac{1}{16}\right\}}$, it follows from homotopy extension property that we may find a $g_1 \in C(|K|,N)$ such that $g_1|_{\left\{|x|_{K_0,k} \ge \frac{1}{16}\right\}} = f|_{\left\{|x|_{K_0,k} \ge \frac{1}{16}\right\}}$ and $g_1 \sim_{\mathrm{rel.}|K_0|} g_0$. By proposition 2.3 of [7] we may find a $g_2 \in \operatorname{Lip}(|K|,N)$ such that $|g_2(x) - g_1(x)| \le \overline{\varepsilon}_0$. Choose a function $\eta \in C^\infty([0,1],\mathbb{R})$ such that $0 \le \eta \le 1$, $\eta|_{\left[\frac{1}{4},1\right]} = 1$ and $\eta|_{\left[0,\frac{1}{8}\right]} = 0$. Let $g_3(x) = \eta\left(|x|_{K_0,k}\right) f(x) + \left(1 - \eta\left(|x|_{K_0,k}\right)\right) g_2(x)$ for $x \in |K|$, then $g_3 \in C\left(|K|,\mathbb{R}^{\overline{l}}\right) \cap \widetilde{W}^{1,p}\left(K,\mathbb{R}^{\overline{l}}\right)$ and $|g_3(x) - g_2(x)| \le \overline{\varepsilon}_0$. Let $g(x) = \pi_N(g_3(x))$, then we get the needed map.

For $0 < \varepsilon \le \frac{1}{2}$, let

$$f_\varepsilon(x) = \begin{cases} f(x), & \varepsilon \le |x|_{K_0,k}; \\ f\left(\phi^k_{K_0,\frac{\varepsilon^2}{|x|_{K_0,k}}}(x)\right), & 2\varepsilon^2 \le |x|_{K_0,k} \le \varepsilon^2; \\ g\left(\phi^k_{K_0,\frac{|x|_{K_0,k}}{4\varepsilon^2}}(x)\right), & 0 < |x|_{K_0,k} \le 2\varepsilon^2; \\ g(x), & |x|_{K_0,k} = 0. \end{cases}$$

Let $u_\varepsilon = f_\varepsilon \circ h^{-1}$, then the same proof as the one for theorem 5.4 in [8], using the deformation with respect to (K, K_0), shows $u_\varepsilon \to u$ in $W^{1,p}(M,N)$. It is clear that $u_\varepsilon \in C(M,N) \cap W^{1,p}_\varphi(M,N)$ and $[u_\varepsilon]_{\text{rel}.\partial M} = \alpha$. □

It follows from Theorem 8 and Theorem 9 that

Theorem 10. *Assume $1 \le p < n$ and $W^{1,p}_\varphi(M,N) \ne \emptyset$, then $H^{1,p}_{\varphi,S}(M,N) = W^{1,p}_\varphi(M,N)$ if and only if $\pi_{[p]}(N) = 0$ and $(M,N;\varphi)$ satisfies the $([p]-1)$-extension property with respect to N.*

Since the proof is almost identical to the one for theorem 6.3 in [7], we omit it here. We remark that theorem 1bis in [1] should be replaced by the above statement, similar to that theorem 1 in [1] should be replaced by theorem 6.3 in [7].

If $1 \le p < n$ and p is not an integer, or $p = 1$, then it follows from similar arguments as in [1, 7, 4] that $H^{1,p}_{\varphi,\text{W}}(M,N) = H^{1,p}_{\varphi,\text{S}}(M,N)$. On the other hand, it follows easily from Theorem 4.1 and Theorem 4.2 that if $1 \le p < n$, $u \in H^{1,p}_{\varphi,\text{W}}(M,N)$, $h : K \to M$ is a bi-Lipschitz rectilinear cell decomposition and $K_0 \subset K$ is a subcomplex with $h(|K_0|) = \partial M$, then $u_{\#,p,|K_0|}(h)$ is extendible to M with respect to N. In particular, u may be connected to some maps in $C(M,N) \cap W^{1,p}_\varphi(M,N)$.

If φ has a continuous extension to M, then it was proved in [11] that for any $u \in W^{1,1}_\varphi(M,N)$, we may find a sequence $u_i \in W^{1,1}_\varphi(M,N) \cap C(M,N)$ such that $du_i \to du$ a.e. and $|u_i - u|_{L^1(M)} \to 0$.

If $n \ge 3$, $h : K \to M$ is a bi-Lipschitz rectilinear cell decomposition and $K_0 \subset K$ is a subcomplex such that $h(|K_0|) = \partial M$, then we have

$$\begin{aligned} & H^{1,2}_{\varphi,\text{W}}(M,N) \\ = & \left\{u \in W^{1,2}_\varphi(M,N) : u_{\#,2,|K_0|}(h) \text{ is extendible to } M \text{ with respect to } N\right\} \\ = & \left\{u \in W^{1,2}_\varphi(M,N) : u \text{ may be connected to a map in } C(M,N) \cap W^{1,2}_\varphi(M,N)\right\}. \end{aligned}$$

Moreover, if $u \in W^{1,2}_\varphi(M,N)$ and $\alpha \in [M,N]_{\text{rel}.\partial M}$ satisfies $\alpha \circ h|_{|K^1 \cup K_0|} = u_{\#,2,|K_0|}(h)$, then we may find a sequence $u_i \in C(M,N) \cap W^{1,2}_\varphi(M,N)$ such that $u_i \rightharpoonup u$ in $W^{1,2}(M,N)$, $du_i \to du$ a.e. and $[u_i]_{\text{rel}.\partial M} = \alpha$. Indeed, by the discussion after the proof of Theorem 8, we may find a smooth cubeulation $\widetilde{h} : \widetilde{K} \to M$ and a subcomplex $\widetilde{K}_0 \subset \widetilde{K}$ with $\widetilde{h}\left(\left|\widetilde{K}_0\right|\right) = \partial M$ such that $u \circ \widetilde{h} \in \mathcal{W}^{1,2}(K,N)$. It follows from the arguments in [5] (based on a local result in [12]) that we may find a sequence $u_i \in C(M,N) \cap W^{1,2}_\varphi(M,N)$ such that $u_i \rightharpoonup u$ in $W^{1,2}(M,N)$, $du_i \to du$ a.e. and $[u_i]_{\text{rel}.\partial M} = \alpha$.

For $3 \le p < n$, $p \in \mathbb{N}$, it is still a hard problem to characterize $H^{1,p}_{\text{W},\varphi}(M,N)$. Nevertheless, we remark that the arguments in [5] reduce the problem to a local one. There is also a very interesting work for $W^{1,3}(B^4, S^2)$ in [9].

REFERENCES

[1] F. Bethuel, *The approximation problem for Sobolev maps between manifolds*, Acta Math., **167** (1991), 153–206.

[2] H. Brezis and Y. Y. Li, *Topology and Sobolev spaces*, J. Func. Anal., **183** (2001), no. 2, 321–369.

[3] H. Brezis and L. Nirenberg, *Degree theory and BMO, part II: compact manifolds with boundaries*, Selecta Math., **2** (1996), 1–60.

[4] F. B. Hang, *Density problems for* $W^{1,1}(M, N)$, Comm. Pure Appl. Math., **55** (2002), no. 7, 937–947.

[5] F. B. Hang, *On the weak limits of smooth maps for the Dirichlet energy between manifolds*, Comm. Anal. Geom., to appear.

[6] F. B. Hang and F. H. Lin, *Topology of Sobolev mappings*, Math. Res. Lett., **8** (2001), no. 3, 321–330.

[7] F. B. Hang and F. H. Lin, *Topology of Sobolev mappings II*, Acta Math., **191** (2003), no. 1, 55–107.

[8] F. B. Hang and F. H. Lin, *Topology of Sobolev mappings III*, Comm. Pure Appl. Math., **56** (2003), no. 10, 1383–1415.

[9] R. Hardt and T. Riviere, *Connecting topological Hopf singularities*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5), **2** (2003), no. 2, 287–344.

[10] J. R. Munkres, Elementary differential topology, Princeton University Press, Princeton, New Jersey, 1963.

[11] M. R. Pakzad, *Weak density of smooth maps in* $W^{1,1}(M, N)$ *for non-abelian* $\pi_1(N)$, Ann. Global Anal. Geom., **23** (2003), no. 1, 1–12.

[12] M. R. Pakzad and T. Riviere, *Weak density of smooth maps for the Dirichlet energy between manifolds*, Geom. Func. Anal., **13** (2003), no. 1, 223–257.

[13] R. Schoen and K. Uhlenbeck, *Approximation theorems for Sobolev mappings*, preprint, (1984).

[14] B. White, *Infima of energy functions in homotopy classes of mappings*, J. Diff. Geom., **23** (1986), 127–142.

[15] B. White, *Homotopy classes in Sobolev spaces and the existence of energy minimizing maps*, Acta Math., **160** (1988), no. 1–2, 1–17.

[16] G. W. Whitehead, Elements of homotopy theory, Springer-Verlag, New York, 1978.

[17] H. Whitney, Geometric integration theory, Princeton University Press, Princeton, NJ, 1957.

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