Generalized Poincaré-Bertrand formula on a hypersurface

Fengbo Hang

Courant Institute of Mathematical Sciences, New York University, New York, NY 10012

Shidong Jiang

Department of Mathematical Sciences, New Jersey Institute of Technology, Newark, NJ 07102

Abstract

The Poincaré-Betrand formula concerning two repeated Cauchy's principal integrals on a smooth curve in the plane is generalized to identities of singular integrals on smooth hypersurfaces in higher dimensions.

Key words: Singular Integrals, Riesz Transforms, Poincaré-Bertrand Formula
1991 MSC: 65R10, 77C05

1 Introduction

The classical Poincaré-Betrand formula states that (see [6])

$$\mathbf{p.v.}\int_{\gamma}\frac{1}{\xi-z}\left(\mathbf{p.v.}\int_{\gamma}\frac{\phi(\xi,w)}{w-\xi}dw\right)d\xi \tag{1.1}$$
$$=-\pi^{2}\phi(z,z)+\int_{\gamma}\left(\mathbf{p.v.}\int_{\gamma}\frac{\phi(\xi,w)}{(\xi-z)(w-\xi)}d\xi\right)dw,$$

Email addresses: `fengbo@cims.nyu.edu` (Fengbo Hang),
`shidong.jiang@njit.edu` (Shidong Jiang).
[1] Supported in part by NSF under Grant DMS-0647010 and a Sloan Research Fellowship.
[2] Supported in part by NSF under Grant DMS-0715121.

where γ is a smooth arc or closed curve in the plane, z, ξ, w are all viewed as complex numbers and ϕ is a Hölder continuous function. The formula has been generalized to accommodate fairly general conditions on γ and ϕ by various researchers (see [4,6] and the references therein).

In this paper we extend the formula to higher dimensions. We first note that if γ is a line segment, Cauchy's principal integral is just the finite Hilbert transform ([6]) of the given function, and the Poincaré-Betrand formula simply computes the iterated finite Hilbert transform of the function ϕ. Since the Riesz transforms ([5]) are natural analogy of the Hilbert transform in higher dimensions. For a bounded smooth domain $\Omega \subset \mathbb{R}^n$, we may consider the iterated principal integral

$$\sum_{j=1}^{n} \mathbf{p.v.} \int_{\Omega} \frac{z_j - x_j}{|z-x|^{n+1}} \left[\mathbf{p.v.} \int_{\Omega} \frac{z_j - y_j}{|z-y|^{n+1}} f(y)\, dy \right] dz.$$

A special case of our main result, Theorem 1.1 below, says for a sufficiently smooth function f,

$$\begin{aligned} &\sum_{j=1}^{n} \mathbf{p.v.} \int_{\Omega} \frac{z_j - x_j}{|z-x|^{n+1}} \left[\mathbf{p.v.} \int_{\Omega} \frac{z_j - y_j}{|z-y|^{n+1}} f(y)\, dy \right] dz \\ &= \int_{\Omega} \left[\mathbf{p.v.} \int_{\Omega} \frac{z-x}{|z-x|^{n+1}} \cdot \frac{z-y}{|z-y|^{n+1}} dz \right] f(y)\, dy + \frac{\pi^{n+1}}{\Gamma\left(\frac{n+1}{2}\right)^2} f(x). \end{aligned}$$

Here Γ represents the Gamma function given by

$$\Gamma(\lambda) = \int_0^{\infty} t^{\lambda-1} e^{-t} dt \quad \text{for } \lambda > 0.$$

We next note that for $n \geq 2$, up to a constant the Green's function of the Laplace operator on $\mathbb{R}^{n+1}$ is given by

$$G(x,y) = \frac{1}{|x-y|^{n-1}}, \tag{1.2}$$

and if we identify $\mathbb{R}^n$ as the hyperplane $\mathbb{R}^n \times \{0\}$ in $\mathbb{R}^{n+1}$, then for $x, z \in \mathbb{R}^n$,

$$\frac{z-x}{|z-x|^{n+1}} = \frac{1}{1-n} \nabla_z^{\mathbb{R}^n} G(x,z).$$

Here $\nabla_z^{\mathbb{R}^n}$ denotes the surface gradient. Hence if $M \subset \mathbb{R}^{n+1}$ is a smooth (curved) n-dimensional hypersurface, we may consider the iterated principal integral

$$\mathbf{p.v.} \int_M \nabla_z^M G(x,z) \cdot \left[\mathbf{p.v.} \int_M \nabla_z^M G(z,y) f(y)\, d\mu(y) \right] d\mu(z), \tag{1.3}$$

where ∇^M denotes the surface gradient. Our main result is stated in the following theorem.

Theorem 1.1 *Assume $n \geq 2$, $M^n \subset \mathbb{R}^{n+1}$ is a smooth compact hypersurface with or without boundary. For $x, y \in \mathbb{R}^{n+1}$, let $G(x,y) = \frac{1}{|x-y|^{n-1}}$, then for $1 < p < \infty$, $f \in L^p(M)$,*

$$\begin{aligned} &\mathbf{p.v.} \int_M \nabla_z^M G(x,z) \cdot \left[\mathbf{p.v.} \int_M \nabla_z^M G(z,y) f(y) \, d\mu(y) \right] d\mu(z) \\ &= \int_M \left[\mathbf{p.v.} \int_M \nabla_z^M G(x,z) \cdot \nabla_z^M G(z,y) \, d\mu(z) \right] f(y) \, d\mu(y) \\ &+ \frac{4\pi^{n+1}}{\Gamma\left(\frac{n-1}{2}\right)^2} f(x) \end{aligned} \tag{1.4}$$

for a.e. $x \in M$. Here μ is the surface measure on M.

Remark 1.1 *Our motivation to consider the extension of the formula in the above form (1.3) comes from the study of scattering problems by open surfaces arising from various areas such as computational electromagnetics, fluid mechanics and elasticity. The original Poincaré-Bertrand formula plays an important role in constructing a second kind integral equation formulation for the Dirichlet problem of the Laplace equation when the boundary consists of a set of open curves in two dimensions (see [1] for details). Here we are looking for stable, efficient, and accurate numerical algorithms for solving the Dirichlet problem on open surfaces in higher dimensions. That is, if M has nonempty boundary, g is a smooth function on M, we are looking for a continuous function u on $\mathbb{R}^{n+1}$ such that $u|_M = g$, u is harmonic in $\mathbb{R}^{n+1} \backslash M$ and $u(x)$ tends to 0 as x tends to infinity. We may assume for $x \in \mathbb{R}^{n+1}$, $u(x)$ is in the form of (1.3) for some unknown f, then Theorem 1.1 together with the kernel estimate in Propositions 3.1 and 3.2 implies the Dirichlet problem is reduced to an integral equation of the second kind.*

A formulation which is more similar to the original Poincaré-Bertrand formula (1.1) is as follows.

Theorem 1.2 *Assume $n \geq 2$, $M^n \subset \mathbb{R}^{n+1}$ is a smooth compact hypersurface with or without boundary. For $x, y \in \mathbb{R}^{n+1}$, let $G(x,y) = \frac{1}{|x-y|^{n-1}}$. Assume $0 < \alpha < 1$, $H \in C^\alpha(M \times M)$, then for every $x \in M \backslash \partial M$,*

$$\begin{aligned} &\mathbf{p.v.} \int_M \nabla_z^M G(x,z) \cdot \left[\mathbf{p.v.} \int_M \nabla_z^M G(z,y) H(z,y) \, d\mu(y) \right] d\mu(z) \\ &= \int_M \left[\mathbf{p.v.} \int_M \nabla_z^M G(x,z) \cdot \nabla_z^M G(z,y) H(z,y) \, d\mu(z) \right] d\mu(y) \\ &+ \frac{4\pi^{n+1}}{\Gamma\left(\frac{n-1}{2}\right)^2} H(x,x). \end{aligned} \tag{1.5}$$

The paper is organized as follows. In Section 2, we first describe some basic notations and formulas to be used subsequently. We then derive some integral identities related to Riesz transforms and some basic facts about singular integrals on hypersurfaces. In Section 3, we show that the kernel on the right hand side of (1.2) is only weakly singular. This estimate is not only needed later in proving (1.2), it also justifies the applicability of the formula in numerical computation. In Section 4, we prove the validity of (1.2) for the case $f = 1$. This is an important step toward the derivation of the general case. This special case is proved by an approximation procedure together with a somewhat lengthy calculation. After all these preparations, in Section 5, we prove the above two main theorems.

2 Some preparations

As usual, if A and B are two quantities, we write $A = O(B)$ to mean there exists a constant $c > 0$ such that $|A| \leq c|B|$, where c depends only on some unimportant ingredients (e.g. the hypersurface M). We also write $A = o_\varepsilon(1)$ to mean $\lim_{\varepsilon\to 0^+} A = 0$. Note here A can depend on some other variables.

For $x_0 \in \mathbb{R}^m$ and $r > 0$, we denote the open ball $B_r(x_0) = \{x \in \mathbb{R}^m : |x - x_0| < r\}$, sometime to emphasize the dimension we use $B_r^m(x_0)$ instead. We also write $B_r = B_r(0)$.

Let $M^n \subset \mathbb{R}^{n+1}$ be a smooth surface, ν be the unit normal direction, the shape operator A is given by

$$A(X) = D_X\nu \quad \text{for } X \in TM,$$

where D denotes the usual directional derivative. The eigenvalues of A are called principle curvatures of M and the trace of A is the mean curvature i.e. $H = \operatorname{tr} A$. If f is a smooth function defined on an open neighborhood of M in $\mathbb{R}^{n+1}$, then on M,

$$\Delta^M f = \Delta f - H\frac{\partial f}{\partial\nu} - \frac{\partial^2 f}{\partial\nu^2}.$$

Here Δ^M is the Laplace operator on M.

Let $x_0 \in M$ be an interior point, then near x_0, M is the graph of a smooth function defined on the tangent space $T_{x_0}M$. More precisely by translation and rotation, we can assume $x_0 = 0$ and the tangent plane is given by $\mathbb{R}^n \times \{0\}$, there exists a $r_0 > 0$ and a smooth function $\varphi : \overline{B}_{2r_0}^n \to \mathbb{R}$ such that near 0, M is just the graph of φ, moreover

$$\varphi(u) = \frac{1}{2}\sum_{i=1}^n \kappa_i u_i^2 + O\left(|u|^3\right) \quad \text{as } u \to 0.$$

Here $\kappa_1, \cdots, \kappa_n$ are the principle curvatures of M at 0.

Let μ be the surface measure on M. If f is a function on M, for every small $\varepsilon > 0$, f is integrable on $M \backslash B_\varepsilon^{n+1}(x_0)$, then we denote

$$\mathbf{p.v.} \int_M f d\mu = \lim_{\varepsilon \to 0^+} \int_{M \backslash B_\varepsilon^{n+1}(x_0)} f d\mu.$$

2.1 Some identities related to Riesz transformation

The Riesz transforms R_j $(j = 1, 2, \cdots, n)$ are defined by the formula ([5, Chapter III])

$$(R_j f)(u) = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\pi^{\frac{n+1}{2}}} \mathbf{p.v.} \int_{\mathbb{R}^n} \frac{u_j - v_j}{|u - v|^{n+1}} f(v)\, dv,$$

for a function f defined on $\mathbb{R}^n$. In terms of Fourier transforms, the Riesz transforms are given by

$$\widehat{R_j f}(\xi) = -\frac{i\xi_j}{|\xi|} \widehat{f}(\xi).$$

If h is a harmonic function on upper half space $\mathbb{R}_+^{n+1}$ with suitable decay condition near infinity, then for $1 \le j \le n$, we have

$$R_j\left(\partial_{n+1} h(u, 0)\right) = \partial_j h(u, 0).$$

For $\varepsilon > 0$, if we choose the harmonic function

$$h(x) = \frac{1}{|x + (0, \cdots, 0, \varepsilon)|^{n-1}},$$

then we get

$$R_j\left(\frac{\varepsilon}{\left(|u|^2 + \varepsilon^2\right)^{\frac{n+1}{2}}}\right) = \frac{u_j}{\left(|u|^2 + \varepsilon^2\right)^{\frac{n+1}{2}}}.$$

Because $\sum_{j=1}^n R_j R_j f = -f$,

$$\sum_{j=1}^n \mathbf{p.v.} \int_{\mathbb{R}^n} \frac{u_j - v_j}{|u - v|^{n+1}} \frac{v_j}{\left(|v|^2 + \varepsilon^2\right)^{\frac{n+1}{2}}} dv = -\frac{\pi^{\frac{n+1}{2}}}{\Gamma\left(\frac{n+1}{2}\right)} \frac{\varepsilon}{\left(|u|^2 + \varepsilon^2\right)^{\frac{n+1}{2}}}.$$

Let $\varepsilon \to 0^+$, then

$$\sum_{j=1}^n \mathbf{p.v.} \int_{\mathbb{R}^n} \frac{u_j - v_j}{|u - v|^{n+1}} \frac{v_j}{|v|^{n+1}} dv = 0 \quad \text{for } u \neq 0. \tag{2.1}$$

Indeed, fix a positive number $\delta < \frac{|u|}{2}$, we have

$$\begin{aligned}
&\mathbf{p.v.}\int_{\mathbb{R}^n} \frac{u_j - v_j}{|u-v|^{n+1}} \frac{v_j}{\left(|v|^2+\varepsilon^2\right)^{\frac{n+1}{2}}} dv \\
&= \int_{B_\delta(u)} \frac{u_j - v_j}{|u-v|^{n+1}} \left(\frac{v_j}{\left(|v|^2+\varepsilon^2\right)^{\frac{n+1}{2}}} - \frac{u_j}{\left(|u|^2+\varepsilon^2\right)^{\frac{n+1}{2}}} \right) dv \\
&+ \int_{B_\delta} \left(\frac{u_j - v_j}{|u-v|^{n+1}} - \frac{u_j}{|u|^{n+1}} \right) \frac{v_j}{\left(|v|^2+\varepsilon^2\right)^{\frac{n+1}{2}}} dv \\
&+ \int_{\mathbb{R}^n \setminus (B_\delta(u) \cup B_\delta)} \frac{u_j - v_j}{|u-v|^{n+1}} \frac{v_j}{\left(|v|^2+\varepsilon^2\right)^{\frac{n+1}{2}}} dv \\
&\to \mathbf{p.v.}\int_{\mathbb{R}^n} \frac{u_j - v_j}{|u-v|^{n+1}} \frac{v_j}{|v|^{n+1}} dv
\end{aligned}$$

as $\varepsilon \to 0^+$, by dominated convergence theorem.

As a consequence we get the following fact which will be useful in Section 3.

Lemma 2.1 *Let* $e_1 = (1, 0, \cdots, 0) \in \mathbb{R}^n$, *then*

$$\int_{\mathbb{R}^{n-1}} \frac{1}{\left(1+|t|^2\right)^n} dt = \int_{\substack{u \in \mathbb{R}^n, \\ u_1 < 1}} \frac{1}{|u|^{n-1}\,|u-2e_1|^{n+1}} du.$$

PROOF. It follows from (2.1) that

$$\mathbf{p.v.}\int_{\mathbb{R}^n} \frac{u}{|u|^{n+1}} \cdot \frac{u-2e_1}{|u-2e_1|^{n+1}} du = 0.$$

By symmetry we see that

$$\mathbf{p.v.}\int_{\substack{u \in \mathbb{R}^n \\ u_1 < 1}} \frac{u}{|u|^{n+1}} \cdot \frac{u-2e_1}{|u-2e_1|^{n+1}} du = \mathbf{p.v.}\int_{\substack{u \in \mathbb{R}^n \\ u_1 > 1}} \frac{u}{|u|^{n+1}} \cdot \frac{u-2e_1}{|u-2e_1|^{n+1}} du = 0.$$

Hence

$$\begin{aligned}
0 &= \lim_{\varepsilon \to 0^+} \int_{\substack{u \in \mathbb{R}^n \setminus B_\varepsilon \\ u_1 < 1}} \frac{u}{|u|^{n+1}} \cdot \frac{u-2e_1}{|u-2e_1|^{n+1}} du \\
&= \lim_{\varepsilon \to 0^+} \frac{1}{1-n} \int_{\substack{u \in \mathbb{R}^n \setminus B_\varepsilon \\ u_1 < 1}} \nabla \frac{1}{|u|^{n-1}} \cdot \frac{u-2e_1}{|u-2e_1|^{n+1}} du \\
&= \frac{1}{n-1} \left(\int_{\mathbb{R}^{n-1}} \frac{1}{\left(1+|t|^2\right)^n} dt - \int_{\substack{u \in \mathbb{R}^n, \\ u_1 < 1}} \frac{1}{|u|^{n-1}\,|u-2e_1|^{n+1}} du \right).
\end{aligned}$$

And the lemma follows.

Another fact which will be useful in Section 4 is the following identity.

Lemma 2.2 *Assume $h \in C^{\infty}(\mathbb{R}^n)$ has compact support, $h(0) = 1$. Let*

$$g(u) = \int_{\mathbb{R}^n} \frac{h(v)}{|u-v|^{n-1}} dv,$$

then

$$\lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^n} \left[\frac{(n+1)\varepsilon^2}{\left(|u|^2+\varepsilon^2\right)^{\frac{n+3}{2}}} - \frac{1}{\left(|u|^2+\varepsilon^2\right)^{\frac{n+1}{2}}} \right] g(u)\, du = \frac{4\pi^{n+1}}{(n-1)\Gamma\left(\frac{n-1}{2}\right)^2}.$$

PROOF. Since

$$\sum_{j=1}^{n} (R_j R_j h)(0) = -h(0) = -1,$$

we have

$$\frac{\Gamma\left(\frac{n+1}{2}\right)^2}{\pi^{n+1}} \sum_{j=1}^{n} \mathbf{p.v.} \int_{\mathbb{R}^n} \frac{u_j}{|u|^{n+1}} \left(\mathbf{p.v.} \int_{\mathbb{R}^n} \frac{u_j - v_j}{|u-v|^{n+1}} h(v)\, dv \right) du = 1.$$

Hence

$$\int_{\mathbb{R}^n} \frac{u}{\left(|u|^2+\varepsilon^2\right)^{\frac{n+1}{2}}} \cdot \left(\mathbf{p.v.} \int_{\mathbb{R}^n} \frac{u-v}{|u-v|^{n+1}} h(v)\, dv \right) du = \frac{\pi^{n+1}}{\Gamma\left(\frac{n+1}{2}\right)^2} + o_{\varepsilon}(1).$$

Using

$$\nabla g(u) = -(n-1)\left(\mathbf{p.v.} \int_{\mathbb{R}^n} \frac{u-v}{|u-v|^{n+1}} h(v)\, dv \right)$$

and integration by parts we get

$$\frac{1}{n-1} \int_{\mathbb{R}^n} \left[\frac{(n+1)\varepsilon^2}{\left(|u|^2+\varepsilon^2\right)^{\frac{n+3}{2}}} - \frac{1}{\left(|u|^2+\varepsilon^2\right)^{\frac{n+1}{2}}} \right] g(u)\, du = \frac{\pi^{n+1}}{\Gamma\left(\frac{n+1}{2}\right)^2} + o_{\varepsilon}(1).$$

And the lemma follows

2.2 Singular integral on a hypersurface

Assume $M^n \subset \mathbb{R}^{n+1}$ is a smooth compact hypersurface without boundary. Denote

$$(Tf)(x) = \mathbf{p.v.} \int_M \nabla_y^M G(x,y)\, f(y)\, d\mu(y). \tag{2.2}$$

In order to find out the most singular part of the kernel, we may assume $x=0$ and near 0, M is the graph of a smooth function φ (as described at the beginning of Section 2), then

$$\left|\nabla_y^M G\left(0,\left(u,\varphi\left(u\right)\right)\right)-\frac{\left(1-n\right)\left(u,0\right)}{\left|u\right|^{n+1}}\right|\leq\frac{c\left(M\right)}{\left|u\right|^{n-1}}.$$

It follows from [3, Chapter IX, X] that for $f\in C^{\alpha}\left(M\right)$, $0<\alpha<1$, $Tf\in C^{\alpha}\left(M\right)$ and

$$\left|Tf\right|_{C^{\alpha}(M)}\leq c\left(M,\alpha\right)\left|f\right|_{C^{\alpha}(M)}.$$

On the other hand, for $f\in L^p\left(M\right)$, $1<p<\infty$, $Tf\in L^p\left(M\right)$ and

$$\left|Tf\right|_{L^p(M)}\leq c\left(M,p\right)\left|f\right|_{L^p(M)}.$$

If $M^n\subset\mathbb{R}^{n+1}$ is a smooth compact hypersurface with nonempty boundary, $M_0^n\subset\mathbb{R}^{n+1}$ is a smooth compact hypersurface without boundary such that M is a domain in M_0. Again we have the operator T defined by (2.2). For a function f on M, we may set

$$f_0\left(x\right)=\begin{cases}f\left(x\right), & x\in M,\\ 0, & x\in M_0\backslash M.\end{cases}$$

Then for $1<p<\infty$,

$$\begin{aligned}\left|Tf\right|_{L^p(M)}&=\left|\mathbf{p.v.}\int_{M_0}\nabla_y^{M_0}G\left(x,y\right)f_0\left(y\right)d\mu\left(y\right)\right|_{L^p_x(M)}\\&\leq c\left(M,p\right)\left|f_0\right|_{L^p(M_0)}=c\left(M,p\right)\left|f\right|_{L^p(M)}.\end{aligned}$$

The next two lemmas are about the validity of the Fubini theorem when singular integrals are involved.

Lemma 2.3 *Assume $M^n\subset\mathbb{R}^{n+1}$ is a smooth compact hypersurface with or without boundary, $1<p<\infty$, $\Phi\in L^p\left(M\times M\right)$, then for a.e. $x\in M$,*

$$\begin{aligned}&\mathbf{p.v.}\int_M\nabla_y^M G\left(x,y\right)\left(\int_M\Phi\left(y,z\right)d\mu\left(z\right)\right)d\mu\left(y\right)\\&=\int_M\left(\mathbf{p.v.}\int_M\nabla_y^M G\left(x,y\right)\Phi\left(y,z\right)d\mu\left(y\right)\right)d\mu\left(z\right).\end{aligned}$$

PROOF. At first we verify the identity for $\Phi\in C^{\infty}\left(M\times M\right)$. Indeed under this assumption, for $x \in M \backslash \partial M$,

$$
\begin{aligned}
&\mathbf{p.v.} \int_M \nabla_y^M G(x,y) \left(\int_M \Phi(y,z)\, d\mu(z) \right) d\mu(y) \\
&= \int_M \nabla_y^M G(x,y) \left(\int_M \Phi(y,z)\, d\mu(z) - \int_M \Phi(x,z)\, d\mu(z) \right) d\mu(y) \\
&+ \int_M \Phi(x,z)\, d\mu(z) \cdot \mathbf{p.v.} \int_M \nabla_y^M G(x,y)\, d\mu(y) \\
&= \int_M d\mu(z) \int_M \nabla_y^M G(x,y) (\Phi(y,z) - \Phi(x,z))\, d\mu(y) \\
&+ \int_M \left(\Phi(x,z) \cdot \mathbf{p.v.} \int_M \nabla_y^M G(x,y)\, d\mu(y) \right) d\mu(z) \\
&= \int_M \left(\mathbf{p.v.} \int_M \nabla_y^M G(x,y)\, \Phi(y,z)\, d\mu(y) \right) d\mu(z).
\end{aligned}
$$

To continue we observe that

$$
\begin{aligned}
&\left| \mathbf{p.v.} \int_M \nabla_y^M G(x,y) \left(\int_M \Phi(y,z)\, d\mu(z) \right) d\mu(y) \right|_{L_x^p(M)} \\
&\leq c(M,p) \left| \int_M \Phi(y,z)\, d\mu(z) \right|_{L_y^p(M)} \\
&\leq c(M,p) \left| \Phi \right|_{L^p(M \times M)},
\end{aligned}
$$

and

$$
\begin{aligned}
&\left| \int_M \left(\mathbf{p.v.} \int_M \nabla_y^M G(x,y)\, \Phi(y,z)\, d\mu(y) \right) d\mu(z) \right|_{L_x^p(M)} \\
&\leq \int_M \left| \mathbf{p.v.} \int_M \nabla_y^M G(x,y)\, \Phi(y,z)\, d\mu(y) \right|_{L_x^p(M)} d\mu(z) \\
&\leq c(M,p) \int_M \left| \Phi(y,z) \right|_{L_y^p(M)} d\mu(z) \\
&\leq c(M,p) \left| \Phi \right|_{L^p(M \times M)}.
\end{aligned}
$$

Because $C^\infty(M \times M)$ is dense in $L^p(M \times M)$, the lemma follows.

Lemma 2.4 *Assume $M^n \subset \mathbb{R}^{n+1}$ is a smooth compact hypersurface with or without boundary, $1 < p < \infty$, $p' = \frac{p}{p-1}$, $f \in L^p(M)$, $g \in L^{p'}(M)$, then*

$$
\begin{aligned}
&\int_M g(x) \left(\mathbf{p.v.} \int_M \nabla_x^M G(x,y)\, f(y)\, d\mu(y) \right) d\mu(x) \\
&= \int_M f(y) \left(\mathbf{p.v.} \int_M \nabla_x^M G(x,y)\, g(x)\, d\mu(x) \right) d\mu(y).
\end{aligned}
$$

PROOF. Denote

$$
\left(\widetilde{T} g \right)(y) = \mathbf{p.v.} \int_M \nabla_x^M G(x,y)\, g(x)\, d\mu(x),
$$

then $\left|\widetilde{T}g\right|_{L^{p'}(M)} \leq c(M,p)|g|_{L^{p'}(M)}$. Hence by approximation to prove the lemma we only need to consider the case when $f, g \in C^{\infty}(M)$. Under this assumption we have for $\varepsilon > 0$,

$$\begin{aligned}
&\int_M g(x)\left(\int_{\substack{y\in M\\ |x-y|>\varepsilon}} \nabla_x^M G(x,y) f(y)\, d\mu(y)\right) d\mu(x) \\
&= \int_M f(y)\left(\int_{\substack{x\in M\\ |x-y|>\varepsilon}} \nabla_x^M G(x,y) g(x)\, d\mu(x)\right) d\mu(y).
\end{aligned}$$

Let $\varepsilon \to 0^+$, we get the needed identity.

Finally we discuss an approximation of the singular integral by integral operators with smooth kernels. If $M^n \subset \mathbb{R}^{n+1}$ is a smooth compact hypersurface without boundary, for $x \in M$, let $\nu(x)$ be the unit normal direction, then for $\varepsilon > 0$ small, we can define

$$(T_\varepsilon f)(x) = \int_M \nabla_y^M G(x + \varepsilon\nu(x), y) f(y)\, d\mu(y).$$

Then for $1 < p < \infty$,

$$|T_\varepsilon f|_{L^p(M)} \leq c(M,p)|f|_{L^p(M)}$$

and

$$T_\varepsilon f \to Tf \quad \text{in } L^p(M)$$

for every $f \in L^p(M)$.

Later on we will need to know the asymptotic formula of $\Delta_y^M G(x + \varepsilon\nu(x), y)$ for y close to x. To this aim we can assume $x = 0$ and near 0, M is the graph of a smooth function φ (as described at the beginning of Section 2). Let $y = (u, \varphi(u))$, then

$$\begin{aligned}
&\Delta_y^M G(x + \varepsilon\nu(x), y) \\
&= \Delta_y G(x + \varepsilon\nu(x), y) - H(y)\frac{\partial G}{\partial \nu}(x + \varepsilon\nu(x), y) - \frac{\partial^2 G}{\partial \nu^2}(x + \varepsilon\nu(x), y) \\
&= \frac{n-1}{\left[|u|^2 + (\varphi+\varepsilon)^2\right]^{\frac{n+1}{2}}} + \frac{(n-1)H(u\cdot\nabla\varphi - \varphi - \varepsilon)}{\sqrt{1+|\nabla\varphi|^2}\left[|u|^2 + (\varphi+\varepsilon)^2\right]^{\frac{n+1}{2}}} \\
&- \frac{(n^2-1)(u\cdot\nabla\varphi - \varphi - \varepsilon)^2}{\left(1+|\nabla\varphi|^2\right)\left[|u|^2 + (\varphi+\varepsilon)^2\right]^{\frac{n+3}{2}}},
\end{aligned}$$

here $\varphi = \varphi(u)$, $H = H(u, \varphi(u))$. Denote $r = \sqrt{|u|^2 + \varepsilon^2}$, then

$$|u|^2 + (\varphi + \varepsilon)^2 = r^2 \left[1 + \frac{\varepsilon \sum_{i=1}^n \kappa_i u_i^2}{r^2} + O\left(r^2\right) \right],$$

$$\frac{1}{\left[|u|^2 + (\varphi + \varepsilon)^2\right]^{\frac{n+1}{2}}} = \frac{1}{r^{n+1}} \left[1 - \frac{n+1}{2} \frac{\varepsilon \sum_{i=1}^n \kappa_i u_i^2}{r^2} + O\left(r^2\right) \right],$$

$$u \cdot \nabla \varphi - \varphi - \varepsilon = -\varepsilon + \frac{1}{2} \sum_{i=1}^n \kappa_i u_i^2 + O\left(r^3\right).$$

Based on these formulas, we get

$$\begin{aligned} &\Delta_y^M G(x + \varepsilon \nu(x), y) \\ &= \frac{n-1}{r^{n+1}} - \frac{(n^2-1)\varepsilon^2}{r^{n+3}} - \frac{(n-1)\varepsilon \sum_{i=1}^n \kappa_i}{r^{n+1}} + \frac{n^2-1}{2} \frac{\varepsilon \sum_{i=1}^n \kappa_i u_i^2}{r^{n+3}} \\ &+ \frac{(n^2-1)(n+3)}{2} \frac{\varepsilon^3 \sum_{i=1}^n \kappa_i u_i^2}{r^{n+5}} + O\left(\frac{1}{r^{n-1}}\right), \end{aligned} \tag{2.3}$$

where $r = \sqrt{|u|^2 + \varepsilon^2}$.

3 Estimate of the kernel

In this section, we consider the kernel K on the right hand side of (1.4) defined by the formula

$$K(x, y) = \mathbf{p.v.} \int_M \nabla_z^M G(x, z) \cdot \nabla_z^M G(z, y)\, d\mu(z).$$

We show that the kernel is only weakly singular along the diagonal $x = y$. The main results are summarized in Proposition 3.1 for the case of a closed surface and Proposition 3.2 for the case of a surface with nonempty boundary.

3.1 Compact hypersurface without boundary

Proposition 3.1 *Assume that $n \geq 2$, $M^n \subset \mathbb{R}^{n+1}$ is a smooth compact hypersurface without boundary. For $x, y \in M$, let*

$$K(x, y) = \mathbf{p.v.} \int_M \nabla_z^M G(x, z) \cdot \nabla_z^M G(z, y)\, d\mu(z).$$

Then

$$|K(x, y)| \leq \begin{cases} c(M)\left(|\log|x-y|| + 1\right), & \text{if } n = 2; \\ c(M)\left(\frac{1}{|x-y|^{n-2}} + 1\right), & \text{if } n \geq 3. \end{cases}$$

Here $c(M)$ is a positive constant depending only on M.

It is clear that we only need to verify the estimate when $|x-y|$ is small. Using notations in Section 2, we may assume $x=0$, $T_0M=\mathbb{R}^n\times\{0\}$ and near 0, M is the graph of a smooth function φ defined on $\overline{B}^n_{2r_0}$. Let $e_1=(1,0,\cdots,0)\in\mathbb{R}^n$, we may assume $y=(2\delta e_1,\varphi(2\delta e_1))$. Using the expansion formula of φ, we see when $|y|$ is small enough,

$$2\delta\le|y|\le 2.1\delta.$$

Note that in the integrand of the expression of $K(x,y)$, we have two singular points as $z=x$ and $z=y$. The main idea of proving the estimate of $K(x,y)$ comes from the proof of Lemma 2.1: we divide the integral into two pieces from the "middle" of x and y, each of which contains a singular point and satisfies the needed estimate. More precisely, let

$$\begin{aligned} D&=\left\{(u,\varphi(u)):u\in B^n_{r_0}(\delta e_1)\right\},\\ D_l&=\left\{(u,\varphi(u)):u\in B^n_{r_0}(\delta e_1),u_1<\delta\right\},\\ D_r&=\left\{(u,\varphi(u)):u\in B^n_{r_0}(\delta e_1),u_1>\delta\right\}.\end{aligned}$$

Since on $M\backslash D$, both $\nabla^M_z G(0,z)$ and $\nabla^M_z G(z,y)$ are uniformly bounded,

$$\begin{aligned} K(0,y)=O(1)&+\mathbf{p.v.}\int_{D_l}\nabla^M_z G(0,z)\cdot\nabla^M_z G(z,y)\,d\mu(z)\\ &+\mathbf{p.v.}\int_{D_r}\nabla^M_z G(0,z)\cdot\nabla^M_z G(z,y)\,d\mu(z).\end{aligned}$$

We claim

$$\left|\mathbf{p.v.}\int_{D_l}\nabla^M_z G(0,z)\cdot\nabla^M_z G(z,y)\,d\mu(z)\right|=\begin{cases}O\left(\log\frac{1}{\delta}\right), & \text{if } n=2,\\ O\left(\frac{1}{\delta^{n-2}}\right), & \text{if } n\ge 3.\end{cases}\tag{3.1}$$

$$\left|\mathbf{p.v.}\int_{D_r}\nabla^M_z G(0,z)\cdot\nabla^M_z G(z,y)\,d\mu(z)\right|=\begin{cases}O\left(\log\frac{1}{\delta}\right), & \text{if } n=2,\\ O\left(\frac{1}{\delta^{n-2}}\right), & \text{if } n\ge 3.\end{cases}\tag{3.2}$$

This clearly implies Proposition 3.1.

Below we only prove (3.1). The verification of (3.2) is almost identical. To derive (3.1), we scale the graph by $\frac{1}{\delta}$. More precisely, let

$$\psi(u)=\frac{\varphi(\delta u)}{\delta},$$

then

$$|\psi(u)|\le c(M)\,\delta\,|u|^2,\quad |\nabla\psi(u)|\le c(M)\,\delta\,|u|,\quad \left|\nabla^2\psi(u)\right|\le c(M)\,\delta.$$

Let

$$N = \left\{ (u, \psi(u)) : u \in B^n_{r_0/\delta}(e_1), u_1 < 1 \right\},$$
$$\overline{y} = (2e_1, \psi(2e_1)).$$

N is simply the $\frac{1}{\delta}$ dilation of D_l. Using integration by parts, we get

$$\begin{aligned}
&\mathbf{p.v.} \int_{D_l} \nabla_z^M G(0,z) \cdot \nabla_z^M G(z,y)\, d\mu(z) \\
&= \frac{1}{\delta^n} \mathbf{p.v.} \int_N \nabla_z^M G(0,z) \cdot \nabla_z^M G(z,\overline{y})\, d\mu_N(z) \\
&= \frac{1}{\delta^n} \int_{\partial N} G(0,z) \frac{\partial G}{\partial \nu_{\partial N}}(z,\overline{y})\, dS(z) - \frac{1}{\delta^n} \int_N G(0,z)\, \Delta_z^N G(z,\overline{y})\, d\mu_N(z).
\end{aligned}$$

Here μ_N is the surface measure on N and $\nu_{\partial N}$ is the outer normal direction of N at ∂N.

We will show

$$\begin{aligned}
&\frac{1}{\delta^n} \int_{\partial N} G(0,z) \frac{\partial G}{\partial \nu_{\partial N}}(z,\overline{y})\, dS(z) \\
&= \frac{n-1}{\delta^n} \int_{\mathbb{R}^{n-1}} \frac{1}{\left(1+|t|^2\right)^n} dt + \begin{cases} O\left(\log \frac{1}{\delta}\right), & \text{if } n = 2, \\ O\left(\frac{1}{\delta^{n-2}}\right), & \text{if } n \geq 3. \end{cases}
\end{aligned} \tag{3.3}$$

and

$$\begin{aligned}
&\frac{1}{\delta^n} \int_N G(0,z)\, \Delta^N G(z,\overline{y})\, d\mu_N(z) \\
&= \frac{n-1}{\delta^n} \int_{\substack{u \in \mathbb{R}^n, \\ u_1 < 1}} \frac{1}{|u|^{n-1} |u-2e_1|^{n+1}} du + \begin{cases} O\left(\log \frac{1}{\delta}\right), & \text{if } n = 2, \\ O\left(\frac{1}{\delta^{n-2}}\right), & \text{if } n \geq 3. \end{cases}
\end{aligned} \tag{3.4}$$

Once we have (3.3) and (3.4), the inequality (3.1) follows from Lemma 2.1 (note that the terms of order $\frac{1}{\delta^n}$ cancels!).

To prove (3.3), we decompose ∂N into two pieces, namely

$$\Sigma_1 = \left\{ (u, \psi(u)) : u \in B^n_{r_0/\delta}(e_1), u_1 = 1 \right\} = \left\{ (1, t, \psi(1,t)) : t \in B^{n-1}_{r_0/\delta} \right\},$$
$$\Sigma_2 = \left\{ (u, \psi(u)) : u \in \partial B^n_{r_0/\delta}(e_1), u_1 \leq 1 \right\}.$$

Using estimates of $|\nabla \psi|$ we see that

$$\left| \frac{1}{\delta^n} \int_{\Sigma_2} G(0,z) \frac{\partial G}{\partial \nu_{\partial N}}(z,\overline{y})\, dS(z) \right| \leq c(M).$$

On the other hand, on Σ_1,

$$
\begin{aligned}
z &= (1, t, \psi(1,t)), \quad t \in B^{n-1}_{r_0/\delta}, \\
dS(z) &= \sqrt{1 + |\nabla \psi|^2 - (\partial_1 \psi)^2}\, dt, \\
G(0,z) &= \left(1 + |t|^2 + \psi^2\right)^{-\frac{n-1}{2}}, \\
\nu_{\partial N}(z) &= \frac{\left(1 + |\nabla\psi|^2 - (\partial_1\psi)^2, -\partial_1\psi\partial_2\psi, \cdots, -\partial_1\psi\partial_n\psi, \partial_1\psi\right)}{\sqrt{1+|\nabla\psi|^2}\sqrt{1+|\nabla\psi|^2 - (\partial_1\psi)^2}}.
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\frac{\partial G}{\partial \nu_{\partial N}}(z, \overline{y}) \\
&= \frac{(n-1)\left[1 + |\nabla\psi|^2 - (\partial_1\psi)^2 + t_1\partial_1\psi\partial_2\psi + \cdots + t_{n-1}\partial_1\psi\partial_n\psi - \partial_1\psi\left(\psi(1,t) - \psi(2e_1)\right)\right]}{\sqrt{1+|\nabla\psi|^2}\sqrt{1+|\nabla\psi|^2-(\partial_1\psi)^2}\left[1+|t|^2 + \left(\psi(1,t)-\psi(2e_1)\right)^2\right]^{\frac{n+1}{2}}}.
\end{aligned}
$$

Based on these formulas and estimates of ψ we see that when $n = 2$,

$$
\begin{aligned}
&\frac{1}{\delta^2}\int_{\Sigma_1} G(0,z)\frac{\partial G}{\partial\nu_{\partial N}}(z,\overline{y})\, dS(z) \\
&= \frac{1}{\delta^2}\int_{-r_0/\delta}^{r_0/\delta} \frac{1}{\sqrt{1+t^2+\psi^2}\sqrt{1+|\nabla\psi|^2}\left[1+t^2+\left(\psi(1,t)-\psi(2,0)\right)^2\right]^{\frac{3}{2}}}dt + O\left(\log\frac{1}{\delta}\right) \\
&= \frac{1}{\delta^2}\int_{-r_0/\delta}^{r_0/\delta}\frac{1}{(1+t^2)^2}dt + O\left(\log\frac{1}{\delta}\right) \\
&= \frac{1}{\delta^2}\int_{-\infty}^{\infty}\frac{1}{(1+t^2)^2}dt + O\left(\log\frac{1}{\delta}\right);
\end{aligned}
$$

and when $n \geq 3$,

$$
\begin{aligned}
&\frac{1}{\delta^2}\int_{\Sigma_1} G(0,z)\frac{\partial G}{\partial\nu_{\partial N}}(z,\overline{y})\, dS(z) \\
&= \frac{n-1}{\delta^n}\int_{B^{n-1}_{r_0/\delta}} \frac{1}{\left(1+|t|^2+\psi^2\right)^{\frac{n-1}{2}}\left[1+|t|^2+\left(\psi(1,t)-\psi(2e_1)\right)^2\right]^{\frac{n+1}{2}}\sqrt{1+|\nabla\psi|^2}}dt \\
&\quad + O\left(\frac{1}{\delta^{n-2}}\right) \\
&= \frac{n-1}{\delta^n}\int_{B^{n-1}_{r_0/\delta}}\frac{1}{\left(1+|t|^2\right)^n}dt + O\left(\frac{1}{\delta^{n-2}}\right) \\
&= \frac{n-1}{\delta^n}\int_{\mathbb{R}^{n-1}}\frac{1}{\left(1+|t|^2\right)^n}dt + O\left(\frac{1}{\delta^{n-2}}\right).
\end{aligned}
$$

Hence we get (3.3).

To prove (3.4), we note that

$$
\begin{aligned}
z &= (u, \psi(u)), \quad u \in B^n_{r_0/\delta}(e_1), u_1 < 1, \\
d\mu_N(z) &= \sqrt{1 + |\nabla\psi|^2} du, \\
\nu_N(z) &= \frac{(\nabla\psi, -1)}{\sqrt{1 + |\nabla\psi|^2}}, \\
G(0, z) &= \left(|u|^2 + \psi^2\right)^{-\frac{n-1}{2}},
\end{aligned}
$$

here ν_N is the normal direction of N. To find the formula of $\Delta^N G(z, \overline{y})$, we note that

$$
\begin{aligned}
&\Delta^N G(z, \overline{y}) \\
&= \Delta G(z, \overline{y}) - H_N \frac{\partial G}{\partial \nu_N}(z, \overline{y}) - \frac{\partial^2 G}{\partial \nu_N^2}(z, \overline{y}) \\
&= \frac{n-1}{\left[|u - 2e_1|^2 + (\psi(u) - \psi(2e_1))^2\right]^{\frac{n+1}{2}}} \\
&- \frac{(n-1) H_N [\psi(u) - \psi(2e_1) - (u - 2e_1) \cdot \nabla\psi]}{\sqrt{1 + |\nabla\psi|^2}\left[|u - 2e_1|^2 + (\psi(u) - \psi(2e_1))^2\right]^{\frac{n+1}{2}}} \\
&- \frac{(n^2 - 1) [\psi(u) - \psi(2e_1) - (u - 2e_1) \cdot \nabla\psi]^2}{\left(1 + |\nabla\psi|^2\right)\left[|u - 2e_1|^2 + (\psi(u) - \psi(2e_1))^2\right]^{\frac{n+3}{2}}}.
\end{aligned}
$$

Based on these formulas, the estimates for ψ and the basic fact $|H_N| \leq c(M)\delta$ we can derive (3.4) by calculations similar as before.

3.2 Compact hypersurface with nonempty boundary

Proposition 3.2 *Assume $n \geq 2$, $M_0^n \subset \mathbb{R}^{n+1}$ is a smooth compact hypersurface without boundary, $M \subset M_0^n$ is a smooth domain in M_0. For $x, y \in M$, let*

$$
K(x, y) = \mathbf{p.v.} \int_M \nabla_z^M G(x, z) \cdot \nabla_z^M G(z, y) \, d\mu(z).
$$

Then

$$
|K(x, y)| \leq \begin{cases} c(M)\left(|\log|x - y|| + \int_{M_0 \backslash M} \frac{1}{|x-z|^2} \frac{1}{|y-z|^2} d\mu(z) + 1\right), & \text{if } n = 2; \\ c(M)\left(\frac{1}{|x-y|^{n-2}} + \int_{M_0 \backslash M} \frac{1}{|x-z|^n} \frac{1}{|y-z|^n} d\mu(z) + 1\right), & \text{if } n \geq 3. \end{cases}
$$

In particular,

$$
\int_M |K(x, y)| \, d\mu(y) \leq c(M)\left(1 + |\log d(x, \partial M)|^2\right)
$$

and

$$\int_M d\mu(x)\int_M |K(x,y)|\,d\mu(y) \le c(M) < \infty.$$

The proposition easily follows from the estimate in the closed surface case. Indeed let

$$K_0(x,y) = \mathbf{p.v.}\int_{M_0} \nabla_z^{M_0} G(x,z)\cdot\nabla_z^{M_0} G(z,y)\,d\mu(z),$$

then

$$\begin{aligned}|K(x,y)| &\le |K_0(x,y)| + \left|\int_{M_0\setminus M} \nabla_z^{M_0} G(x,z)\cdot\nabla_z^{M_0} G(z,y)\,d\mu(z)\right| \\ &\le |K_0(x,y)| + c(M)\int_{M_0\setminus M} \frac{1}{|x-z|^n\,|y-z|^n}\,d\mu(z).\end{aligned}$$

The first inequality in the proposition follows from this and Proposition 3.1. Next we have

$$\begin{aligned}&\int_M |K(x,y)|\,d\mu(y) \\ &\le c(M) + c(M)\int_{M_0\setminus M} d\mu(z)\int_M \frac{1}{|x-z|^n\,|y-z|^n}\,d\mu(y) \\ &\le c(M) + c(M)\int_{M_0\setminus M} \frac{1+|\log d(z,M)|}{|x-z|^n}\,d\mu(z) \\ &\le c(M)\left(1+|\log d(x,\partial M)|^2\right).\end{aligned}$$

The last two inequalities in the Proposition 3.2 follows.

4 Poincaré-Betrand formula with a constant density

4.1 Compact hypersurface without boundary

The main aim of this subsection is to prove the special case of (1.4) when the density function $f = 1$. It is an important step in deriving the general Poincaré-Betrand formula.

Proposition 4.1 *Assume $n \ge 2$, $M^n \subset \mathbb{R}^{n+1}$ is a smooth compact hypersurface without boundary. Then for $x \in M$,*

$$\begin{aligned}&\mathbf{p.v.}\int_M \nabla_z^M G(x,z)\cdot\left[\mathbf{p.v.}\int_M \nabla_z^M G(z,y)\,d\mu(y)\right]d\mu(z) \\ &= \int_M\left[\mathbf{p.v.}\int_M \nabla_z^M G(x,z)\cdot\nabla_z^M G(z,y)\,d\mu(z)\right]d\mu(y) + \frac{4\pi^{n+1}}{\Gamma\left(\frac{n-1}{2}\right)^2}.\end{aligned}$$

Our main strategy to prove the identity is by an approximation procedure, more precisely we will replace $\nabla_z^M G(x,z)$ by $\nabla_z^M G(x+\varepsilon\nu(x),z)$ and check what happens when $\varepsilon \to 0^+$. The advantage of this procedure is that we can reduce the number of principal integrals in the expressions and apply the Fubini type results in Lemma 2.3 and Lemma 2.4. After that by a lengthy and careful local analysis we derive the needed identity through a limiting process.

Without losing of generality we may assume $x=0$ and near 0, M is the graph of a smooth function $\varphi : B_{2r_0}^n \to \mathbb{R}$ (as described at the beginning of Section 2). Let $e_{n+1} = (0,\cdots,0,1) \in \mathbb{R}^{n+1}$ and

$$K_\varepsilon(y) = \mathbf{p.v.}\int_M \nabla_z^M G(-\varepsilon e_{n+1},z)\cdot\nabla_z^M G(z,y)\,d\mu(z),$$
$$K(y) = \mathbf{p.v.}\int_M \nabla_z^M G(0,z)\cdot\nabla_z^M G(z,y)\,d\mu(z).$$

It follows form Proposition 3.1 that $K \in L^1(M)$. By Lemma 2.4,

$$\begin{aligned}&\int_M K_\varepsilon(y)\,d\mu(y)\\&=\int_M \nabla_z^M G(-\varepsilon e_{n+1},z)\cdot\left[\mathbf{p.v.}\int_M \nabla_z^M G(z,y)\,d\mu(y)\right]d\mu(z)\\&=\mathbf{p.v.}\int_M \nabla_z^M G(0,z)\cdot\left[\mathbf{p.v.}\int_M \nabla_z^M G(z,y)\,d\mu(y)\right]d\mu(z)+o_\varepsilon(1).\end{aligned}$$

On the other hand we will show

$$\int_M K_\varepsilon(y)\,d\mu(y) = \int_M K(y)\,d\mu(y) + \frac{4\pi^{n+1}}{\Gamma\left(\frac{n-1}{2}\right)^2} + o_\varepsilon(1), \tag{4.1}$$

and the proposition clearly follows from the above two equalities by letting $\varepsilon \to 0^+$.

To prove (4.1), first we claim for every $\delta > 0$,

$$\int_{M\backslash B_\delta^{n+1}} |K_\varepsilon(y) - K(y)|\,d\mu(y) \to 0 \quad \text{as } \varepsilon \to 0^+. \tag{4.2}$$

This claim means interesting things happen only near 0. To prove the claim we note that

$$\begin{aligned}&\int_{M\cap B_{\delta/2}^{n+1}} \nabla_z^M G(-\varepsilon e_{n+1},z)\cdot\nabla_z^M G(z,y)\,d\mu(z)\\&\to \mathbf{p.v.}\int_{M\cap B_{\delta/2}^{n+1}} \nabla_z^M G(0,z)\cdot\nabla_z^M G(z,y)\,d\mu(z) \quad \text{in } L_y^\infty(M\backslash B_\delta^n),\end{aligned}$$

and for any $1 < p < \infty$,

$$\begin{aligned}&\mathbf{p.v.}\int_{M\backslash B^{n+1}_{\delta/2}} \nabla^M_z G\left(-\varepsilon e_{n+1}, z\right)\cdot\nabla^M_z G\left(z,y\right)d\mu\left(z\right)\\ &\to \mathbf{p.v.}\int_{M\backslash B^{n+1}_{\delta/2}} \nabla^M_z G\left(0, z\right)\cdot\nabla^M_z G\left(z,y\right)d\mu\left(z\right) \quad \text{in } L^p_y\left(M\backslash B^n_\delta\right)\end{aligned}$$

as $\varepsilon\to 0^+$. (4.2) follows.

Next we study $K_\varepsilon\left(y\right)$ for y near 0. For $0<\delta<2r_0$, we write

$$D_\delta = \left\{\left(u,\varphi\left(u\right)\right) : u\in B^n_\delta\right\}.$$

Fix a cutoff function $\eta\in C^\infty\left(\mathbb{R}^n\right)$ such that $\eta|_{B_{1/2}} = 1$ and $\eta|_{\mathbb{R}^n\backslash B_1} = 0$. For $\delta>0$ small, define a function h_δ on M by

$$h_\delta\left(y\right) = \begin{cases}\eta\left(\frac{u}{\delta}\right), & \text{if } y=\left(u,\varphi\left(u\right)\right), u\in B_\delta;\\ 0, & \text{otherwise.}\end{cases}$$

For $\delta>0$ small and $y\in D_\delta$, we have

$$\begin{aligned}K_\varepsilon\left(y\right) = &\ \mathbf{p.v.}\int_{D_{r_0}} \nabla^M_z G\left(-\varepsilon e_{n+1}, z\right)\cdot\nabla^M_z G\left(z,y\right)d\mu\left(z\right)\\ &+\int_{M\backslash D_{r_0}} \nabla^M_z G\left(-\varepsilon e_{n+1}, z\right)\cdot\nabla^M_z G\left(z,y\right)d\mu\left(z\right).\end{aligned}$$

Note that

$$\begin{aligned}&\int_{M\backslash D_{r_0}} \nabla^M_z G\left(-\varepsilon e_{n+1}, z\right)\cdot\nabla^M_z G\left(z,y\right)d\mu\left(z\right)\\ &\to \int_{M\backslash D_{r_0}} \nabla^M_z G\left(0, z\right)\cdot\nabla^M_z G\left(z,y\right)d\mu\left(z\right)\end{aligned} \tag{4.3}$$

uniformly for $y\in D_\delta$ as $\varepsilon\to 0^+$. Hence we only need to know what happens for

$$\mathbf{p.v.}\int_{D_{r_0}} \nabla^M_z G\left(-\varepsilon e_{n+1}, z\right)\cdot\nabla^M_z G\left(z,y\right)d\mu\left(z\right)$$

as $\varepsilon\to 0^+$.

To continue we observe that

$$\begin{aligned}
&\int_M \left[\mathbf{p.v.} \int_{D_{r_0}} \nabla_z^M G\left(-\varepsilon e_{n+1}, z\right) \cdot \nabla_z^M G\left(z, y\right) d\mu\left(z\right)\right] h_\delta\left(y\right) d\mu\left(y\right) \qquad (4.4)\\
&= \int_{D_{r_0}} \nabla_z^M G\left(-\varepsilon e_{n+1}, z\right) \cdot \left[\mathbf{p.v.} \int_M \nabla_z^M G\left(z, y\right) h_\delta\left(y\right) d\mu\left(y\right)\right] d\mu\left(z\right)\\
&= \int_{D_{r_0}} \nabla_z^M G\left(-\varepsilon e_{n+1}, z\right) \cdot \nabla_z^M f_\delta\left(z\right) d\mu\left(z\right)\\
&= -\int_{D_{r_0}} f_\delta\left(z\right) \Delta_z^M G\left(-\varepsilon e_{n+1}, z\right) d\mu\left(z\right)\\
&+ \int_{\partial D_{r_0}} f_\delta\left(z\right) \frac{\partial G}{\partial \nu_{\partial D_{r_0}}} \left(\left(-\varepsilon e_{n+1}, z\right)\right) dS\left(z\right),
\end{aligned}$$

where

$$f_\delta\left(z\right) = \int_M G\left(z, y\right) h_\delta\left(y\right) d\mu\left(y\right)$$

is the single layer potential and $\nu_{\partial D_{r_0}}$ is the unit outer normal direction on ∂D_{r_0}. For $z \in D_{\frac{3}{2}r_0}$, we can write $z = \left(u, \varphi\left(u\right)\right)$ and

$$\begin{aligned}
f_\delta\left(z\right) &= \int_{D_\delta} G\left(z, y\right) h_\delta\left(y\right) d\mu\left(y\right) \qquad (4.5)\\
&= \int_{B_\delta} \frac{\eta\left(\frac{v}{\delta}\right) \sqrt{1 + \left|\nabla \varphi\left(v\right)\right|^2}}{\left[\left|u - v\right|^2 + \left(\varphi\left(u\right) - \varphi\left(v\right)\right)^2\right]^{\frac{n-1}{2}}} dv\\
&= F_\delta\left(u\right).
\end{aligned}$$

The last equality is the definition of F_δ. In particular, for $z \in D_{r_0}$, $\left|f_\delta\left(z\right)\right| \leq c\left(M\right)\delta$, hence

$$\int_{\partial D_{r_0}} f_\delta\left(z\right) \frac{\partial G}{\partial \nu_{\partial D_{r_0}}} \left(\left(-\varepsilon e_{n+1}, z\right)\right) dS\left(z\right) = O\left(\delta\right). \qquad (4.6)$$

On the other hand, by (2.3),

$$
\begin{aligned}
&-\int_{D_{r_0}} f_\delta(z)\,\Delta_z^M G(-\varepsilon e_{n+1}, z)\,d\mu(z) \qquad (4.7)\\
&=(n-1)\int_{B_{r_0}}\left[\frac{(n+1)\varepsilon^2}{\left(|u|^2+\varepsilon^2\right)^{\frac{n+3}{2}}}-\frac{1}{\left(|u|^2+\varepsilon^2\right)^{\frac{n+1}{2}}}\right]F_\delta(u)\sqrt{1+|\nabla\varphi|^2}du\\
&+(n-1)\int_{B_{r_0}}\frac{\varepsilon\sum_{i=1}^n\kappa_i}{\left(|u|^2+\varepsilon^2\right)^{\frac{n+1}{2}}}F_\delta(u)\sqrt{1+|\nabla\varphi|^2}du\\
&-\frac{n^2-1}{2}\int_{B_{r_0}}\frac{\varepsilon\sum_{i=1}^n\kappa_i u_i^2}{\left(|u|^2+\varepsilon^2\right)^{\frac{n+3}{2}}}F_\delta(u)\sqrt{1+|\nabla\varphi|^2}du\\
&-\frac{(n^2-1)(n+3)}{2}\int_{B_{r_0}}\frac{\varepsilon^3\sum_{i=1}^n\kappa_i u_i^2}{\left(|u|^2+\varepsilon^2\right)^{\frac{n+5}{2}}}F_\delta(u)\sqrt{1+|\nabla\varphi|^2}du\\
&+O(\delta).
\end{aligned}
$$

Calculation shows that

$$
\int_{\mathbb{R}^n}\frac{1}{\left(|u|^2+1\right)^{\lambda}}du=\pi^{\frac{n}{2}}\frac{\Gamma\left(\lambda-\frac{n}{2}\right)}{\Gamma(\lambda)}\quad \text{for } \lambda>\frac{n}{2}.
$$

Hence

$$
\begin{aligned}
&\int_{B_{r_0}}\frac{\varepsilon}{\left(|u|^2+\varepsilon^2\right)^{\frac{n+1}{2}}}F_\delta(u)\sqrt{1+|\nabla\varphi|^2}du \qquad (4.8)\\
&=F_\delta(0)\int_{\mathbb{R}^n}\frac{1}{\left(|u|^2+1\right)^{\frac{n+1}{2}}}du+o_\varepsilon(1)\\
&=\frac{\pi^{\frac{n+1}{2}}}{\Gamma\left(\frac{n+1}{2}\right)}F_\delta(0)+o_\varepsilon(1).
\end{aligned}
$$

Similarly, for $1\le i\le n$,

$$
\begin{aligned}
&\int_{B_{r_0}}\frac{\varepsilon u_i^2}{\left(|u|^2+\varepsilon^2\right)^{\frac{n+3}{2}}}F_\delta(u)\sqrt{1+|\nabla\varphi|^2}du \qquad (4.9)\\
&=\frac{1}{n+1}\frac{\pi^{\frac{n+1}{2}}}{\Gamma\left(\frac{n+1}{2}\right)}F_\delta(0)+o_\varepsilon(1),
\end{aligned}
$$

and

$$\int_{B_{r_0}} \frac{\varepsilon^3 u_i^2}{\left(|u|^2+\varepsilon^2\right)^{\frac{n+5}{2}}} F_\delta(u)\sqrt{1+|\nabla\varphi|^2}du \tag{4.10}$$
$$= \frac{1}{(n+1)(n+3)} \frac{\pi^{\frac{n+1}{2}}}{\Gamma\left(\frac{n+1}{2}\right)} F_\delta(0) + o_\varepsilon(1).$$

Note that (4.8)-(4.10) implies the summation of those terms on the right hand side of (4.7) containing principal curvature κ_i's is equal to $o_\varepsilon(1)$.

We still need to study the term

$$\int_{B_{r_0}} \left[\frac{(n+1)\varepsilon^2}{\left(|u|^2+\varepsilon^2\right)^{\frac{n+3}{2}}} - \frac{1}{\left(|u|^2+\varepsilon^2\right)^{\frac{n+1}{2}}}\right] F_\delta(u)\sqrt{1+|\nabla\varphi|^2}du$$

which is "hypersingular". We will show

$$\int_{B_{r_0}} \left[\frac{(n+1)\varepsilon^2}{\left(|u|^2+\varepsilon^2\right)^{\frac{n+3}{2}}} - \frac{1}{\left(|u|^2+\varepsilon^2\right)^{\frac{n+1}{2}}}\right] F_\delta(u)\sqrt{1+|\nabla\varphi|^2}du \tag{4.11}$$
$$= \frac{4\pi^{n+1}}{(n-1)\Gamma\left(\frac{n-1}{2}\right)^2} + o_\varepsilon(1) + O(\delta) + O\left(\frac{\delta^2}{\varepsilon_1}\right) + O\left(\varepsilon_1^\alpha\right).$$

For this purpose we will compare $F_\delta(u)$ with the single layer potential $g_\delta(u)$ of the flat surface case given by

$$g_\delta(u) = \int_{B_\delta} \frac{\eta\left(\frac{v}{\delta}\right)}{|u-v|^{n-1}}du. \tag{4.12}$$

First note that

$$\int_{B_{r_0}} \left[\frac{(n+1)\varepsilon^2}{\left(|u|^2+\varepsilon^2\right)^{\frac{n+3}{2}}} - \frac{1}{\left(|u|^2+\varepsilon^2\right)^{\frac{n+1}{2}}}\right] F_\delta(u)\sqrt{1+|\nabla\varphi|^2}du \tag{4.13}$$
$$= \int_{B_{r_0}} \left[\frac{(n+1)\varepsilon^2}{\left(|u|^2+\varepsilon^2\right)^{\frac{n+3}{2}}} - \frac{1}{\left(|u|^2+\varepsilon^2\right)^{\frac{n+1}{2}}}\right] F_\delta(u)du + O(\delta).$$

Next we observe that for a fixed $0<\alpha<1$,

$$|f_\delta|_{C^{1,\alpha}(M)} \le c(M,\alpha)|h_\delta|_{C^\alpha(M)} \le \frac{c(M,\alpha)}{\delta^\alpha}.$$

Hence

$$|F_\delta|_{C^{1,\alpha}\left(\overline{B}_{r_0}\right)} \leq \frac{c\,(M,\alpha)}{\delta^\alpha}.$$

On the other hand, let g_δ be the single layer potential in the flat case defined in (4.12), then

$$|g_\delta|_{C^{1,\alpha}\left(\overline{B}_{r_0}\right)} \leq \frac{c\,(n)}{\delta^\alpha}.$$

It follows that

$$(F_\delta - g_\delta)(u) = (F_\delta - g_\delta)(0) + \sum_{i=1}^{n} \partial_i (F_\delta - g_\delta)(0)\, u_i + O\left(\frac{|u|^{1+\alpha}}{\delta^\alpha}\right).$$

On the other hand using the Taylor expansion of φ, we have

$$\begin{aligned}(F_\delta - g_\delta)(u) &= \int_{B_\delta} \frac{\eta\left(\frac{v}{\delta}\right)\sqrt{1+|\nabla\varphi(v)|^2}}{\left[|u-v|^2 + (\varphi(u)-\varphi(v))^2\right]^{\frac{n-1}{2}}} dv - \int_{B_\delta} \frac{\eta\left(\frac{v}{\delta}\right)}{|u-v|^{n-1}} dv \\ &= O\left(\delta\,|u|^2 + \delta^3\right).\end{aligned}$$

In particular,

$$(F_\delta - g_\delta)(0) = O\left(\delta^3\right).$$

Hence for $\varepsilon_1 > 0$ small,

$$\begin{aligned}&\int_{B_{r_0}\setminus B_{\varepsilon_1\delta}} \left[\frac{(n+1)\,\varepsilon^2}{\left(|u|^2+\varepsilon^2\right)^{\frac{n+3}{2}}} - \frac{1}{\left(|u|^2+\varepsilon^2\right)^{\frac{n+1}{2}}}\right] (F_\delta(u) - g_\delta(u))\, du \qquad (4.14)\\ &= O(\delta) + O\left(\frac{\delta^2}{\varepsilon_1}\right),\end{aligned}$$

and

$$\begin{aligned}&\int_{B_{\varepsilon_1\delta}} \left[\frac{(n+1)\,\varepsilon^2}{\left(|u|^2+\varepsilon^2\right)^{\frac{n+3}{2}}} - \frac{1}{\left(|u|^2+\varepsilon^2\right)^{\frac{n+1}{2}}}\right] (F_\delta(u) - g_\delta(u))\, du \qquad (4.15)\\ &= (F_\delta - g_\delta)(0) \int_{B_{\varepsilon_1\delta}} \left[\frac{(n+1)\,\varepsilon^2}{\left(|u|^2+\varepsilon^2\right)^{\frac{n+3}{2}}} - \frac{1}{\left(|u|^2+\varepsilon^2\right)^{\frac{n+1}{2}}}\right] du + O\left(\varepsilon_1^\alpha\right) \\ &= (F_\delta - g_\delta)(0) \int_{B_{\varepsilon_1\delta}} \operatorname{div}\left(\frac{u}{\left(|u|^2+\varepsilon^2\right)^{\frac{n+1}{2}}}\right) du + O\left(\varepsilon_1^\alpha\right) \\ &= O\left(\frac{\delta^2}{\varepsilon_1}\right) + O\left(\varepsilon_1^\alpha\right).\end{aligned}$$

Finally by Lemma 2.2,

$$\int_{B_{r_0}}\left[\frac{(n+1)\,\varepsilon^2}{\left(|u|^2+\varepsilon^2\right)^{\frac{n+3}{2}}}-\frac{1}{\left(|u|^2+\varepsilon^2\right)^{\frac{n+1}{2}}}\right]g_\delta\left(u\right)du \tag{4.16}$$
$$=\int_{\mathbb{R}^n}\left[\frac{(n+1)\,\varepsilon^2}{\left(|u|^2+\varepsilon^2\right)^{\frac{n+3}{2}}}-\frac{1}{\left(|u|^2+\varepsilon^2\right)^{\frac{n+1}{2}}}\right]g_\delta\left(u\right)du+O\left(\delta\right)$$
$$=\frac{4\pi^{n+1}}{(n-1)\,\Gamma\left(\frac{n-1}{2}\right)^2}+o_\varepsilon\left(1\right)+O\left(\delta\right).$$

Combining (4.13)-(4.16), (4.11) follows.

In view of (4.2)-(4.11) and Proposition 3.1 we have

$$\begin{aligned}
&\int_M K_\varepsilon\left(y\right)d\mu\left(y\right)\\
&=\int_M K_\varepsilon\left(y\right)\left(1-h_\delta\left(y\right)\right)d\mu\left(y\right)+\int_M K_\varepsilon\left(y\right)h_\delta\left(y\right)d\mu\left(y\right)\\
&=\int_M K\left(y\right)d\mu\left(y\right)+\frac{4\pi^{n+1}}{\Gamma\left(\frac{n-1}{2}\right)^2}+o_\varepsilon\left(1\right)+O\left(\delta\right)+O\left(\frac{\delta^2}{\varepsilon_1}\right)+O\left(\varepsilon_1^\alpha\right).
\end{aligned}$$

Hence

$$\begin{aligned}
&\limsup_{\varepsilon\to 0^+}\left|\int_M K_\varepsilon\left(y\right)d\mu\left(y\right)-\int_M K\left(y\right)d\mu\left(y\right)-\frac{4\pi^{n+1}}{\Gamma\left(\frac{n-1}{2}\right)^2}\right|\\
&\le c\left(M\right)\left(\delta+\frac{\delta^2}{\varepsilon_1}+\varepsilon_1^\alpha\right).
\end{aligned}$$

Let $\delta\to 0^+$, then $\varepsilon_1\to 0^+$, we get

$$\int_M K_\varepsilon\left(y\right)d\mu\left(y\right)\to\int_M K\left(y\right)d\mu\left(y\right)+\frac{4\pi^{n+1}}{\Gamma\left(\frac{n-1}{2}\right)^2}\quad\text{as }\varepsilon\to 0^+,$$

which is exactly (4.1). And Proposition 4.1 follows.

4.2 Compact hypersurface with nonempty boundary

Here we derive the same identity for the case of a surface with nonempty boundary.

Proposition 4.2 *Assume $n \geq 2$, $M^n \subset \mathbb{R}^{n+1}$ is a smooth compact hypersurface with nonempty boundary. Then for*

$$\begin{aligned}&\mathbf{p.v.}\int_M \nabla_z^M G(x,z)\cdot\left[\mathbf{p.v.}\int_M \nabla_z^M G(z,y)\,d\mu(y)\right]d\mu(z)\\&=\int_M\left[\mathbf{p.v.}\int_M \nabla_z^M G(x,z)\cdot\nabla_z^M G(z,y)\,d\mu(z)\right]d\mu(y)+\frac{4\pi^{n+1}}{\Gamma\left(\frac{n-1}{2}\right)^2}\end{aligned}$$

for $x \in M\backslash\partial M$.

Basically Proposition 4.2 follows from Proposition 4.1. Indeed we may find a smooth compact hypersurface without boundary, namely $M_0^{n+1} \subset \mathbb{R}^{n+1}$ such that M is a domain in M_0. For $x \in M\backslash\partial M$, the same argument as in the proof of Lemma 2.3 gives us

$$\begin{aligned}&\mathbf{p.v.}\int_M \nabla_z^{M_0} G(x,z)\cdot\left[\int_{M_0\backslash M} \nabla_z^{M_0} G(z,y)\,d\mu(y)\right]d\mu(z)\\&=\int_{M_0\backslash M}\left[\mathbf{p.v.}\int_M \nabla_z^{M_0} G(x,z)\cdot\nabla_z^{M_0} G(z,y)\,d\mu(z)\right]d\mu(y).\end{aligned}\tag{4.17}$$

By the Fubini theorem we have

$$\begin{aligned}&\int_{M_0\backslash M} \nabla_z^{M_0} G(x,z)\cdot\left[\int_M \nabla_z^{M_0} G(z,y)\,d\mu(y)\right]d\mu(z)\\&=\int_M\left[\int_{M_0\backslash M} \nabla_z^{M_0} G(x,z)\cdot\nabla_z^{M_0} G(z,y)\,d\mu(z)\right]d\mu(y).\end{aligned}\tag{4.18}$$

Using Lemma 2.4 we have

$$\begin{aligned}&\int_{M_0\backslash M} \nabla_z^{M_0} G(x,z)\cdot\left[\mathbf{p.v.}\int_{M_0\backslash M} \nabla_z^{M_0} G(z,y)\,d\mu(y)\right]d\mu(z)\\&=\int_{M_0\backslash M}\left[\mathbf{p.v.}\int_{M_0\backslash M} \nabla_z^{M_0} G(x,z)\cdot\nabla_z^{M_0} G(z,y)\,d\mu(z)\right]d\mu(y).\end{aligned}\tag{4.19}$$

Finally, it follows from Proposition 4.1 that

$$\begin{aligned}&\mathbf{p.v.}\int_{M_0} \nabla_z^{M_0} G(x,z)\cdot\left[\mathbf{p.v.}\int_{M_0} \nabla_z^{M_0} G(z,y)\,d\mu(y)\right]d\mu(z)\\&=\int_{M_0}\left[\mathbf{p.v.}\int_{M_0} \nabla_z^{M_0} G(x,z)\cdot\nabla_z^{M_0} G(z,y)\,d\mu(z)\right]d\mu(y)+\frac{4\pi^{n+1}}{\Gamma\left(\frac{n-1}{2}\right)^2}.\end{aligned}\tag{4.20}$$

Substracting (4.17)-(4.19) from (4.20), we obtain the identity in Proposition 4.2.

5 Proof of the Poincaré-Betrand formula

After all the preparations in Sections 2-4, we can now easily verify Theorem 1.1 and Theorem 1.2.

PROOF. (Proof of Theorem 1.2) Since H is Holder continuous, we can show that both sides of (1.5) are continuous functions in x when $x \in M \backslash \partial M$. Hence we only need to verify (1.5) for a.e. $x \in M$. We have for a.e. $x \in M$,

$$
\begin{aligned}
&\mathbf{p.v.}\int_M \nabla_z^M G(x,z)\cdot\left[\mathbf{p.v.}\int_M \nabla_z^M G(z,y)\,H(z,y)\,d\mu(y)\right]d\mu(z)\\
&=\mathbf{p.v.}\int_M \nabla_z^M G(x,z)\cdot\left[\int_M \nabla_z^M G(z,y)\,(H(z,y)-H(z,z))\,d\mu(y)\right]d\mu(z)\\
&+\int_M \nabla_z^M G(x,z)\,(H(z,z)-H(x,x))\left[\mathbf{p.v.}\int_M \nabla_z^M G(z,y)\,d\mu(y)\right]d\mu(z)\\
&+H(x,x)\,\mathbf{p.v.}\int_M \nabla_z^M G(x,z)\cdot\left[\mathbf{p.v.}\int_M \nabla_z^M G(z,y)\,d\mu(y)\right]d\mu(z)\\
&=\int_M\left[\mathbf{p.v.}\int_M \nabla_z^M G(x,z)\cdot\nabla_z^M G(z,y)\,(H(z,y)-H(z,z))\,d\mu(z)\right]d\mu(y)\\
&+\int_M\left[\mathbf{p.v.}\int_M \nabla_z^M G(x,z)\cdot\nabla_z^M G(z,y)\,(H(z,z)-H(x,x))\,d\mu(z)\right]d\mu(y)\\
&+H(x,x)\int_M\left[\mathbf{p.v.}\int_M \nabla_z^M G(x,z)\cdot\nabla_z^M G(z,y)\,d\mu(z)\right]d\mu(y)\\
&+\frac{4\pi^{n+1}}{\Gamma\left(\frac{n-1}{2}\right)^2}H(x,x)\\
&=\int_M\left[\mathbf{p.v.}\int_M \nabla_z^M G(x,z)\cdot\nabla_z^M G(z,y)\,H(z,y)\,d\mu(z)\right]d\mu(y)\\
&+\frac{4\pi^{n+1}}{\Gamma\left(\frac{n-1}{2}\right)^2}H(x,x)\,.
\end{aligned}
$$

Note that in the second step we have used Lemma 2.3, Lemma 2.4, Proposition 4.1 and Proposition 4.2.

Theorem 1.1 is basically a corollary of Theorem 1.2.

PROOF. (Proof of Theorem 1.1) Since smooth functions are dense in $L^p(M)$, we only need to verify (1.4) for $f \in C^\infty(M)$. Under this assumption, (1.4) follows from (1.5) by choosing $H(z,y) = f(y)$.

6 Conclusions

We have generalized the classical Poincaré-Bertrand formula to the case of singular integrals on smooth hypersurfaces in higher dimensions. The classical Poincaré-Bertrand formula is fairly easy to prove since one may either apply techniques from complex analysis or take advantage of the fact that the kernel is a simple rational function. Our proof of the generalized formula is rather lengthy. The main difficulties lie on the fact that we are dealing with much more complicated kernels on curved hypersurfaces instead of flat Euclidean spaces. The proof is based entirely on the local analysis and thus can be easily extended to handle other kernels with similar singularities. Specifically, we may replace the Green's function of the Laplace equation by the Green's function of other elliptic partial differential equations (for example, the Helmholtz equation). Finally, the formula is expected to be used in the construction of second kind integral equation formulations for the open surface problems in higher dimensions (≥ 3). This application is currently under investigation and will be reported in a later date.

References

[1] S. Jiang and V. Rokhlin, *Second kind integral equations for the classical potential theory on open surfaces I: analytical apparatus*, J. Comput. Phys. **191** (2003) 40-74.

[2] S. Jiang, *Jump relations of the quadruple layer potential on a regular surface in three dimensions*, Appl. Comput. Harmon. Anal. **21** (2006) 395-403.

[3] S. G. Mikhlin and S. Prossdorf, *Singular Integral Operators*, Springer, 1986.

[4] N. I. Muskhelishvili, *Singular integral equations*, Dover, New York, 1991.

[5] E. M. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton University Press, Princeton NJ, 1970.

[6] F. G. Tricomi, *Integral equations*, Dover, New York, 1985.