THE SOBOLEV INEQUALITY FOR PANEITZ OPERATOR ON THREE MANIFOLDS

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1. Introduction

The Paneitz operator introduced in [P] has attracted some attention in conformal geometry recently. In particular its associated Q-curvature equation has demonstrated its importance in four dimensional conformal geometry (cf. [CGY]). According to Branson ([B]), this operator enjoys similar conformal covariance properties in dimenion three as well. However there are a number of features of this equation that are quite distinct from its analogues in dimensions four and above. In this dimension, the positivity of the Paneitz operator is an important issue. In a previous paper [XY1], the analogue of the Yamabe equation is verified for the conformal classes for which both the conformal Laplacian operator and the Paneitz operator are positive. However the conformal classes of interests namely those that are near the standard 3-sphere fail to be in this class. The main purpose of this article is to study the positivity issue for the family of conformal structures provided by the Berger metrics.

Let (M,g) be a three dimensional smooth compact Riemannian manifold. The Q curvature is defined by

$$Q=-\frac{1}{4}\Delta R-2\left|Rc\right|^{2}+\frac{23}{32}R^{2}. \tag{1.1}$$

The Paneitz operator is defined by

$$P\varphi=\Delta^{2}\varphi+4\operatorname{div}\left(Rc\left(\nabla\varphi,e_{i}\right)e_{i}\right)-\frac{5}{4}\operatorname{div}\left(R\nabla\varphi\right)-\frac{1}{2}Q\varphi \tag{1.2}$$

for any $\varphi\in C^{\infty}\left(M,\mathbb{R}\right)$. Here Rc denotes the Ricci tensor, e_1,e_2,e_3 is any local orthonormal frame. This operator satisfies the conformal covariant property. That is, for any $\rho\in C^{\infty}\left(M,\mathbb{R}\right),\rho>0$, and $\varphi\in C^{\infty}\left(M,\mathbb{R}\right)$,

$$P_{\rho^{-4}g}\varphi=\rho^{7}P_{g}\left(\rho\varphi\right). \tag{1.3}$$

In particular, if we let $\widetilde{g}=u^{-4}g$ for some $u\in C^{\infty}\left(M\right),u>0$, then $\widetilde{Q}=-2u^{7}P_{g}u$. The analogue of the Yamabe problem is to find a conformal metric with constant Q-curvature. The equation may be written as

$$\begin{cases} P_{g}u=\text{const}\cdot u^{-7}, & \text{on } M,\\ u\in C^{\infty}\left(M\right),u>0. \end{cases} \tag{1.4}$$

(1.4) has a variational structure. To describe it, let us introduce some notations. For any $u,v\in H^{2}\left(M\right)$, denote

$$E\left(u,v\right)=\int_{M}\left(\Delta u\Delta v-4Rc\left(\nabla u,\nabla v\right)+\frac{5}{4}R\nabla u\cdot\nabla v-\frac{1}{2}Quv\right)d\mu, \tag{1.5}$$

here μ is the measure corresponding to g. For convenience, we also write $E(u) = E(u,u)$ for $u \in H^2(M)$. It is easy to see that for any $u, v \in C^\infty(M)$, $E(u,v) = \int_M Pu \cdot v d\mu = \int_M uPvd\mu$. Let

$$V = \left\{ u \in H^2(M) : u > 0 \right\}. \tag{1.6}$$

Since the Sobolev embedding theorem implies that $H^2(M) \subset C^{1/2}(M)$, it is meaningful to say $u > 0$ pointwisely. For any $u \in V$, let

$$I(u) = I_4^3(u) = \frac{E(u)}{|u|^2_{L^{-6}(M)}} = E(u) \left|u^{-1}\right|^2_{L^6(M)}.$$

It is clear that (1.4) is the Euler-Lagrange for I. A natural way to find critical point is to minimize the functional I. Define

$$Y_4^3(M,g) = \inf_{u \in V} I_4^3(u). \tag{1.7}$$

It follows easily from (1.3) that $Y_4^3(M,g)$ is a conformal invariant quantity, that is for any $\rho \in C^\infty(M), \rho > 0$, $Y_4^3\left(M, \rho^{-4} g\right) = Y_4^3(M,g)$. Unlike the Yamabe constant, it is no longer trivial whether we have $Y_4^3(M,g) > -\infty$. In contrast to the usual Sobolev inequality, which is derived from the corresponding inequality in Euclidean space by a partition of unity argument; such a process cannot proceed due to the negative exponent in this inequality.

In dealing with this question, we find it convenient to introduce the following

Definition 1.1. *If for any $u \in H^2(M), u \geq 0$ and u vanishes somewhere, we have $E(u) \geq 0$, then we say (M,g) satisfies the condition (NN^+). If for any $u \in H^2(M) \setminus \{0\}, u \geq 0$ and u vanishes somewhere, we have $E(u) > 0$, then we say (M,g) satisfies the condition (P^+).*

Simple consideration shows that if $Y_4^3(M,g) > -\infty$, then the metric must satisfy the condition (NN^+) (see Remark 3.1). It follows easily from (1.3) that the condition (NN^+) (or the condition (P^+)) is conformal invariant. In general, condition (NN^+) is difficult to verify and also difficult to use. We shall introduce in Section 5 the stronger condition (NN) (and condition (P)), which will be important in identifying certain blowup limits as Green's function when the $\ker P_g = \{0\}$. It also reveals the crucial role of the value of the Green's function at pole. Now we may state the following result (see Section 3).

Theorem 1.1. *Assume a three dimensional smooth compact Riemannian manifold (M,g) satisfies the condition (P^+), then there exists a $u \in V$ such that $I_4^3(u) = Y_4^3(M,g)$, in particular $Y_4^3(M,g) > -\infty$.*

We remark that the condition (P^+) is clearly satisfied when $P_g > 0$. This case was studied earlier in [XY2]. In fact for this case, modulo a positive constant factor, the critical function is unique (cf. Corollary 6.2). In Section 6, we will give some curvature conditions under which the Paneitz operator is positive definite. But the standard sphere does not satisfy the condition (P^+), since the value of the Green's function at the pole is always 0 (see Section 7). Nevertheless, using symmetrization methods, [YZ] is able to prove the Sobolev inequality in this very singular case. Here we give a proof without using the symmetrization argument (see Section 7).

Theorem 1.2 ([YZ])**.** *Let S^3 be equipped with the standard metric, then there exists a $u \in V$, such that $I_4^3(u) = Y_4^3\left(S^3, g_{S^3}\right)$.*

It then follows from a theorem of [CX], that $u = cv$, for some constant $c > 0$ and there exists a conformal transformation σ of S^3 such that $\sigma^* g_{S^3} = v^{-4} g_{S^3}$. In particular, this will show that $Y_4^3\left(S^3, g_{S^3}\right) = I_4^3(1) = -\frac{15}{16} \cdot 2^{4/3}\pi^{8/3}$ by a direct computation.

In order to understand what happens near the standard 3-sphere, we examine the family of Berger spheres. The Lie group

$$SU(2) = \left\{ \begin{pmatrix} a & -\overline{b} \\ b & \overline{a} \end{pmatrix} : a, b \in \mathbb{C}, |a|^2 + |b|^2 = 1 \right\},$$

is naturally identified with S^3. Its Lie algebra is

$$\mathfrak{su}(2) = \left\{ \begin{pmatrix} it_1 & it_2 - t_3 \\ it_2 + t_3 & -it_1 \end{pmatrix} : t_1, t_2, t_3 \in \mathbb{R} \right\}.$$

Choose the following basis elements for $\mathfrak{su}(2)$,

$$X_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad X_3 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

For $t > 0$, we may define an inner product on $\mathfrak{su}(2)$ such that $t^{-1}X_1, X_2, X_3$ is an orthonormal base. Then the Berger sphere has the metric g_t which is the left invariant metric extending the above inner product. Let P_t be the Paneitz operator of g_t, then we have (see Section 8)

Theorem 1.3. *For* $t \in \left(0, \frac{-2+4\sqrt{10}}{13}\right) \cup \left(\frac{2+4\sqrt{10}}{13}, \infty\right)$, $P_t > 0$, *and we know*

$$Y_4^3\left(S^3, g_t\right) = I_4^3(1) = -\frac{169}{4} 2^{1/3}\pi^{8/3} t^{16/3} + 82 \cdot 2^{1/3}\pi^{8/3} t^{10/3} - 36 \cdot 2^{1/3}\pi^{8/3} t^{4/3}.$$

In addition, the constant multiple of g_t *is the only metric having constant* Q *curvature in the conformal class.*

For $t = \frac{-2+4\sqrt{10}}{13}$ *or* $\frac{2+4\sqrt{10}}{13}$, $P_t \geq 0$, *and the kernel of* P_t *consists of constant functions. Again, the constant multiple of* g_t *are the unique constant* Q *curvature metrics in the conformal class.*

For $t \in \left(\frac{-2+4\sqrt{10}}{13}, 1\right) \cup \left(1, \frac{2+4\sqrt{10}}{13}\right)$, *the first eigenvalue is the only negative eigenvalue of* P_t, *the corresponding eigenfunctions are constant functions. In addition,* P_t *satisfies the condition* (P), *and the Greens's function of* P_t *are strictly negative, we also know* $Y_4^3\left(S^3, g_t\right) > -\infty$ *and it is achieved by some metrics in the conformal class.*

The proof will be based on a careful analysis for the value of the Green's function at the pole and a continuity argument. For $t \in \left(\frac{-2+4\sqrt{10}}{13}, 1\right) \cup \left(1, \frac{2+4\sqrt{10}}{13}\right)$, we do not know whether $Y_4^3\left(S^3, g_t\right) = I_4^3(1, g_t)$. One may expect this to be true and positive constants are the only minimizers.

The plan of the paper is as follows. In Section 2, we will prove some elementary lemmas which will be used frequently later. In Section 3, we shall apply the direct method of calculus of variations to the minimizing problem (1.7) and prove Theorem 1.1. In Section 4, we shall discuss the basic properties of Green's function, in particular, we shall derive an equation for the evolution of the Green function pole's value. In Section 5, we propose the important condition (NN) and show the important role of Green function pole's value under this condition. In Section 6, we shall first show certain pointwise curvature condition would guarantee the positivity of the Paneitz operator, then we will discuss the existence and uniqueness of the minimizing problem (1.7) when the Paneitz operator is nonnegative. In Section 7, we will give another proof of Theorem 1.2 from [YZ]. In Section 8, we shall prove that for Berger spheres, the minimizing problem (1.7) always has a solution. The main effort is to show the value of the Green's function at the pole does not vanish when it is not the standard sphere. Then a continuity argument gives us Theorem 1.3.

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2. Some calculus lemmas

Lemma 2.1. *If* $u \in H^2\left(B_1^3\right), \left|u^{-1}\right|_{L^6(B_1)} \leq 1, |u|_{H^2\left(B_1^3\right)} \leq A$, *then* $|u(0)| \geq Ae^{-cA^6}$, *here* c *is an absolute constant.*

Proof. Using Sobolev embedding theorem we see $|u|_{C^{1/2}\left(\overline{B_1}\right)} \leq cA$, and hence for any $x \in B_1$, $|u(x)| \leq |u(0)| + cA|x|^{1/2}$. This implies $|u(x)|^6 \leq c\left(|u(0)|^6 + A^6|x|^3\right)$. Then we may compute

$$1 \geq \int_{B_1} |u(x)|^{-6}\,dx \geq c\int_{B_1}\left(|u(0)|^6 + A^6|x|^3\right)^{-1} dx = \frac{c}{A^6}\log\frac{|u(0)|^6 + A^6}{|u(0)|^6}.$$

This shows $|u(0)| \geq Ae^{-cA^6}$. ∎

Corollary 2.1. *If* $u \in H^2(M)$ *such that* $\left|u^{-1}\right|_{L^6(M)} \leq 1$ *and* $|u|_{H^2(M)} \leq A$, *then we have* $|u(p)| \geq c(g)\,Ae^{-c(g)A^6}$ *for any* $p \in M$.

We also need the following approximation lemma.

Lemma 2.2. *Let* $u \in H^2\left(B_1^3\right)$, $u(0) = 0$, $\eta \in C^\infty\left(\mathbb{R}^3\right)$, $\left.\eta\right|_{B_1} = 1$, $\left.\eta\right|_{\mathbb{R}^3\setminus B_2} = 0$, $0 \leq \eta \leq 1$, $\eta_\varepsilon(x) = \eta\left(\frac{x}{\varepsilon}\right)$. *Then* $\eta_\varepsilon u \to 0$ *in* $H^2\left(B_1^3\right)$ *as* $\varepsilon \to 0^+$.

Proof. By the fact that for any $u \in H^2\left(B_1^3\right)$, any $\alpha > 0$, $|u|_{H^1(B_1)} \leq \alpha\left|D^2u\right|_{L^2(B_1)} + c(\alpha)|u|_{L^2(B_1)}$, we see we only need to prove $|\eta_\varepsilon u|_{L^2(B_1)}, \left|D^2(\eta_\varepsilon u)\right|_{L^2(B_1)} \to 0$ as $\varepsilon \to 0^+$. Since

$$\partial_{ij}(\eta_\varepsilon u) = \frac{1}{\varepsilon^2}\partial_{ij}\eta\left(\frac{x}{\varepsilon}\right)u(x) + \frac{1}{\varepsilon}\partial_i\eta\left(\frac{x}{\varepsilon}\right)\partial_j u(x) + \frac{1}{\varepsilon}\partial_j\eta\left(\frac{x}{\varepsilon}\right)\partial_i u(x) + \eta\left(\frac{x}{\varepsilon}\right)\partial_{ij}u(x),$$

and

$$\begin{aligned}
\left|\frac{1}{\varepsilon^2}\partial_{ij}\eta\left(\frac{x}{\varepsilon}\right)u(x)\right|_{L^2(B_1)} &\leq \frac{c}{\varepsilon^2}|u|_{L^2(B_{2\varepsilon})} \leq c[u]_{C^{\frac{1}{2}}(B_{2\varepsilon})} \leq c|\nabla u|_{L^6(B_{2\varepsilon})} \to 0,\\
\left|\frac{1}{\varepsilon}\partial_i\eta\left(\frac{x}{\varepsilon}\right)\partial_j u(x)\right|_{L^2(B_1)} &\leq \frac{c}{\varepsilon}|\partial_j u|_{L^2(B_{2\varepsilon})} \leq c|\partial_j u|_{L^6(B_{2\varepsilon})} \to 0,
\end{aligned}$$

the lemma follows. ∎

It follows easily from Lemma 2.2 that (M, g) satisfies the condition (NN^+) if and only if for any $u \in C^\infty(M), u \geq 0$ and u vanishes on a nonempty open subset implies $E(u) \geq 0$.

The following interpolation inequality follows from the usual interpolation inequality and a covering argument, a proof may be found in [DHL].

Lemma 2.3. *For $\varepsilon > 0$ and $u \in H^2(M)$,*

$$|u|_{H^1(M)} \leq \varepsilon \left|D^2 u\right|_{L^2(M)} + c(g,\varepsilon)|u|_{L^2(M)}.$$

By elliptic estimates (cf. [GT], chapter 8) and a covering argument, we have

Lemma 2.4. *For any $u \in H^2(M)$, $\int_M \left((\Delta u)^2 + u^2\right) d\mu \geq c(g)|u|^2_{H^2(M)}$, here $c(g) > 0$.*

As a simple corollary of Lemma 2.3 and 2.4, we have

Corollary 2.2. *For $u \in H^2(M)$, $E(u) + c_1(g)|u|^2_{L^2(M)} \geq c(g)|u|^2_{H^2(M)}$, here $c(g) > 0$.*

3. A preliminary discussion

We shall apply the direct method of calculus of variations to the minimizing problem (1.7). Choose a sequence $u_i \in V$ such that $I_4^3(u_i) \to Y_4^3(M,g)$. By scaling, we may assume $\max_M u_i = 1$. Then we know $\left|u_i^{-1}\right|_{L^6(M)} \geq (\mu(M))^{1/6}$ and $\left|u_i^{-1}\right|^2_{L^6(M)} E(u_i) \leq c_1$ for some constant c_1 independent of i. In view of Corollary 2.2, we have $c_1 \geq E(u_i) \geq c(g)|u_i|^2_{H^2(M)} - c(g)|u_i|^2_{L^2(M)}$, and hence $|u_i|_{H^2(M)} \leq c_1$ independent of i. After passing to a subsequence, we have $u_i \rightharpoonup u$ in $H^2(M)$. It follows from Sobolev's embedding theorem that $u_i \to u$ uniformly on M. Hence $u \geq 0$ on M and $\max_M u = 1$. Now we proceed into two separate cases.

Case 3.1. *$u > 0$ on M.*

In this case, we know $u_i^{-1} \to u^{-1}$ uniformly on M, and hence $\left|u_i^{-1}\right|_{L^6(M)} \to \left|u^{-1}\right|_{L^6(M)}$. Since $E(u) \leq \liminf_{i\to\infty} E(u_i)$, we have

$$I_4^3(u) = \left|u^{-1}\right|^2_{L^6(M)} E(u) \leq \liminf_{i\to\infty} \left|u_i^{-1}\right|^2_{L^6(M)} E(u_i) = Y_4^3(M,g).$$

This shows $I_4^3(u) = Y_4^3(M,g)$. Hence $u \in C^\infty(M)$.

Case 3.2. *u vanishes somewhere.*

In this case it follows from Corollary 2.1 that $\left|u^{-1}\right|_{L^6(M)} = \infty$. Using Fatou's lemma we have $\left|u^{-1}\right|_{L^6(M)} \leq \liminf_{i\to\infty} \left|u_i^{-1}\right|_{L^6(M)}$. Hence $\left|u_i^{-1}\right|_{L^6(M)} \to \infty$ as $i \to \infty$. This shows $\limsup_{i\to\infty} E(u_i) \leq 0$. Hence $E(u) \leq 0$.

Remark 3.1. *If we have some $u \in H^2(M), u \geq 0, E(u) < 0$ and u vanishes somewhere, then for $\varepsilon > 0$, we have $u + \varepsilon \in V$ and $\left|(u+\varepsilon)^{-1}\right|_{L^6(M)} \to \infty$, $E(u+\varepsilon) \to E(u)$ as $\varepsilon \to 0^+$. This implies $I_4^3(u+\varepsilon) = \left|(u+\varepsilon)^{-1}\right|^2_{L^6(M)} E(u+\varepsilon) \to -\infty$ and hence $Y_4^3(M,g) = -\infty$. This shows that a necessary condition for $Y_4^3(M,g)$ to be finite is that (M,g) satisfies the condition (NN^+).*

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. Following the above discussions, we only need to show the Case 3.2 does not happen. If it does happen, then we may find a $u \in H^2(M)$ with $u \geq 0, \max_M u = 1$ and $E(u) \leq 0$. This contradicts with the condition (P^+). ∎

4. The Green's function

For basics about the Green's function of elliptic operators, one may refer to [G]. Assume $\ker P_g = \{0\}$ and $\rho \in C^\infty(M), \rho > 0$, then we have $\ker P_{\rho^{-4}g} = \{0\}$ too. In fact, if $P_{\rho^{-4}g} u = 0$, then it follows from (1.3) that $\rho^7 P_g(\rho u) = 0$, and hence $\rho u = 0$. This implies $u = 0$. The claim follows.

Let G_p be the Green's function of P_g at p, then the Green's function for $\widetilde{g} = \rho^{-4} g$ at p is

$$\widetilde{G}_p = \rho(p)^{-1} \rho^{-1} G_p. \tag{4.1}$$

In fact, for any $u \in C^\infty(M)$,

$$\rho(p) u(p) = \int_M G_p P_g(\rho u) d\mu_g = \int_M G_p \rho^{-7} P_{\rho^{-4}g} u d\mu_g = \int_M \rho^{-1} G_p P_{\rho^{-4}g} u d\mu_{\rho^{-4}g},$$

hence $u(p) = \int_M \rho(p)^{-1} \rho^{-1} G_p P_{\rho^{-4}g} u d\mu_{\rho^{-4}g}$ and (4.1) follows. Note that it follows from (4.1) that the positivity of Green's function is a property of the conformal class.

Lemma 4.1. *Assume* $\ker P_g = \{0\}$. *Then for any* $p \in M$, *the Green's function* G_p *exists and* $G_p \in H^2(M)$. *In addition, if for some* $p \in M$, *the exponential map* $\exp_p : M_p \to M$ *is locally volume preserving near* $0 \in M_p$ *(for example relative to the conformal normal coordinate at* p, *cf.* [LP]*), then on a neighborhood of* p, *we have* $G_p = A - \frac{r}{8\pi} + O_4(r)$. *Here* A *is a constant,* $r(\cdot) = d_g(p, \cdot)$ *and* $O_4(r)$ *means a* C^4 *function on a neighborhood of* p *which lies in* $O(r)$.

Proof. Since $H^2(M) \subset C^{1/2}(M)$, we see easily that the Dirac function at p, $\delta_p \in H^{-2}(M)$, hence it follows from standard elliptic theory (the Hilbert space theory) that $G_p \in H^2(M)$.

Now assume the map $\exp_p$ preserves the volume near $0 \in M_p$, then choose an orthonormal base for M_p, namely e_1, e_2, e_3. Let e^1, e^2, e^3 be the dual base. Define $x^j = e^j \circ \exp_p^{-1}$ near p. Then x^1, x^2, x^3 forms a conformal normal coordinate at p. If we write $g = g_{ij} dx^i \otimes dx^j$, then we know (see [LP])

$$\begin{aligned} G &= \det(g_{ij})_{1 \le i,j \le 3} = 1, \quad g_{ij}(p) = \delta_{ij}, \quad \partial_k g_{ij}(p) = 0, \quad R(p) = 0, \\ \partial_i R(p) &= 0, \quad \Delta R(p) = 0, \quad Rc_{ij}(p) = 0, \quad Rc_{ijk}(p) y^i y^j y^k = 0 \text{ for } y \in \mathbb{R}^3, \end{aligned}$$

and $r = |x|$. By these identities, one may deduce that

$$4 \operatorname{div}(Rc(\nabla r, e_i) e_i) - \frac{5}{4} \operatorname{div}(R \nabla r) - \frac{1}{2} Q r \in \operatorname{Lip}(U(p))$$

for some small neighborhood $U(p)$ of p. This implies $P_g r = -8\pi \delta_p + \operatorname{Lip}(U(p))$. A simple cutoff argument plus the standard elliptic theory tells us $G_p = -\frac{r}{8\pi} + C^4(U(p))$. The lemma follows. ∎

Corollary 4.1. *Assume* $\ker P_g = 0$. *Then for any* $p \in M$, $u \in H^2(M)$, $E(G_p, u) = u(p)$. *In particular, this implies that for any* $p, q \in M$, $G_p(q) = E(G_p, G_q) = G_q(p)$.

Proof. If $u \in C^\infty(M)$, then it follows from the definition of G_p that $E(G_p, u) = \int_M G_p P u d\mu = u(p)$. Since $C^\infty(M)$ is dense in $H^2(M)$, the above corollary follows from an approximation argument. ∎

In the future, we shall see that the value of the Green's function at the pole is very important. The following lemma will be useful.

Lemma 4.2. *Let M^3 be a smooth compact manifold and $g(t)$ be a smooth family of metrics on M for t near 0. Denote $\partial_t|_{t=0}\, g = h$. If $\ker P_{g(0)} = 0$, then for any $p, q \in M$, we have*

$$\begin{aligned}\partial_t|_{t=0}\, G_p(q,t) &= -\int_M G_p(\cdot,0)\left(\partial_t|_{t=0}\, P\right) G_q(\cdot,0)\, d\mu - \frac{1}{2} G_p(q,0) \operatorname{tr} h(q) \\ &= -\,\partial_t|_{t=0}\, E\left(G_p(\cdot,0), G_q(\cdot,0)\right).\end{aligned}$$

Proof. For any $u \in C^\infty(M)$, $u(p) = \int_M G_p(x,t) \cdot Pu(x)\, d\mu(x,t)$. By differentiation with respect to t, we have

$$\begin{aligned}0 &= \int_M \partial_t|_{t=0}\, G_p(x,t) \cdot Pu(x)\, d\mu(x,0) + \int_M G_p(x,0) \cdot \left(\partial_t|_{t=0}\, P\right) u(x)\, d\mu(x,0) \\ &\quad + \frac{1}{2} \int_M G_p(x,0) \cdot Pu(x) \cdot \operatorname{tr} h(x)\, d\mu(x,0).\end{aligned}$$

By approximation, we know the above equation is still true for $u \in H^2(M)$. To get the lemma, we only need to let $u = G_q(\cdot,0)$. ∎

5. The condition (NN)

In this section, we formulate some nonnegativity conditions which will be relevant to determine the limit function in Case 3.2 of Section 3. At the center of attention is the value of the Green's function at poles in case $\ker P_g = 0$.

Definition 5.1. *We say (M,g) satisfies the condition (NN) if for any $u \in H^2(M)$ and u vanishes somewhere, we have $E(u) \geq 0$. We say (M,g) satisfies the condition (P) if for any $u \in H^2(M) \setminus \{0\}$ and u vanishes somewhere, we have $E(u) > 0$.*

It follows from Lemma 2.2 that (M,g) satisfies the condition (NN) if and only if for any $u \in C^\infty(M)$, u vanishes on an nonempty open subset implies $E(u) \geq 0$. By the conformal covariant property of the Paneitz operator (1.3), we see that the condition (NN) (or the condition (P)) depends only on the conformal class. We will see in Section 7 that S^3 with the standard metric, satisfies the condition (NN). The following example shows that the condition (NN) is not always satisfied.

Example 5.1. *Let Σ be a 2 dimensional smooth compact Riemannian manifold with Gauss curvature $K \equiv -1$, this can happen when $\chi(\Sigma) < 0$. Then the Paneitz operator on the product manifold $S^1 \times \Sigma$ is $P\varphi = \Delta^2\varphi - 4\Delta_\Sigma\varphi + \frac{5}{2}\Delta\varphi + \frac{9}{16}\varphi$. Let θ be the angle function on S^1, then $P(\sin\theta) = -\frac{15}{16}\sin\theta$. This implies $\int_{S^1\times\Sigma} P(\sin\theta) \cdot \sin\theta d\mu = -\frac{15}{16}\pi \operatorname{Area}(\Sigma) < 0$. Note that $\{\sin\theta = 0\} = \{(1,0),(-1,0)\} \times \Sigma$ is nonempty. Hence $S^1 \times \Sigma$ does not satisfy the condition (NN).*

Since $P1 = \frac{9}{16}$, $P(\sin\theta) = -\frac{15}{16}\sin\theta$, and $P(\cos\theta) = -\frac{15}{16}\cos\theta$, we see that the first eigenfunction must change sign (because it has to be orthogonal to 1). This is very different from second order elliptic operators, for which the first eigenfunction never changes sign.

Lemma 5.1. *If (M,g) satisfies the condition (P), then we may find a constant $c(g) > 0$ such that for any $u \in H^2(M)$, which vanishes somewhere, we have $E(u) \geq c(g) |u|^2_{H^2(M)}$.*

Proof. We first prove that there exists a positive constant $c(g)$ such that for any $u \in H^2(M)$,

$$u \text{ vanishes somewhere} \Rightarrow E(u) \geq c(g) |u|^2_{L^2(M)}. \tag{5.1}$$

In fact, if this is not the case, then we may find a sequence $u_j \in H^2(M)$ such that u_j vanishes somewhere and $E(u_j) < j^{-1} |u_j|^2_{L^2(M)}$. By scaling, we may assume $|u_j|_{L^2(M)} = 1$. Then it follows from Corollary 2.2 that $c(g) |u_j|^2_{H^2(M)} - c(g) |u_j|^2_{L^2(M)} \le j^{-1}$. This implies $\sup_j |u_j|_{H^2(M)} < \infty$. Hence we may find a subsequence $u_{j'}$ and a $u \in H^2(M)$ such that $u_{j'} \rightharpoonup u$ in $H^2(M)$. This implies $u_{j'} \to u$ in $C(M, \mathbb{R})$. Hence u vanishes somewhere and $|u|_{L^2(M)} = 1$. On the other hand, it follows from lower semicontinuity that $E(u) \le 0$. This shows $u = 0$, by the condition (P). A contradiction.

Now for any $u \in H^2(M)$, which vanishes somewhere, by Corollary 2.2 and (5.1) we have, $c(g) |u|^2_{H^2(M)} \le E(u) + c(g) |u|^2_{L^2(M)} \le c(g) E(u)$, and the lemma follows. ∎

In particular, it follows from Lemma 5.1 that the condition (P) is preserved under a small smooth perturbation of the metric. Similarly, we have

Lemma 5.2. *If (M, g) satisfies the condition (P^+), then we may find a constant $c(g) > 0$ such that for any $u \in H^2(M)$, which is nonnegative and vanishes somewhere, we have $E(u) \ge c(g) |u|^2_{H^2(M)}$.*

An important consequence of the condition (NN) is that we may identify the limit function in the Case 3.2 of Section 3. Indeed, we have the following simple lemma.

Lemma 5.3. *If (M, g) satisfies the condition (NN), $u \in H^2(M)$, $E(u) = 0$, and $u(p) = 0$ for some $p \in M$, then u is smooth on $M \backslash \{p\}$, in addition, $P_g u = \text{const} \cdot \delta_p$.*

Proof. For any $\varphi \in C^\infty(M)$ with $\varphi(p) = 0$, clearly $E(u + t\varphi)$ takes a minimum at $t = 0$. Hence $E(u, \varphi) = 0$.

Fix a $\varphi_0 \in C^\infty(M)$ such that $\varphi_0(p) = 1$. Then for any $\varphi \in C^\infty(M)$, it follows from $E(u, \varphi - \varphi(p)\varphi_0) = 0$ that $E(u, \varphi) = E(u, \varphi_0)\varphi(p)$. Therefore we conclude $P_g u = E(u, \varphi_0)\delta_p$. ∎

As a remark, we see if M satisfies the condition (NN), $u \in H^2(M)$, $p_1, p_2 \in M$, $p_1 \ne p_2$, $u(p_1) = u(p_2) = 0$ and $E(u) = 0$, then $u \in C^\infty(M)$ and $P_g u = 0$.

Lemma 5.4. *Assume M satisfies the condition (NN) and $\ker P = \{0\}$. If for some $p \in M$, $G_p(p) = 0$, then $G_p|_{M \backslash \{p\}} < 0$.*

Proof. We first note that $G_p|_{M \backslash \{p\}}$ can not vanish anywhere. Hence either it is always positive or it is always negative. Now observe that under conformal normal coordinate at p, we have (after conformally changing the metric, but we may still denote it as the original one in view of (4.1)) $G_p = -\frac{r}{8\pi} + O_4(r)$ near p, hence $\frac{1}{4\pi\delta^3} \int_{\partial B_\delta(p)} G_p d\mu_g \to -\frac{1}{8\pi}$ as $\delta \to 0^+$. This implies $G_p|_{M \backslash \{p\}} < 0$. ∎

Corollary 5.1. *Assume (M, g) satisfies the condition (NN) and $\ker P = \{0\}$. Then g satisfies the condition (P^+) if and only if g satisfies the condition (P), it is also equivalent to the condition that for any $p \in M$, $G_p(p) \ne 0$.*

Proof. If g satisfies the condition (P^+), then for any $p \in M$, we must have $G_p(p) \ne 0$. Otherwise, suppose for some $p \in M$, $G_p(p) = 0$, then it follows from Lemma 5.4

that $G_p \le 0$. Hence $E(G_p) = G_p(p) > 0$ by the condition (P^+). This gives us a contradiction.

On the other hand, assume for any $p \in M$, $G_p(p) \neq 0$. If for some $u \in H^2(M)$, we have $u(p) = 0$ for some $p \in M$ and $E(u) = 0$, then it follows from Lemma 5.3 that for some constant $c \in \mathbb{R}$, we have $Pu = c\delta_p$. If $c \neq 0$, then $u = cG_p$ and this shows $u(p) = cG_p(p) \neq 0$, a contradiction. Hence $c = 0$. This shows $u = 0$ because $\ker P = \{0\}$. It follows that g satisfies the condition (P) and in particular (P^+). ∎

Finally, we state the following simple observation about the condition (NN).

Lemma 5.5. *Assume (M, g) satisfies the condition (NN) and P has some negative eigenvalues, then P has exactly one negative eigenvalue, and the corresponding eigenfunction can not change sign, in addition,* $\ker P = \{0\}$.

Proof. Let $(\lambda_j)_{j=1}^{\infty}$ be the eigenvalues of P (counting the multiplicity), and $(\phi_j)_{j=1}^{\infty}$ be the corresponding eigenfunctions with $|\phi_j|_{L^2(M)} = 1$. Then it follows from the assumption that $\lambda_1 < 0$. Since $E(\phi_1) = \lambda_1 < 0$, we see ϕ_1 can not touch 0 anywhere. Without losing of generality, we may assume $\phi_1 > 0$. Since $\int_M \phi_1 \phi_2 d\mu = 0$, we see ϕ_2 must change sign. This shows for $\varepsilon > 0$ small enough, $\varepsilon\phi_1 + \phi_2$ must change sign too. Hence $E(\varepsilon\phi_1 + \phi_2) = \varepsilon^2\lambda_1 + \lambda_2 \ge 0$, and this implies that $\lambda_2 > 0$. The lemma follows. ∎

6. The case $P \ge 0$

In this section we shall study the specific case when the Paneitz operator is nonnegative. At first, let us show under certain pointwise curvature condition, the Paneitz operator is positive definite. The proof will be based on certain Bochner type technique. One should compare with [XY1].

Proposition 6.1. *Let $A = Rc - \frac{R}{4}g$ be the Schouten tensor. Assume the Yamabe constant $Y(M, g) > 0, \sigma_2(A) > 0$ and $Q \le 0$.*

1. *If Q is not identically zero, then $P > 0$.*
2. *If $Q \equiv 0$, then $P \ge 0$, in addition,* $\ker P = \{\text{const}\}$.

We need the following elementary algebraic lemma.

Lemma 6.1. *Let B be a 3×3 symmetric real matrix. Assume* $\operatorname{tr} B > 0$ *and*

$$\sigma_2\left(B - \frac{\operatorname{tr} B}{4} \cdot I\right) = \frac{1}{48}(\operatorname{tr} B)^2 - \frac{1}{2}\left|B - \frac{\operatorname{tr} B}{3} I\right|^2 > 0,$$

here I denotes the identity matrix, then we have $\frac{\operatorname{tr} B}{6} \cdot I < B < \frac{\operatorname{tr} B}{2} \cdot I$. In particular, this implies $b_{11} + b_{22} - b_{33} = \operatorname{tr} B - 2b_{33} > 0$.

Proof. Without losing of generality, we may assume $B = \operatorname{diag}(b_1, b_2, b_3)$, then

$$\frac{(\operatorname{tr} B)^2}{24} > \left|B - \frac{\operatorname{tr} B}{3} I\right|^2 \ge \frac{1}{2}\left(b_1 + b_2 - \frac{2\operatorname{tr} B}{3}\right)^2 + \left(b_3 - \frac{\operatorname{tr} B}{3}\right)^2 = \frac{3}{2}\left(b_3 - \frac{\operatorname{tr} B}{3}\right)^2,$$

hence $\left|b_3 - \frac{\operatorname{tr} B}{3}\right| < \frac{\operatorname{tr} B}{6}$. Similar inequalities are also true for b_1 and b_2. The lemma follows. ∎

Proof of Proposition 6.1. For any $u\in C^{\infty}(M)$, a simple Bochner technique shows

$$\int_M (\Delta u)^2 d\mu=\int_M \left|D^2u\right|^2 d\mu+\int_M Rc(\nabla u,\nabla u)\,d\mu.$$

Since $\left|D^2u\right|^2=\left|D^2u-\frac{\Delta u}{3}g\right|^2+\frac{(\Delta u)^2}{3}$, we see

$$\int_M (\Delta u)^2 d\mu=\frac{3}{2}\int_M \left|D^2u-\frac{\Delta u}{3}g\right|^2 d\mu+\frac{3}{2}\int_M Rc(\nabla u,\nabla u)\,d\mu. \tag{6.1}$$

A simple approximation procedure shows (6.1) is still true for $u\in H^2(M)$. It follows from this and (1.5) that

$$\begin{aligned} E(u) &= \frac{3}{2}\int_M \left|D^2u-\frac{\Delta u}{3}g\right|^2 d\mu+\frac{5}{2}\int_M \left(\frac{R}{2}g-Rc\right)(\nabla u,\nabla u)\,d\mu \\ &\quad -\frac{1}{2}\int_M Qu^2 d\mu. \end{aligned} \tag{6.2}$$

Note that $\sigma_2(A)=\frac{1}{48}R^2-\frac{1}{2}|E|^2$, where $E=Rc-\frac{R}{3}g$ is the traceless Ricci tensor. Since $\sigma_2(A)>0$, we see $R^2>0$. This shows $R>0$ or $R<0$ on M. Using $Y(M,g)>0$, we see $R>0$. It follows from Lemma 6.1 that $\frac{R}{6}g<Rc<\frac{R}{2}g$. Hence using (6.2), we see $Q\le 0$ would imply $P\ge 0$. In addition, if $E(u)=0$, then we have $\int_M\left(\frac{R}{2}g-Rc\right)(\nabla u,\nabla u)\,d\mu=0$ and $\int_M Qu^2d\mu=0$. The first equation implies $u\equiv$ const. When Q is not identically zero, it follows from the second equation that this constant must be zero. ∎

Remark 6.1. *Examples of metrics satisfying the conditions in Proposition 6.1 can be found in Berger spheres, see Remark 8.1.*

Now we proceed to general metrics with $P\ge 0$.

Lemma 6.2. *Assume $P_g\ge 0$. Then the following three statements are equivalent.*
1. *There exists a $u\in C^{\infty}(M), u>0$ such that $I_4^3(u)=Y_4^3(M,g)$.*
2. *There exists a $u\in C^{\infty}(M), u>0$ such that $Q_{u^{-4}g}\equiv$ const.*
3. *Either there exists a $u\in\ker P_g\backslash\{0\}$ such that $u>0$ on M, or for any $u\in\ker P_g\backslash\{0\}$, u changes sign.*

Proof. We prove it in three steps.

(1)⇒(2) : It follows from the Euler-Lagrange equation.

(2)⇒(3) : Let $\widetilde{g}=u^{-4}g$, then $Q_{\widetilde{g}}\equiv$ const, $P_{\widetilde{g}}\ge 0$ and $\ker P_{\widetilde{g}}=\left\{u^{-1}v: v\in\ker P_g\right\}$.

If $Q_{\widetilde{g}}=0$, then $1\in\ker P_{\widetilde{g}}$. This implies $u\in\ker P_g\backslash\{0\}, u>0$ on M.

If $Q_{\widetilde{g}}<0$, then 1 is an eigenfunction of $P_{\widetilde{g}}$ with eigenvalue $-\frac{1}{2}Q_{\widetilde{g}}\neq 0$. If $v\in\ker P_g\backslash\{0\}$, then $u^{-1}v\in\ker P_{\widetilde{g}}$. This implies $\int_M u^{-1}v\,d\mu_{\widetilde{g}}=0$. Hence v must change sign.

(3)⇒(1) : If there exists a $u\in\ker P_g\backslash\{0\}, u>0$, then $I_4^3(u)=0$. Hence $Y_4^3(M,g)=0=I_4^3(u)$. If we know for any $u\in\ker P_g\backslash\{0\}$, u changes sign, then we claim that Case 3.2 could not happen. In fact, if it does happen, then the limit u satisfies $E(u)\le 0, u\ge 0$ and $\max_M u=1$. Since $P_g\ge 0$, we see $u\in\ker P_g\backslash\{0\}$. On the other hand, it never changes sign, a contradiction. Hence we may find a minimizer.

∎

Corollary 6.1. *Assume $P_g \geq 0$. If for any $u \in \ker P_g \backslash \{0\}$, u vanishes somewhere, and there exists a $u \in \ker P_g \backslash \{0\}$ such that $u \geq 0$, then there does not exist any $u \in C^\infty(M), u > 0$ such that $Q_{u^{-4}g} \equiv \text{const}$. Note that in this case, we must have $Y_4^3(M, g) = 0$.*

In the special case when $P_g > 0$, we can always find a positive minimizer. We also have a somewhat direct argument for this specific case..

Lemma 6.3. *Assume $P_g > 0$, then there exists a $u \in C^\infty(M), u > 0$ such that $I_4^3(u) = Y_4^3(M, g)$.*

Proof. Clearly $Y_4^3(M, g) \geq 0$. Choose a sequence $u_i \in V$ such that $|u_i|_{L^{-6}(M)} = 1$, $I_4^3(u_i) = E(u_i) \to Y_4^3(M, g)$. Since $E(u_i) \geq \lambda |u_i|^2_{L^2(M)}$ for some $\lambda > 0$, and $E(u_i) \geq c(g, M)|u_i|^2_{H^2(M)} - c(g, M)|u_i|^2_{L^2(M)}$, we see $|u_i|^2_{H^2(M)} \leq c(g, M) E(u_i)$. Hence $\sup_i |u_i|_{H^2(M)} < \infty$. It follows from Corollary 2.1 that for some $c_1 > 0, u_i \geq c_1$. We may find a $u \in H^2(M)$ such that $u_i \rightharpoonup u$ in $H^2(M)$. Then $u_i \to u$ uniformly on M. This plus the lower bound shows $u_i^{-1} \to u^{-1}$ uniformly on M, and hence $\left|u^{-1}\right|_{L^6(M)} = 1$. Then $Y_4^3(M, g) \leq E(u) \leq \liminf_{i\to\infty} E(u_i) \leq Y_4^3(M, g)$. The regularity of u follows from the Euler-Lagrange equation. ∎

For the uniqueness of solutions, we have the following very simple lemma.

Lemma 6.4. *Assume $P_g \geq 0, u, v \in C^\infty(M), u > 0, v > 0$ satisfying $P_g u = u^{-7}$ and $P_g v = v^{-7}$,then $u = v$.*

Proof. By substraction, we have $P_g(u - v) = u^{-7} - v^{-7}$, hence

$$\int_M P_g(u - v) \cdot (u - v)\, d\mu = \int_M \left(u^{-7} - v^{-7}\right)(u - v)\, d\mu.$$

By the fact $P_g \geq 0$ we see LHS ≥ 0. On the other hand, it is clear that RHS ≤ 0. Hence they are both equal to zero. It follows that $u \equiv v$ by looking at the right hand side of the above integral. ∎

Corollary 6.2. *Assume $P_g > 0$. Then there exists exactly one $\widetilde{g}$ conformal to g, such that $Q_{\widetilde{g}} \equiv \text{const}$ and $\mu_{\widetilde{g}}(M) = 1$.*

Proof. Assume we have two positive smooth functions, namely u_1, u_2, such that for some constants c_1, c_2,

$$P_g u_1 = c_1 u_1^{-7}, \quad P_g u_2 = c_2 u_2^{-7}, \quad \int_M u_1^{-6} d\mu = \int_M u_2^{-6} d\mu = 1.$$

Since $P_g > 0$, we know both c_1 and c_2 are strictly positive. It follows from the above claim that $c_1^{-1/8} u_1 = c_2^{-1/8} u_2$. Using the volume constrain, we see $c_1 = c_2$, hence $u_1 = u_2$. ∎

7. The sphere S^3

For the sphere S^3, we have

$$R_{ijkl} = g_{ik}g_{jl} - g_{il}g_{jk}, \ Rc = 2g, \ R = 6, \ Q = \frac{15}{8}, \ Pu = \Delta^2 u + \frac{1}{2}\Delta u - \frac{15}{16}u.$$

To proceed, we need to introduce more notations. For any $\xi \in S^3$, let π_ξ be the stereographic projection from $S^3 \backslash \{\xi\}$ to the hyperplane $\xi^\perp = \left\{x \in \mathbb{R}^4 : x \cdot \xi = 0\right\}$.

For $\lambda > 0$, we define a map $\sigma_{\xi,\lambda} : S^3 \to S^3$ by $\sigma_{\xi,\lambda}(\zeta) = \pi_\xi^{-1}(\lambda \cdot \pi_\xi(\zeta))$. $\sigma_{\xi,\lambda}$ is a conformal map and in fact

$$\sigma_{\xi,\lambda}^* g_{S^3} = \frac{\lambda^2 \left(1 + |\pi_\xi|^2\right)^2}{\left(1 + \lambda^2 |\pi_\xi|^2\right)^2} g_{S^3}.$$

By using the fact $\Delta^2 r = -8\pi\delta$ on $\mathbb{R}^3$ and (1.3) we easily deduce that the Green's function for the Paneitz operator at any point $\xi \in S^3$ is

$$G_\xi = -\frac{1}{4\pi}\left(1 + |\pi_{-\xi}|^2\right)^{-\frac{1}{2}} |\pi_{-\xi}| = -\frac{1}{4\pi}\left(1 + |\pi_\xi|^2\right)^{-\frac{1}{2}}.$$

In particular, $G_\xi(\xi) = 0$ for any $\xi \in S^3$. By Corollary 4.1, we see

$$E(G_\xi, G_\xi) = G_\xi(\xi) = 0. \tag{7.1}$$

This shows the standard S^3 does not satisfy the condition (P^+). Nevertheless, it was proved in [YZ] that $Y_4^3(S^3, g_{S^3})$ is still achieved (see Theorem 1.2). We are aiming to give a proof of this result without using the symmetrization. At first, we prove an identity from which one sees easily that the condition (NN) is true for S^3.

Lemma 7.1. *Let $\xi \in S^3, u \in H^2(S^3)$ such that $u(\xi) = 0$. Denote*

$$\rho_\xi = \frac{\sqrt{2}}{2}\left(1 + |\pi_\xi|^2\right)^{1/2}.$$

Then we know $\Delta\left((\rho_\xi \cdot u) \circ \pi_\xi^{-1}\right) \in L^2\left(\xi^\perp\right)$ and

$$E(u) = \int_{\xi^\perp} \left|\Delta\left((\rho_\xi \cdot u) \circ \pi_\xi^{-1}\right)\right|^2 d\mathcal{H}^3,$$

here Δ is the Euclidean Laplacian.

Proof. By Lemma 2.2 we may find a sequence $u_i \in C^\infty(S^3)$, such that u_i equals to 0 near ξ and $u_i \to u$ in $H^2(S^3)$. By (1.3), we see

$$\int_{\xi^\perp} \left|\Delta\left((\rho_\xi \cdot u_i - \rho_\xi \cdot u_j) \circ \pi_\xi^{-1}\right)\right|^2 d\mathcal{H}^3 = E(u_i - u_j) \to 0$$

as $i, j \to \infty$. Hence we may find a $f \in L^2\left(\xi^\perp\right)$ such that

$$\Delta\left((\rho_\xi \cdot u_i) \circ \pi_\xi^{-1}\right) \to f \quad \text{in } L^2\left(\xi^\perp\right).$$

This clearly implies $\Delta\left((\rho_\xi \cdot u) \circ \pi_\xi^{-1}\right) = f \in L^2\left(\xi^\perp\right)$. On the other hand, we have

$$E(u_i) = \int_{\xi^\perp} \left|\Delta\left((\rho_\xi \cdot u_i) \circ \pi_\xi^{-1}\right)\right|^2 d\mathcal{H}^3.$$

Letting $i \to \infty$, we get the needed inequality. ∎

Corollary 7.1. *If $u \in H^2(S^3)$ such that $u(\xi) = 0$ for some $\xi \in S^3$, then $E(u) \geq 0$ and $E(u) = 0$ if and only if $u = \text{const} \cdot G_\xi$.*

Proof. If $E(u)=0$, then it follows from the above claim that $\left(\rho_\xi\cdot u\right)\circ\pi_\xi^{-1}$ is a harmonic function. But it follows from the fact $u\in C^{1/2}\left(S^3\right)$ and $u(\xi)=0$ that $\left|\left(\rho_\xi\cdot u\right)\left(\pi_\xi^{-1}(x)\right)\right|\leq c(u)\sqrt{|x|}$ for $|x|\geq 1$. It follows from Liouville theorem that $\left(\rho_\xi\cdot u\right)\circ\pi_\xi^{-1}$ is equal to a constant. One direction of the equivalence follows. The other direction follows from (7.1). ∎

Now we are ready to prove Theorem 1.2.

Proof of Theorem 1.2. Clearly $Y_4^3\left(S^3,g_{S^3}\right)\leq I_4^3(1)<0$. Choose a sequence $u_i\in V$ such that $I_4^3(u_i)\to Y_4^3\left(S^3,g_{S^3}\right)$. By scaling and rotation, we may assume $\max_{S^3}u_i=u_i(\mathrm{N})=1$. Here N is the north pole. For i large enough, we have $E(u_i)<0$. But since $E(u_i)\geq c\left|u_i\right|^2_{H^2(S^3)}-c\left|u_i\right|^2_{L^2(S^3)}$, we see $\left|u_i\right|_{H^2(S^3)}\leq c$. Hence after passing to a subsequence, we may find a $u\in H^2\left(S^3\right)$ such that $u_i\rightharpoonup u$ in $H^2\left(S^3\right)$. This implies in particular that $u_i\to u$ uniformly on S^3 and hence $u\geq 0$.

Case 7.1. *$u>0$ on S^3.*

In this case, we see $u_i^{-1}\to u^{-1}$ uniformly on S^3. Hence $\left|u_i^{-1}\right|_{L^6(S^3)}\to\left|u^{-1}\right|_{L^6(S^3)}$. By lower semicontinuity we have $E(u)\leq\liminf_{i\to\infty}E(u_i)$. Hence

$$Y_4^3\left(S^3,g_{S^3}\right)\leq I_4^3(u)\leq\lim\inf_{i\to\infty}I_4^3(u_i)\leq Y_4^3\left(S^3,g_{S^3}\right).$$

Case 7.2. *u vanishes at some point.*

In this case, on one hand, it follows from the lower semicontinuity that $E(u)\leq 0$. On the other hand, it follows from Corollary 7.1 that $E(u)=0$ and $u=c_1\left(1+\left|\pi_\xi\right|^2\right)^{-1/2}$ for some $\xi\in S^3$ and $c_1\in\mathbb{R}$. It follows from the uniform convergence and $\max_{S^3}u_i=u_i(\mathrm{N})=1$ that $\max_{S^3}u=u(\mathrm{N})=1$. This implies $\xi=-\mathrm{N}=\mathrm{S}$, the south pole. Choose $\xi_i\in S^3$ such that $\min_{S^3}u_i=u_i(\xi_i)=\lambda_i$. Then clearly $\xi_i\to\mathrm{S}$ and $\lambda_i\to 0$. Choose a $O_i\in O(4)$ such that $O_i\mathrm{S}=\xi_i$ and $\left|O_i-\mathrm{id}\right|\to 0$. Let $v_i(\xi)=u_i(O_i\xi)$ for $\xi\in S^3$, then $v_i\in V$ is another minimizing sequence with $\min_{S^3}v_i=v_i(\mathrm{S})=\lambda_i$, $\max_{S^3}v_i=1$ and $v_i(\mathrm{N})\to 1$. Now we define

$$w_i=\left(\frac{1+\lambda_i^2\left|\pi_\mathrm{N}\right|^2}{\lambda_i\left(1+\left|\pi_\mathrm{N}\right|^2\right)}\right)^{1/2}\cdot v_i\circ\sigma_{\mathrm{N},\lambda_i},$$

then w_i is also a minimizing sequence. In addition, $w_i(\mathrm{S})=\sqrt{\lambda_i}$ and $w_i(\mathrm{N})/\sqrt{\lambda_i}\to 1$. Let $\nu_i=\max_{S^3}w_i$, then as before we know $\frac{w_i}{\nu_i}\rightharpoonup w$ for some $w\in H^2\left(S^3\right)$.

We claim $w>0$ on S^3. Indeed, if w vanishes somewhere, then the arguments before shows $w=\left(1+\left|\pi_\xi\right|^2\right)^{-1/2}$ for some $\xi\in S^3$. Since it is clear that $w(\mathrm{N})=w(\mathrm{S})$, we see $w(\mathrm{N})=w(\mathrm{S})>0$. Hence $\frac{\sqrt{\lambda_i}}{\nu_i}\to w(\mathrm{N})>0$. But on $S^3\backslash\{\mathrm{N},\mathrm{S}\}$,

$$\frac{w_i}{\nu_i}\geq\frac{\lambda_i}{\nu_i}\left(\frac{1+\lambda_i^2\left|\pi_\mathrm{N}\right|^2}{\lambda_i\left(1+\left|\pi_\mathrm{N}\right|^2\right)}\right)^{1/2}\to\frac{w(\mathrm{N})}{\left(1+\left|\pi_\mathrm{N}\right|^2\right)^{1/2}},$$

this shows $w>0$ too, a contradiction. This shows $w>0$ on S^3 and hence it must be a minimizer. ∎

8. The Berger spheres

The Lie group

$$SU(2)=\left\{\begin{pmatrix} a & -\overline{b} \\ b & \overline{a}\end{pmatrix}: a,b\in\mathbb{C}, |a|^2+|b|^2=1\right\},$$

may be identified with $SU(2)\cong S^3$. Its Lie algebra is

$$\mathfrak{su}(2)=\left\{\begin{pmatrix} it_1 & it_2-t_3 \\ it_2+t_3 & -it_1\end{pmatrix}: t_1,t_2,t_3\in\mathbb{R}\right\}.$$

Choosing basis elements in $\mathfrak{su}(2)$, namely

$$X_1=\begin{pmatrix} i & 0 \\ 0 & -i\end{pmatrix},\quad X_2=\begin{pmatrix} 0 & i \\ i & 0\end{pmatrix},\quad X_3=\begin{pmatrix} 0 & -1 \\ 1 & 0\end{pmatrix},$$

then $[X_1,X_2]=2X_3$, $[X_2,X_3]=2X_1$, $[X_3,X_1]=2X_2$.

Let (cf. section 3.1 of [H])

$$X=X_1,\quad Z^+=X_2+iX_3,\quad Z^-=X_2-iX_3,$$

then

$$X_1=X,\quad X_2X_2+X_3X_3=Z^+Z^-+2iX.$$

Let $SU(2)\times\mathbb{C}^2\to\mathbb{C}^2$ be the standard representation, denote

$$x=\begin{pmatrix}1\\0\end{pmatrix},\quad y=\begin{pmatrix}0\\1\end{pmatrix},$$

then we have

$$\begin{cases} X(x^my^n)=i(m-n)x^my^n, \\ Z^+(x^my^n)=2imx^{m-1}y^{n+1}, \\ Z^-(x^my^n)=2inx^{m+1}y^{n-1}, \end{cases}\qquad \text{for } m,n\geq 0.$$

For $t>0$, let $e_1=t^{-1}X_1$, $e_2=X_2$, $e_3=X_3$, then the Berger sphere has the metric g_t on S^3 such that e_1,e_2,e_3 is an orthonormal base. Under the metric g_t, in terms of e_1,e_2,e_3, we have (see [Pe], p81)

$$R_{1212}=t^2,\ R_{1313}=t^2,\ R_{2323}=4-3t^2,\ R_{1213}=R_{1223}=R_{1323}=0.$$

Hence

$$\begin{aligned} R &= 8-2t^2, \\ Q &= -\frac{169}{8}t^4+41t^2-18 \\ &= -\frac{169}{8}\left(t+\frac{2-4\sqrt{10}}{13}\right)\left(t-\frac{2-4\sqrt{10}}{13}\right)\left(t+\frac{2+4\sqrt{10}}{13}\right)\left(t-\frac{2+4\sqrt{10}}{13}\right). \end{aligned}$$

We remark that $\frac{-2+4\sqrt{10}}{13}\approx 0.8192$, $\frac{2+4\sqrt{10}}{13}\approx 1.1269$.

The Shouten tensor is $A=Rc-\frac{R}{4}g$, in index form,

$$A_{11}=\frac{5}{2}t^2-2,\ A_{22}=2-\frac{3}{2}t^2,\ A_{33}=2-\frac{3}{2}t^2,\ A_{12}=A_{13}=A_{23}=0.$$

Hence $\sigma_2(A)=-\frac{21}{4}\left(t^2-\frac{4}{7}\right)\left(t^2-\frac{4}{3}\right)$.

The Cotton tensor is $C_{ijk}=A_{ijk}-A_{ikj}$, more precisely,

$$\begin{aligned} C_{123} &= 8t^3-8t,\ C_{231}=4t-4t^3,\ C_{312}=4t-4t^3,\ C_{ijk}=-C_{ikj}, \\ C_{iij} &= C_{iji}=C_{jii}=0. \end{aligned}$$

In particular, this shows $|C|^2 = 192t^2\left(t^2-1\right)^2$. As a corollary, we see for $t \neq 1$, $\mathrm{Conf}\left(S^3, g_t\right) = \mathrm{Isom}\left(S^3, g_t\right)$. In fact, given $\sigma \in \mathrm{Conf}\left(S^3, g_t\right)$, we have $\sigma^* g_t = \rho^2 g_t$ for some $\rho \in C^\infty\left(S^3\right), \rho > 0$. Hence by the conformal transformation law for Cotton tensor, we have $\left|C_{\sigma^* g_t}\right|^2 = \left|C_{\rho^2 g_t}\right|^2 = \rho^{-6}|C|^2 = 192t^2\left(t^2-1\right)^2\rho^{-6}$. On the other hand, we know $\left|C_{\sigma^* g_t}\right|^2 = |C|^2 \circ \sigma = 192t^2\left(t^2-1\right)^2$. Hence $\rho \equiv 1$. That is σ is an isometry.

For the Laplace operator, we have

$$\Delta_t = e_1e_1 + e_2e_2 + e_3e_3 = t^{-2}XX + 2iX + Z^+Z^-,$$

hence we have

Lemma 8.1. *The eigenvalues of Δ_t are*

$$\lambda_{m,n} = 4mn + 2(m+n) + (m-n)^2 t^{-2}, \quad m \geq 0, n \geq 0,$$

here each $\lambda_{m,n}$ should be counted $m+n+1$ times.

Proof. This follows from the fact

$$\Delta_t\left(x^m y^n\right) = \left(4mn + 2(m+n) + (m-n)^2 t^{-2}\right) x^m y^n$$

for $m, n \geq 0$. The similar calculation was used in section 3.1 of [H] to compute the eigenvalues of the Dirac's operator on Berger spheres. ∎

For the Paneitz operator, we have

$$\begin{aligned} P_t &= \Delta^2 + \left(\frac{5}{2}t^2 - 10\right)\Delta + 8t^2 e_1e_1 + 8\left(2-t^2\right)\left(e_2e_2 + e_3e_3\right) \\ &\quad + \left(\frac{169}{16}t^4 - \frac{41}{2}t^2 + 9\right) \\ &= \Delta^2 + \left(\frac{5}{2}t^2 - 10\right)\Delta + \left(\frac{169}{16}t^4 - \frac{41}{2}t^2 + 9\right) + 8XX \\ &\quad + 16\left(2-t^2\right)iX + 8\left(2-t^2\right)Z^+Z^-. \end{aligned}$$

From this we deduce

Lemma 8.2. *The eigenvalues of P_t are*

$$\begin{aligned} \lambda_{m,n} &= \frac{169}{16}t^4 + \left(22mn + 11(m+n) - \frac{41}{2}\right)t^2 + 16m^2n^2 \\ &\quad + 16mn(m+n) - \frac{13}{2}(m-n)^2 - 8mn - 12(m+n) + 9 \\ &\quad + (8mn + 4(m+n) + 10)(m-n)^2 t^{-2} + (m-n)^4 t^{-4}, \end{aligned}$$

for $m, n \geq 0$, here each $\lambda_{m,n}$ should be counted $m+n+1$ times.

Now we proceed to carefully study these eigenvalues.

Lemma 8.3. *For $m \geq 0, n \geq 0, m+n \geq 1$, we have $\lambda_{m,n}(t) > 0$ for $t > 0$.*

Proof. First we observe that for any $m, n \geq 0$, $\lambda_{m,n} = \lambda_{n,m}$.

If $m \geq 1$ and $n \geq 1$, then we may assume $m \geq n$. In this case,

$$22mn + 11(m+n) - \frac{41}{2} \geq \frac{47}{2} > 0,$$

and

$$\begin{aligned}
& 16m^2n^2+16mn\left(m+n\right)-\frac{13}{2}\left(m-n\right)^2-8mn-12\left(m+n\right)+9 \\
\geq\ & 16m^2+16\left(m+n\right)-\frac{29}{2}m^2-12\left(m+n\right)+9\geq 9>0,
\end{aligned}$$

hence $\lambda_{m,n}\left(t\right)>0$ for any $t>0$.

If $n=0$, $m\geq 3$, then

$$\begin{aligned}
& \lambda_{m,0}\left(t\right) \\
= & \frac{169}{16}t^4+\left(11m-\frac{41}{2}\right)t^2-\frac{13}{2}m^2-12m+9+\left(4m^3+10m^2\right)t^{-2}+m^4t^{-4} \\
= & \left(\frac{13}{4}t^2-m^2t^{-2}\right)^2+\left(11m-\frac{41}{2}\right)t^2-12m+9+\left(4m^3+10m^2\right)t^{-2} \\
\geq & \frac{25}{6}mt^2-12m+66mt^{-2}+9\geq 9>0
\end{aligned}$$

for $t>0$.

In addition,

$$\lambda_{2,0}\left(t\right)=\frac{169}{16}t^4+\frac{3}{2}t^2-41+72t^{-2}+16t^{-4}\geq\frac{3}{2}t^2+72t^{-2}-15>0$$

for $t>0$, and

$$\begin{aligned}
\lambda_{1,0}\left(t\right) &= \frac{169}{16}t^4-\frac{19}{2}t^2-\frac{19}{2}+14t^{-2}+t^{-4}>10t^4-10t^2-10+10t^{-2} \\
&= 10t^{-2}\left(t^2+1\right)\left(t^2-1\right)^2\geq 0
\end{aligned}$$

for $t>0$. The lemma follows. ∎

Lemma 8.4. *For $m\geq 0,n\geq 0,m+n\geq 1$, we have $\lambda_{m,n}\left(t\right)>\lambda_{0,0}\left(t\right)$ for $t>0$.*

Proof. Note that

$$\begin{aligned}
& \lambda_{m,n}\left(t\right)-\lambda_{0,0}\left(t\right) \\
= & \left(22mn+11\left(m+n\right)\right)t^2+16m^2n^2+16mn\left(m+n\right)-\frac{13}{2}\left(m-n\right)^2-8mn \\
& -12\left(m+n\right)+\left(8mn\left(m-n\right)^2+4\left(m+n\right)\left(m-n\right)^2+10\left(m-n\right)^2\right)t^{-2} \\
& +\left(m-n\right)^4t^{-4}.
\end{aligned}$$

If $m\geq n\geq 1$, then again, using

$$\begin{aligned}
& 16m^2n^2+16mn\left(m+n\right)-\frac{13}{2}\left(m-n\right)^2-8mn-12\left(m+n\right) \\
\geq\ & 16m^2+16\left(m+n\right)-\frac{29}{2}m^2-12\left(m+n\right)\geq 9>0
\end{aligned}$$

we deduce the conclusion.

If $m\geq 1,n=0$, then

$$\begin{aligned}
\lambda_{m,0}\left(t\right)-\lambda_{0,0}\left(t\right) &= 11mt^2-\frac{13}{2}m^2-12m+\left(10m^2+4m^3\right)t^{-2}+m^4t^{-4} \\
&> m\left(3t^2-\frac{13}{2}m+4m^2t^{-2}+8t^2-12+10mt^{-2}\right)\geq 0.
\end{aligned}$$

The lemma follows. ∎

Corollary 8.1. *We have the following*

- $\lambda_{0,0}(t)$ *is the first eigenvalue of* P_t*, with multiplicity one, and the corresponding eigenspace is the set of constant functions.*
- *For* $t \in \left(0, \frac{-2+4\sqrt{10}}{13}\right) \cup \left(\frac{2+4\sqrt{10}}{13}, \infty\right)$, $P_t > 0$ *and*

$$\begin{aligned} & Y_4^3\left(S^3, g_t\right) \\ = \; & I_4^3(1) = -\frac{169}{4} 2^{1/3}\pi^{8/3}t^{16/3} + 82 \cdot 2^{1/3}\pi^{8/3}t^{10/3} - 36 \cdot 2^{1/3}\pi^{8/3}t^{4/3}, \end{aligned}$$

  in addition, the constant multiple of g_t *is the only metric having constant* Q *curvature in the conformal class.*
- $P_{\frac{-2+4\sqrt{10}}{13}} \geq 0$, $\ker\left(P_{\frac{-2+4\sqrt{10}}{13}}\right) = \{\text{const}\}$.
- $P_{\frac{2+4\sqrt{10}}{13}} \geq 0$, $\ker\left(P_{\frac{2+4\sqrt{10}}{13}}\right) = \{\text{const}\}$.
- *For* $t \in \left(\frac{-2+4\sqrt{10}}{13}, \frac{2+4\sqrt{10}}{13}\right)$, P_t *has exactly one negative eigenvalue* $\lambda_{0,0}(t)$.

Remark 8.1. *It is worth noticing that for* $t \in \left(\sqrt{\frac{4}{7}}, \frac{-2+4\sqrt{10}}{13}\right) \cup \left(\frac{2+4\sqrt{10}}{13}, \sqrt{\frac{4}{3}}\right)$, $\sigma_2(A) = -\frac{21}{4}\left(t^2 - \frac{4}{7}\right)\left(t^2 - \frac{4}{3}\right) > 0$, $P_t > 0$, $R = 8 - 2t^2 > 0$.

Now we state the following

Lemma 8.5. *For any* $m \geq 0, n \geq 0, (m,n) \neq (0,0), (1,1)$*, we have* $\lambda_{m,n}(t)\lambda_{1,1}(t)^{-1}$ *is strictly decreasing for* $t \in (0,1]$.

Proof. Note that $\lambda_{1,1}(t) = \frac{169}{16}t^4 + \frac{47}{2}t^2 + 25$. Computations show that

$$-\left(\lambda'_{m,n}(t)\lambda_{1,1}(t) - \lambda_{m,n}(t)\lambda'_{1,1}(t)\right) = c_5t^5 + c_3t^3 + c_1t + c_{-1}t^{-1} + c_{-3}t^{-3} + c_{-5}t^{-5},$$

where

$$\begin{aligned} c_5 &= \frac{1859}{8}(2mn + m + n - 4), \\ c_3 &= 676m^2n^2 + 676m^2n + 676mn^2 - \frac{2197}{8}m^2 - \frac{2197}{8}n^2 + \frac{845}{4}mn \\ &\quad -507m - 507n - 676, \\ c_1 &= 507m^3n - 262m^2n^2 + 507mn^3 + \frac{507}{2}m^3 + \frac{997}{2}m^2n + \frac{997}{2}mn^2 + \frac{507}{2}n^3 \\ &\quad + \frac{1313}{4}m^2 - \frac{4265}{2}mn + \frac{1313}{4}n^2 - 1114m - 1114n + 1448, \\ c_{-1} &= \left(\frac{169}{2}m^2 + 583mn + \frac{169}{2}n^2 + 376m + 376n + 940\right)(m-n)^2, \\ c_{-3} &= \left(141m^2 + 118mn + 141n^2 + 200m + 200n + 500\right)(m-n)^2, \\ c_{-5} &= 100(m-n)^4. \end{aligned}$$

Assume $m \geq n \geq 1$ and $m \neq 1$, then one shows easily as before that $c_5 > 0$, $c_3 > 0$ and $c_1 > 0$. Hence $\lambda'_{m,n}(t)\lambda_{1,1}(t) - \lambda_{m,n}(t)\lambda'_{1,1}(t) < 0$ for $0 < t \leq 1$.

On the other hand,

$$-\left(\lambda'_{m,0}(t)\lambda_{1,1}(t) - \lambda_{m,0}(t)\lambda'_{1,1}(t)\right) = d_5t^5 + d_3t^3 + d_1t + d_{-1}t^{-1} + d_{-3}t^{-3} + d_{-5}t^{-5},$$

where

$$\begin{aligned}
d_5 &= \frac{1859}{8}(m-4), \\
d_3 &= -\frac{2197}{8}m^2 - 507m - 676, \\
d_1 &= \frac{507}{2}m^3 + \frac{1313}{4}m^2 - 1114m + 1448, \\
d_{-1} &= \frac{169}{2}m^4 + 376m^3 + 940m^2, \\
d_{-3} &= 141m^4 + 200m^3 + 500m^2, \\
d_{-5} &= 100m^4.
\end{aligned}$$

If $m \geq 4$, then $d_5 \geq 0$, $d_1 \geq 0$, in addition

$$d_{-5} + d_{-3} + d_{-1} + d_3 = \frac{651}{2}m^4 + 576m^3 + \frac{9323}{8}m^2 - 507m - 676 > 0.$$

This implies $\lambda'_{m,0}(t)\lambda_{1,1}(t) - \lambda_{m,0}(t)\lambda'_{1,1}(t) < 0$ for $0 < t \leq 1$.

For $m = 3$,

$$\begin{aligned}
& \lambda'_{3,0}(t)\lambda_{1,1}(t) - \lambda_{3,0}(t)\lambda'_{1,1}(t) \\
= & \frac{1859}{8}t^5 + \frac{37349}{8}t^3 - \frac{31619}{4}t - \frac{50913}{2}t^{-1} - 21321t^{-3} - 8100t^{-5} \\
< & \ 0 \quad \text{when } 0 < t \leq 1.
\end{aligned}$$

For $m = 2$,

$$\begin{aligned}
& \lambda'_{2,0}(t)\lambda_{1,1}(t) - \lambda_{2,0}(t)\lambda'_{1,1}(t) \\
= & \frac{1859}{4}t^5 + \frac{5577}{2}t^3 - 2561t - 8120t^{-1} - 5856t^{-3} - 1600t^{-5} \\
< & \ 0 \quad \text{when } 0 < t \leq 1.
\end{aligned}$$

For $m = 1$,

$$\begin{aligned}
& \lambda'_{1,0}(t)\lambda_{1,1}(t) - \lambda_{1,0}(t)\lambda'_{1,1}(t) \\
= & \frac{5577}{8}t^5 + \frac{11661}{8}t^3 - \frac{3663}{4}t - \frac{2801}{2}t^{-1} - 841t^{-3} - 100t^{-5} \\
< & \ 0 \quad \text{when } 0 < t \leq 1.
\end{aligned}$$

The lemma follows. ∎

By direct calculation, we have

Lemma 8.6. *We have*

$$\frac{\lambda_{0,0}(t)}{\lambda_{1,1}(t)} \begin{cases} > \frac{\lambda_{0,0}(1)}{\lambda_{1,1}(1)}, & \text{when } 0 < t < \frac{2\sqrt{37}}{13} \approx 0.9358, \\ < \frac{\lambda_{0,0}(1)}{\lambda_{1,1}(1)}, & \text{when } \frac{2\sqrt{37}}{13} < t < 1. \end{cases}$$

Corollary 8.2. *For any* $t \in \left(0, \frac{2\sqrt{37}}{13}\right]$, $P_t \geq \left(\frac{169}{945}t^4 + \frac{376}{945}t^2 + \frac{80}{189}\right)P_1$ *on the space* $L^2\left(S^3, d\mathcal{H}^3_{S^3}\right)$. *In addition, for any* $t \in \left(\frac{2\sqrt{37}}{13}, 1\right)$, *there is no any real number* c *such that* $P_t \geq c \cdot P_1$.

Proof. Assume $t \in \left(0, \frac{2\sqrt{37}}{13}\right]$, then for any $m \geq 0, n \geq 0$, we have $\frac{\lambda_{m,n}(t)}{\lambda_{1,1}(t)} \geq \frac{\lambda_{m,n}(1)}{\lambda_{1,1}(1)}$. Hence $\lambda_{m,n}(t) \geq \frac{\lambda_{1,1}(t)}{\lambda_{1,1}(1)} \lambda_{m,n}(1)$. This implies $P_t \geq \frac{\lambda_{1,1}(t)}{\lambda_{1,1}(1)} P_1$. Computation shows $\frac{\lambda_{1,1}(t)}{\lambda_{1,1}(1)} = \frac{169}{945} t^4 + \frac{376}{945} t^2 + \frac{80}{189}$.

Assume $t \in \left(\frac{2\sqrt{37}}{13}, 1\right)$. If for some real number c we have $P_t \geq c \cdot P_1$, then we have $\lambda_{0,0}(t) \geq c\lambda_{0,0}(1)$ and $\lambda_{1,1}(t) \geq c\lambda_{1,1}(1)$ .Since $\lambda_{0,0}(1) < 0, \lambda_{1,1}(1) > 0$, we get $\frac{\lambda_{0,0}(t)}{\lambda_{0,0}(1)} \leq \frac{\lambda_{1,1}(t)}{\lambda_{1,1}(1)}$. This implies $\frac{\lambda_{0,0}(t)}{\lambda_{1,1}(t)} \geq \frac{\lambda_{0,0}(1)}{\lambda_{1,1}(1)}$ and it contradicts with Lemma 8.6. ∎

Corollary 8.3. *For any* $t \in \left(0, \frac{2\sqrt{37}}{13}\right)$, $\left(S^3, g_t\right)$ *satisfies the condition* (P).

Proof. It clear that g_t satisfies the condition (NN). In addition, if for some $u \in H^2\left(S^3\right)$, u vanishes somewhere and $E_t(u) = 0$, then it follows from Corollary 8.2 that $E_1(u) \leq 0$. Using Corollary 7.1 we see for some $\xi \in S^3$, and some constant c, that $u = c\left(1 + |\pi_\xi|^2\right)^{-1/2}$. It follows easily from Lemma 8.6 and the proof of Corollary 8.2 that

$$E_t\left(\left(1 + |\pi_\xi|^2\right)^{-1/2}\right) > \left(\frac{169}{945} t^4 + \frac{376}{945} t^2 + \frac{80}{189}\right) E_1\left(\left(1 + |\pi_\xi|^2\right)^{-1/2}\right) = 0.$$

Hence we get $c = 0$. This implies $u = 0$ and the corollary follows. Another way to see $E_t\left(\left(1 + |\pi_\xi|^2\right)^{-1/2}\right) > 0$ is by the formula

$$E_t\left(\left(1 + |\pi_\xi|^2\right)^{-1/2}\right) = \frac{\pi^2}{12}\left(507t^5 - 940t^3 + 366t + 60t^{-1} + 7t^{-3}\right), \tag{8.1}$$

which follows from a direct computation. Let $\phi(t) = 507t^5 - 940t^3 + 366t + 60t^{-1} + 7t^{-3}$, then one easily verifies that $\phi(1) = \phi'(1) = 0$, and $\phi''(t) > 0$ for $t > 0$. This implies $\phi(t) > 0$ for $t \neq 1$. ∎

Lemma 8.7. *For* $m \geq 0, n \geq 0, (m,n) \neq (0,0)$, $t^4 \lambda_{m,n}(t)$ *is strictly increasing on* $[1, \infty)$.

Proof. Note that

$$\begin{aligned} t^4 \lambda_{m,n}(t) &= \frac{169}{16} t^8 + \left(22mn + 11(m+n) - \frac{41}{2}\right) t^6 \\ &\quad + \left(16m^2n^2 + 16mn(m+n) - \frac{13}{2}(m-n)^2 - 8mn - 12(m+n) + 9\right) t^4 \\ &\quad + \left(8mn + 4(m+n) + 10\right)(m-n)^2 t^2 + (m-n)^4. \end{aligned}$$

If $m \geq n \geq 1$, then since the coefficients of t^6 and t^4 are positive, as proved in Lemma 8.3, we see $t^4 \lambda_{m,n}(t)$ is strictly increasing on $(0, \infty)$.

Assume $n = 0$, then

$$\left(t^4 \lambda_{m,0}(t)\right)' = \frac{169}{2} t^7 + (66m - 123) t^5 + \left(-26m^2 - 48m + 36\right) t^3 + \left(8m^3 + 20m^2\right) t.$$

Note that for $t > 0$,

$$\frac{169}{2} t^7 + 8m^3 t = \frac{169}{2} t^7 + 4m^3 t + 4m^3 t \geq 6\sqrt[3]{169} m^2 t^3 \geq 33 m^2 t^3,$$

hence for $t \geq 1$,

$$\left(t^4\lambda_{m,0}(t)\right)' \geq (66m-123)t^5 + \left(7m^2-48m+36\right)t^3 \geq \left(7m^2+18m-87\right)t^3.$$

This implies if $m \geq 3$, then $t^4\lambda_{m,0}(t)$ is strictly increasing on $[1,\infty)$. On the other hand, for $t \geq 1$,

$$\left(t^4\lambda_{2,0}(t)\right)' = \frac{169}{2}t^7 + 9t^5 - 164t^3 + 144t \geq 81t^5 - 164t^3 + 144t \geq 52t^3 > 0,$$

and

$$\begin{aligned}\left(t^4\lambda_{1,0}(t)\right)' &= \frac{169}{2}t^7 - 57t^5 - 38t^3 + 28t = \frac{169}{4}t^7 + \frac{169}{4}t^7 + 28t - 57t^5 - 38t^3 \\ &\geq 3\sqrt[3]{\frac{199927}{4}}t^5 - 57t^5 - 38t^3 \geq 53t^5 - 38t^3 > 0.\end{aligned}$$

The lemma follows. ∎

Lemma 8.8. *There exists a unique number $t_* > 1$ such that*

$$t^4\lambda_{0,0}(t)\begin{cases} < \lambda_{0,0}(1), & \text{when } 1 < t < t_*, \\ > \lambda_{0,0}(1), & \text{when } t_* < t < \infty. \end{cases}$$

In addition, $t_* \approx 1.0468$.

Proof. Let $\phi(t) = t^4\lambda_{0,0}(t) = \frac{169}{16}t^8 - \frac{41}{2}t^6 + 9t^4$. Since $\phi'(t) = \frac{169}{2}t^7 - 123t^5 + 36t^3$, we see

$$\phi'(t)\begin{cases} < 0, & \text{when } t \in \left(1, \frac{\sqrt{123+3\sqrt{329}}}{13}\right), \\ > 0, & \text{when } t \in \left(\frac{\sqrt{123+3\sqrt{329}}}{13}, \infty\right). \end{cases}$$

Note that $\frac{\sqrt{123+3\sqrt{329}}}{13} \approx 1.0246$. In view of the fact $\phi(\infty) = \infty$, the lemma follows. ∎

Corollary 8.4. *Let t_* be the number in Lemma 8.8, then for any $t \in [t_*,\infty)$, we have $P_t \geq t^{-4}P_1$. In addition, for $t \in (1,t_*)$, one could not find any number c such that $P_t \geq cP_1$.*

Proof. For any $t \in [t_*,\infty)$, since $t^4\lambda_{m,n}(t) \geq \lambda_{m,n}(1)$ for $m,n \geq 0$, we see $\lambda_{m,n}(t) \geq t^{-4}\lambda_{m,n}(1)$. Hence $P_t \geq t^{-4}P_1$.

For any $t \in (1,t_*)$, if for some real number c we have $P_t \geq cP_1$, then for each $m \geq 0, n \geq 0, \lambda_{m,n}(t) \geq c\lambda_{m,n}(1)$. In particular, for any $m \geq 0, \lambda_{m,0}(t) \geq c\lambda_{m,0}(1)$. Since

$$\lambda_{m,0}(t) = \frac{169}{16}t^4 + \left(11m - \frac{41}{2}\right)t^2 - \frac{13}{2}m^2 - 12m + 9 + \left(4m^3+10m^2\right)t^{-2} + m^4t^{-4},$$

by letting $m \to \infty$, we see $t^{-4} \geq c$. On the other hand, $\lambda_{0,0}(t) \geq c\lambda_{0,0}(1) \geq t^{-4}\lambda_{0,0}(1)$, here we have used the fact $\lambda_{0,0}(1) < 0$. Hence $t^4\lambda_{0,0}(t) \geq \lambda_{0,0}(1)$, but this contradicts with Lemma 8.8. ∎

Similar to Corollary 8.3, we have

Corollary 8.5. *For $t \in (t_*,\infty)$, with t_* being the number in Lemma 8.8, $\left(S^3, g_t\right)$ satisfies the condition (P).*

Lemma 8.9. *For* $t \in \left(\frac{-2+4\sqrt{10}}{13}, \frac{2+4\sqrt{10}}{13}\right)$, *let* $G_t(x,y)$ *be the Green's function for* P_t, *then we have*

$$G_t(x,x) = \frac{1}{2\pi^2 t} \sum_{m,n\geq 0} \frac{m+n+1}{\lambda_{m,n}(t)}$$

for any $x \in S^3$.

Proof. Let $(\lambda_i)_{i=1}^{\infty}$ be the eigenvalues of P_t, and ϕ_i be the (real) orthonormal base for $L^2(S^3, d\mu)$ such that $P_t\phi_i = \lambda_i \phi_i$. Then $G_t(x,y) = \sum_{i=1}^{\infty} \frac{1}{\lambda_i}\phi_i(x)\phi_i(y)$. This shows $G_t(x,x) = \sum_{i=1}^{\infty} \frac{1}{\lambda_i}\phi_i(x)^2$. Hence

$$\sum_{i=1}^{\infty} \frac{1}{\lambda_i} = \int_{S^3} G_t(x,x)\, d\mu = t\int_{S^3} G_t(x,x)\, d\mathcal{H}^3.$$

Since $SU(2)$ acts transitively on S^3 and they are all isometric actions, we see $G_t(x,x)$ does not depend on the choice of $x \in S^3$. Hence $G_t(x,x) = \frac{1}{2\pi^2 t}\sum_{i=1}^{\infty}\frac{1}{\lambda_i}$. The lemma follows. ∎

Lemma 8.10. *Let* $\zeta(t) = \sum_{m,n\geq 0} \frac{m+n+1}{t^4 \lambda_{m,n}(t)}$, *then* $\zeta''(t) < 0$ *for* $0.93 \leq t \leq 1.05$.

Proof. Let $p_{m,n}(t) = t^4 \lambda_{m,n}(t)$. Then we have the following basic facts.

(1) For $m \geq 0$, $n \geq 0$, $m+n \geq 1$, $0.93 \leq t \leq 1.05$, we have $p'_{m,n}(t) > 0$, $\frac{tp'_{m,n}(t)}{p_{m,n}(t)} \leq 6$, in particular, this shows $\frac{p'_{m,n}(t)}{p_{m,n}(t)} \leq 6.5$.
(2) For $m \geq 0$, $n \geq 0$, $m+n \geq 2$, $0.93 \leq t \leq 1.05$, we have $p''_{m,n}(t) \geq 0$.
(3) For $m \geq 0$, $n \geq 0$, $m+n \geq 1$, $0.93 \leq t \leq 1.05$, $p_{m,n}(t) \geq 0.67 p_{m,n}(1)$.
(4) For $0.93 \leq t \leq 1.05$, $\left(\frac{1}{p_{0,0}(t)}\right)'' \leq -110$.
(5) For $0.93 \leq t \leq 1.05$, $\left(\frac{1}{p_{1,0}(t)}\right)'' \leq -\frac{5}{4}$.

We note that (1) can be proved exactly the same as before by considering $6p_{m,n}(t) - tp'_{m,n}(t)$. (2) can be proved in the same way before too. (3) can be proved in view of the fact $p_{m,n}(t)\, p_{1,1}(t)^{-1}$ is decreasing on $(0,1]$. (4) and (5) can be proved by usual calculus. We omit the long but not illuminating details here. Hence for $m \geq 0, n \geq 0, k = m+n \geq 2, 0.93 \leq t \leq 1.05$, we have

$$\begin{aligned}
\left(\frac{1}{p_{m,n}(t)}\right)'' &= \frac{2p'_{m,n}(t)^2}{p_{m,n}(t)^3} - \frac{p''_{m,n}(t)}{p_{m,n}(t)^2} \\
&\leq 2 \times \frac{1}{0.67} \times 6.5^2 \times \frac{1}{(k+5/2)(k+3/2)(k+1/2)(k-1/2)} \\
&\leq \frac{127}{(k+5/2)(k+3/2)(k+1/2)(k-1/2)}.
\end{aligned}$$

Now for $0.93 \leq t \leq 1.05$,

$$\begin{aligned}
\zeta''(t) &\leq \left(\frac{1}{p_{0,0}(t)}\right)'' + \left(\frac{4}{p_{1,0}(t)}\right)'' + \sum_{k=2}^{\infty} \frac{127(k+1)^2}{(k+5/2)(k+3/2)(k+1/2)(k-1/2)} \\
&\leq -115 + 127\sum_{k=2}^{\infty}\frac{1}{(k+1/2)(k-1/2)} = -115 + 127 \times 2/3 < 0.
\end{aligned}$$

The lemma follows. ∎

Corollary 8.6. *For $t \in [0.93, 1.05]$, the Green's function for the Paneitz operator P_t satisfies $G_t(x,x) < 0$ for $x \in S^3$.*

Proof. First we observe that $G_1(x,x) = 0$ and $\partial_t|_{t=1} G_t(x,x) = 0$. Note that the second equation follows from the direct computation (8.1) and Lemma 4.2. Since $\zeta(t) = \frac{2\pi^2}{t^3} G_t(x,x)$, we see $\zeta(1) = \zeta'(1) = 0$. It follows from this and Lemma 8.10 that $\zeta(t) < 0$ for $t \in [0.93, 1.05]$. ∎

Now we are ready to prove Theorem 1.3.

Proof of Theorem 1.3. The case $t \in \left(0, \frac{-2+4\sqrt{10}}{13}\right) \cup \left(\frac{2+4\sqrt{10}}{13}, \infty\right)$ has been handled in Corollary 8.1.

Assume $t = \frac{-2+4\sqrt{10}}{13}$ or $\frac{2+4\sqrt{10}}{13}$. Given a metric in the conformal class of g_t with constant Q curvature, it may be written as $u^{-4} g_t$ for some $u \in C^\infty(S^3), u > 0$. Then $P_t u = c u^{-7}$ for some $c \in \mathbb{R}$. Hence $0 = \int_{S^3} P_t u \, d\mu_t = c \int_{S^3} u^{-7} d\mu_t$ and this implies $c = 0$. We have $u \in \ker P_t = \{\text{const}\}$. This gives the needed conclusion.

Now we concentrate on the case $t \in \left(\frac{-2+4\sqrt{10}}{13}, 1\right)$. The case $t \in \left(1, \frac{2+4\sqrt{10}}{13}\right)$ may be proved in a similar way. By Lemma 8.9 and Lemma 8.5, we have

$$
\begin{aligned}
G_t(x,x) &= \frac{1}{2\pi^2 t} \sum_{m,n \geq 0} \frac{m+n+1}{\lambda_{m,n}(t)} \\
&\leq \frac{1}{2\pi^2 t} \frac{1}{\lambda_{0,0}(t)} + \frac{\lambda_{1,1}(t)}{2\pi^2 t \lambda_{1,1}(1)} \sum_{\substack{m,n \geq 0 \\ m+n \geq 1}} \frac{m+n+1}{\lambda_{m,n}(1)} \\
&= \frac{1}{2\pi^2 t} \frac{1}{\lambda_{0,0}(t)} + \frac{\lambda_{1,1}(t)}{2\pi^2 t \lambda_{1,1}(1)} \sum_{k=1}^{\infty} \frac{(k+1)^2}{(k+5/2)(k+3/2)(k+1/2)(k-1/2)}.
\end{aligned}
$$

From this we see easily that $\lim_{t \to \frac{-2+4\sqrt{10}}{13}+0} G_t(x,x) = -\infty$.

We claim $G_t(x,x) < 0$ for any $t \in \left(\frac{-2+4\sqrt{10}}{13}, \frac{2\sqrt{37}}{13}\right)$. Indeed, if this is not the case, then for some $t \in \left(\frac{-2+4\sqrt{10}}{13}, \frac{2\sqrt{37}}{13}\right)$, we have $G_t(x,x) = 0$. But since g_t satisfies the condition (P) (see Corollary 8.3), we see $G_t(x,x) = E_t(G_t(x,\cdot), G_t(x,\cdot)) > 0$, a contradiction. Hence, in view of Corollary 8.6, we know $G_t(x,x) < 0$ for all $t \in \left(\frac{-2+4\sqrt{10}}{13}, 1\right)$ (note that $\frac{2\sqrt{37}}{13} \approx 0.9358$).

We claim that for any $t \in (0.93, 1)$, g_t satisfies the condition (P). If this is not the case, by Corollary 8.3, Lemma 5.1 and a continuity argument, we may find some $t \in (0.93, 1)$ such that g_t satisfies the condition (NN) but does not satisfies the condition (P). Hence we may find some $u \in H^2(S^3) \setminus \{0\}$ such that u vanishes somewhere and $E_t(u) = 0$. Assume $u(x) = 0$ for some $x \in S^3$, then it follows from Lemma 5.3 that this u must be the nonzero constant multiple of the Green's function $G_t(x,\cdot)$. In particular, this shows $0 = E_t(G_t(x,\cdot)) = G_t(x,x)$ a contradiction. Hence g_t does satisfy the condition (P) for $t \in (0.93, 1)$. Now we know for any $t \in \left(\frac{-2+4\sqrt{10}}{13}, 1\right)$, $G_t(x,x) < 0$ and g_t satisfies the condition (P). It follows easily that $G_t(x,y) < 0$ on $S^3 \times S^3$, because otherwise the value of the Green's function at the pole would be positive. By Theorem 1.1, we conclude the finiteness of $Y_4^3(S^3, g_t)$ and the existence of minimizing function in V. ∎

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