A Remark on the Jacobians [1]

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1 Introduction

Let $\Omega \subset \mathbb{R}^n$ be an open subset and suppose $u : \Omega \to \mathbb{R}^n$ is a differentiable map. Following [M1] we denote

(1.1) $$\det(Du) = \text{the usual determinant of the Jacobian matrix } Du,$$

(1.2) $$\mathrm{Det}(Du) = \partial_j(u^i(\mathrm{adj}Du)_i^j).$$

Here we use the standard summation convention and $\mathrm{adj}Du$ means the adjoint matrix of Du. If $u \in W^{1,n}(\Omega, \mathbb{R}^n)$, then $\det(Du)$ is a L^1 function on Ω and $\mathrm{Det}(Du)$ defined by (1.2) is well-defined in the sense of distribution. It is a well known fact that $\det(Du) = \mathrm{Det}(Du)$ when $u \in W^{1,n}(\Omega, \mathbb{R}^n)$, which can be proved by the smooth approximations. However the formula (1.2) is well-defined in the sense of distribution even for $u \in W^{1,n-1}(\Omega, \mathbb{R}^n) \cap L^\infty(\Omega)$ or $u \in W^{1,\frac{n^2}{n+1}}(\Omega, \mathbb{R}^n)$. An important question is to understand the range of the map $u \mapsto \mathrm{Det}(Du)$. If $\mathrm{Det}(Du)$ is a Radon measure, then from [M1] we know $\det(Du)$ is the regular part of $\mathrm{Det}(Du)$ and $\mathrm{Det}(Du) - \det(Du)$ is the singular part. It follows from the constructions in [M2] that the support of the singular part could be a closed set of arbitrary Hausdorff dimension between 0 and n. In other words there is essentially no restriction on the support of the singular part of this measure. But the situation is quite different when we consider maps $u \in W^{1,n-1}(\Omega, S^{n-1})$. In this case one may show that if $\mathrm{Det}(Du)$ is a signed measure of finite total mass, then it is a finite integer combination of Dirac masses, see also [BN1,2]. Hence it defines an integer multiplicity current. A higher dimensional version of this latter fact is the recent theorem of Jerrard and Soner [JS].

Theorem of Jerrard and Soner *For any $u \in W^{1,n-1}(\Omega, S^{n-1})$, if $Jac(u)$ is a vector-valued measure with finite total mass, then it is an integer multiplicity rectifiable current.*

The proof of this theorem follows essentially from the fact just mentioned above for the maps $u \in W^{1,n-1}_{loc}(\mathbb{R}^n, S^{n-1})$ and the slicing criterion for rectifiability of normal currents established recently by Ambrosio and Kirchheim [AK]. There is also a simple and direct proof due to

[1] Newly modified in April, 2001

Giaquinta and Modica. For detailed discussion of all these results we refer to [JS]. Note that $\det(Du) = 0$ in the case that the image of the maps are in the unit sphere. $\mathrm{Det}(Du)$ is therefore singular respect to Lebesgue measure and will be denoted as $\mathrm{Sing}(u)$ later. In this paper we will generalize the above fact to higher codimension case for a class of fractional order Sobolev space. We are interested in this class also because it is related to the study of asymptotic of Ginzburg-Landau minimizers. Let us describe the set up of the problem. Assume $m \geq n \geq 2$, $m, n \in \mathbb{N}$, $\mathbb{R}_+^{m+1} = \mathbb{R}^m \times (0, \infty)$, $g \in W_{loc}^{1-\frac{1}{n},n}(\mathbb{R}^m, S^{n-1})$. We define $\mathrm{Sing}(g)$ to be a $(m-n)$ current as follows. We choose an extension $u \in W_{loc}^{1,n}(\mathbb{R}_+^{m+1}, \mathbb{R}^n)$ such that $u|_{\mathbb{R}^m} = g$ and for any τ, a smooth $(m-n)$ form on $\mathbb{R}^m$ with compact support, we choose $\tilde{\tau}$, a smooth $(m-n)$ form on $\mathbb{R}^{m+1}$ with compact support, such that $\tilde{\tau}|_{\mathbb{R}^m} = \tau$, then one defines

$$\langle \mathrm{Sing}(g), \tau \rangle = \frac{1}{\omega_n} \int_{\mathbb{R}_+^{m+1}} d\tilde{\tau} \wedge u^*(dy^1 \wedge \cdots \wedge dy^n), \tag{1.3}$$

where ω_n denotes the volume of the unit ball in $\mathbb{R}^n$. Using the Stokes' theorem, one can easily verify that $\mathrm{Sing}(g)$ is a well-defined $(m-n)$ current. The Similar definition as (1.3) was introduced already in the earlier work of Brezis-Coron-Lieb [BCL] for finite energy maps from a three dimensional domain into the standard 2-sphere. Our definition was motivated by the recent work of Riviere [Ri] and that of Bourgain-Brezis-Mironescu [BBM]. For any $p \in [1, \infty)$, $s \in (0,1)$ and f a measurable function, one defines

$$[f]_{s,p,\mathbb{R}^m} = \Big(\int_{\mathbb{R}^m} \int_{\mathbb{R}^m} \frac{|f(x) - f(y)|^p}{|x-y|^{m+sp}} dx dy \Big)^{\frac{1}{p}}. \tag{1.4}$$

In fact this is the seminorm of fractional order Sobolev spaces, see p214 of [A]. We have the following theorem,

Theorem 1.1 *Suppose $m \geq n \geq 2$, $g : \mathbb{R}^m \to S^{n-1}$ such that $[g]_{1-\frac{1}{n},n,\mathbb{R}^m} < \infty$, then there exists an integer multiplicity $m-n+1$ current J in $\overline{\mathbb{R}_+^{m+1}}$ such that $\partial J = Sing(g)$ and the mass of J is bounded by $c(m,n)[g]^n_{1-\frac{1}{n},n,\mathbb{R}^m}$.*

We call J in the theorem the connecting current. The theorem has the following important corollary which implies the rectifiability of the support of the $\mathrm{Sing}(g)$,

Corollary 1.1 *Under the assumption of the theorem, if Sing(g) is a vector-valued measure with finite total mass, then it is an integer multiplicity rectifiable current.*

We remark that all the proofs given in the paper could be localized, so we may have a local version of the corollary too. But we will not elaborate further these minor changes. We note that although our result has lower order requirement on differentiability, Jerrard and Soner's result is not contained in ours in view of the embedding theorem in chapter 7 of [A]. On the other hand, our result is more natural for the boundary value problems.

Added in April, 2001 : It follows from the generalized Gagliardo-Nirenberg inequality (Lemma D.1 and Remark D.2 in [BBM]) that for $\Omega \subset \mathbb{R}^m$*, an open bounded smooth subset,* $n \geq 3$*, we have*

$$W^{1,n-1}(\Omega, S^{n-1}) \subset W^{1-\frac{1}{n},n}(\Omega, S^{n-1}),$$

hence when $n \geq 3$*, the result of Jerrard and Soner mentioned above is contained in Corollary 1.1. Unfortunately the inclusion above is not true for the case* $n = 2$*, that is when the target is* S^1*. But we have,*

$$W^{1,1+\varepsilon}(\Omega, S^1) \subset W^{\frac{1}{2},2}(\Omega, S^1) \quad \textit{for any } \varepsilon > 0.$$

The paper is written as follows. In section 2 below, we describe a few facts related to the harmonic extensions of functions. Section 3 gives the proof of theorem 1.1 by an elementary method, which only uses the co-area formula. In section 4, we give another proof which fits in the degree theory developed in [BN1,2]. There we present an adaption of a proof from [Ri].

2 Harmonic functions on the upper half space

In this section we will present some basic properties of harmonic extensions to the upper half space of certain fractional order Sobolev functions. These results are elementary, but we give the proofs here for reader's convenience. Let $n \in \mathbb{N}$, recall for any $t > 0$, $x \in \mathbb{R}^n$ the Poisson kernel

$$P(x,t) = P_t(x) = \frac{c_n t}{(|x|^2 + t^2)^{\frac{n+1}{2}}}, \quad c_n = \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}}. \tag{2.1}$$

Here Γ denotes the special gamma function. We know $\int_{\mathbb{R}^n} P_t(x)dx = 1$, see chapter 3 of [St]. For $f \in L^1(\mathbb{R}^n) + L^\infty(\mathbb{R}^n)$, we call $u(x,t) = \int_{\mathbb{R}^n} P_t(x-y)f(y)dy$ the Poisson integral of f.

Proposition 2.1 *Suppose* $1 < p < \infty$*,* $g \in L^1(\mathbb{R}^n) + L^\infty(\mathbb{R}^n)$ *satisfies* $[g]_{1-\frac{1}{p},p,\mathbb{R}^n} < \infty$*, then* $|P_t * g - g|_{L^p(\mathbb{R}^n)} = o(t^{1-\frac{1}{p}})$ *as* $t \to 0+0$*.*

Proof. First denote for any $y \in \mathbb{R}^n$

$$\omega_g(y) = \Big(\int_{\mathbb{R}^n} |g(x+y) - g(x)|^p dx \Big)^{\frac{1}{p}}. \tag{2.2}$$

We have

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|g(x)-g(y)|^p}{|x-y|^{n+p-1}}dxdy=\int_{\mathbb{R}^n}dx\int_{\mathbb{R}^n}\frac{|g(x+y)-g(x)|^p}{|y|^{n+p-1}}dy \tag{2.3}$$

$$=\int_{\mathbb{R}^n}\Big(\frac{1}{|y|^{n+p-1}}\int_{\mathbb{R}^n}|g(x+y)-g(x)|^pdx\Big)dy=\int_{\mathbb{R}^n}\frac{\omega_g(y)^p}{|y|^{n+p-1}}dy.$$

$$|(P_t*g)(x)-g(x)|=|\int_{\mathbb{R}^n}P_t(y)g(x-y)dy-\int_{\mathbb{R}^n}P_t(y)g(x)dy| \tag{2.4}$$

$$\le\int_{\mathbb{R}^n}P_t(y)|g(x-y)-g(x)|dy.$$

So

$$|P_t*g-g|_{L^p}\le\int_{\mathbb{R}^n}P_t(y)\Big(\int_{\mathbb{R}^n}|g(x-y)-g(x)|^pdx\Big)^{\frac{1}{p}}dy \tag{2.5}$$

$$=\int_{\mathbb{R}^n}P_t(y)\omega_g(-y)dy=\int_{\mathbb{R}^n}P_t(y)\omega_g(y)dy.$$

Denote $e_1=(1,0,\cdots,0)\in\mathbb{R}^n$, the first direction.

$$\int_{\mathbb{R}^n}P_t(y)\omega_g(y)dy=\sum_{j\in\mathbb{Z}}\int_{B_{2^{j+1}}\setminus B_{2^j}}P_t(y)\omega_g(y)dy \tag{2.6}$$

$$\le\sum_{j\in\mathbb{Z}}P_t(2^je_1)\int_{B_{2^{j+1}}\setminus B_{2^j}}\omega_g(y)dy\le\sum_{j\in\mathbb{Z}}P_t(2^je_1)\Big(\int_{B_{2^{j+1}}\setminus B_{2^j}}\omega_g(y)^pdy\Big)^{\frac{1}{p}}(c(n)2^{jn})^{\frac{1}{p'}},$$

here $p'=\frac{p}{p-1}$. For any $r>0$, put

$$\varepsilon(r)=\frac{1}{r^{n+p-1}}\int_{B_{2r}\setminus B_r}\omega_g(y)^pdy, \tag{2.7}$$

then

$$\varepsilon(r)\le 2^{n+p-1}\int_{B_{2r}\setminus B_r}\frac{\omega_g(y)^p}{|y|^{n+p-1}}dy\le 2^{n+p-1}[g]^p_{1-\frac{1}{p},p,\mathbb{R}^n}. \tag{2.8}$$

Hence $\varepsilon(r)$ is bounded and $\varepsilon(r)\to 0$ as $r\to 0+0$ or $r\to\infty$. From (2.6) we know

$$\int_{\mathbb{R}^n}P_t(y)\omega_g(y)dy\le c(n,p)\sum_{j\in\mathbb{Z}}P_t(2^je_1)\varepsilon(2^j)^{\frac{1}{p}}2^{jn}\Big(\frac{2^j}{t}\Big)^{1-\frac{1}{p}}t^{1-\frac{1}{p}}. \tag{2.9}$$

Put

$$Q(x) = P_1(x)|x|^{1-\frac{1}{p}} = \frac{c_n|x|^{1-\frac{1}{p}}}{(1+|x|^2)^{\frac{n+1}{2}}}, \tag{2.10}$$

then $\int_{\mathbb{R}^n} Q(x)dx < \infty$. $Q_t(x) = \frac{1}{t^n}Q(\frac{x}{t}) = P_t(x)|\frac{x}{t}|^{1-\frac{1}{p}}$. From (2.9) we get

$$\int_{\mathbb{R}^n} P_t(y)\omega_g(y)dy \le c(n,p)\sum_{j\in\mathbb{Z}} \varepsilon(2^j)^{\frac{1}{p}} 2^{jn} Q_t(2^j e_1) t^{1-\frac{1}{p}} \tag{2.11}$$

$$\le c(n,p)t^{1-\frac{1}{p}} \sum_{j\in\mathbb{Z}} \varepsilon(2^j)^{\frac{1}{p}} \int_{B_{2^j}\setminus B_{2^{j-1}}} Q_t(y)dy = c(n,p)t^{1-\frac{1}{p}} \int_{\mathbb{R}^n} Q_t(y)\varepsilon(2^{[\log_2|y|]+1})^{\frac{1}{p}}dy$$

$$= c(n,p)t^{1-\frac{1}{p}} \int_{\mathbb{R}^n} Q(y)\varepsilon(2^{[\log_2|ty|]+1})^{\frac{1}{p}}dy.$$

Now using the fact $\varepsilon(2^{[\log_2|ty|]+1}) \le 2^{n+p-1}[g]^p_{1-\frac{1}{p},p,\mathbb{R}^n}$ and $\varepsilon(2^{[\log_2|ty|]+1}) \to 0$ as $t \to 0+0$, from the Lebesgue dominated convergence theorem we know $|P_t * g - g|_{L^p} = o(t^{1-\frac{1}{p}})$ as $t \to 0+0$. **Q.E.D.**

Proposition 2.2 *Suppose* $1 < p < \infty$, $g \in L^1(\mathbb{R}^n) + L^\infty(\mathbb{R}^n)$ *satisfies* $[g]_{1-\frac{1}{p},p,\mathbb{R}^n} < \infty$, *let* u *be the Poisson integral of* g, *then we have* $(\int_{\mathbb{R}^{n+1}_+} |\nabla u(x,t)|^p dxdt)^{\frac{1}{p}} \le c(n,p)[g]_{1-\frac{1}{p},p,\mathbb{R}^n}$.

Proof. At first we have

$$u(x,t) = \int_{\mathbb{R}^n} P(y,t)g(x-y)dy = \int_{\mathbb{R}^n} P(y,t)(g(x-y)-g(x))dy + g(x). \tag{2.12}$$

So

$$\partial_{n+1}u(x,t) = \int_{\mathbb{R}^n} \partial_{n+1}P(y,t)(g(x-y)-g(x))dy \tag{2.13}$$

$$= c_n \int_{\mathbb{R}^n} \frac{|y|^2 - nt^2}{(|y|^2+t^2)^{\frac{n+3}{2}}}(g(x-y)-g(x))dy.$$

By simple computation we know $\frac{np-n+1}{(n+1)p} > \frac{n}{(n+1)p'}$, we may choose $\theta \in (\frac{n}{(n+1)p'}, \frac{np-n+1}{(n+1)p})$, say the middle point of the interval.

$$|\partial_{n+1}u(x,t)| \le c(n) \int_{\mathbb{R}^n} \frac{||y|^2 - nt^2|}{(|y|^2+t^2)^{\frac{n+3}{2}}}|g(x-y)-g(x)|dy \tag{2.14}$$

$$\le c(n)\Big(\int_{\mathbb{R}^n}\Big(\frac{||y|^2-nt^2|}{(|y|^2+t^2)^{\frac{n+3}{2}}}\Big)^{(1-\theta)p}|g(x-y)-g(x)|^p dy\Big)^{\frac{1}{p}}\Big(\int_{\mathbb{R}^n}\Big(\frac{||y|^2-nt^2|}{(|y|^2+t^2)^{\frac{n+3}{2}}}\Big)^{p'\theta}dy\Big)^{\frac{1}{p'}}.$$

Now

$$\int_{\mathbb{R}^n}\Big(\frac{||y|^2-nt^2|}{(|y|^2+t^2)^{\frac{n+3}{2}}}\Big)^{p'\theta}dy = t^{n-(n+1)p'\theta}\int_{\mathbb{R}^n}\Big(\frac{||y|^2-n|}{(|y|^2+1)^{\frac{n+3}{2}}}\Big)^{p'\theta}dy \tag{2.15}$$

$$= c(n,p)t^{n-(n+1)p'\theta}.$$

Here we use the fact $(n+1)p'\theta > n$. So

$$|\partial_{n+1}u(x,t)| \tag{2.16}$$

$$\le c(n,p)t^{\frac{n}{p'}-(n+1)\theta}\Big(\int_{\mathbb{R}^n}\Big(\frac{||y|^2-nt^2|}{(|y|^2+t^2)^{\frac{n+3}{2}}}\Big)^{(1-\theta)p}|g(x-y)-g(x)|^p dy\Big)^{\frac{1}{p}}.$$

$$\int_{\mathbb{R}^n}dx\int_0^\infty|\partial_{n+1}u(x,t)|^p dt \tag{2.17}$$

$$\le c(n,p)\int_{\mathbb{R}^n}dx\int_0^\infty dt\int_{\mathbb{R}^n}\Big(\frac{||y|^2-nt^2|}{(|y|^2+t^2)^{\frac{n+3}{2}}}\Big)^{p(1-\theta)}t^{\frac{np}{p'}-(n+1)p\theta}|g(x-y)-g(x)|^p dy$$

$$= c(n,p)\int_{\mathbb{R}^n}\frac{\omega_g(-y)^p}{|y|^{n+p-1}}\Big(\int_0^\infty t^{\frac{np}{p'}-(n+1)p\theta}\Big(\frac{|1-nt^2|}{(1+t^2)^{\frac{n+3}{2}}}\Big)^{p(1-\theta)}dt\Big)dy,$$

By the choice of θ we know

$$\int_0^\infty t^{\frac{np}{p'}-(n+1)p\theta}\Big(\frac{|1-nt^2|}{(1+t^2)^{\frac{n+3}{2}}}\Big)^{p(1-\theta)}dt<\infty, \tag{2.18}$$

From (2.3), (2.17) and (2.18) we get

$$\int_{\mathbb{R}^n}dx\int_0^\infty|\partial_{n+1}u(x,t)|^p dt \le c(n,p)\int_{\mathbb{R}^n}\frac{\omega_g(-y)^p}{|y|^{n+p-1}}dy \tag{2.19}$$

$$= c(n,p)\int_{\mathbb{R}^n}\frac{\omega_g(y)^p}{|y|^{n+p-1}}dy = c(n,p)[g]^p_{1-\frac{1}{p},p,\mathbb{R}^n}.$$

On the other hand,

$$\nabla' P(x,t) = -(n+1)c_n\frac{tx}{(|x|^2+t^2)^{\frac{n+3}{2}}}, \tag{2.20}$$

where ∇' means the derivatives which only involve the first n directions.

$$\nabla' u(x,t) = \int_{\mathbb{R}^n} \nabla' P(x-y,t) g(y) dy \tag{2.21}$$

$$= \int_{\mathbb{R}^n} \nabla' P(y,t) g(x-y) dy = \int_{\mathbb{R}^n} \nabla' P(y,t)(g(x-y)-g(x)) dy.$$

Hence

$$|\nabla' u(x,t)| \le c(n) \int_{\mathbb{R}^n} \frac{t|x|}{(|y|^2+t^2)^{\frac{n+3}{2}}} |g(x-y)-g(x)| dy. \tag{2.22}$$

Choose the same θ as before, then

$$|\nabla' u(x,t)| \tag{2.23}$$

$$\le c(n,p) \Big(\int_{\mathbb{R}^n} \Big(\frac{t|y|}{(|y|^2+t^2)^{\frac{n+3}{2}}} \Big)^{p(1-\theta)} |g(x-y)-g(x)|^p dy \Big)^{\frac{1}{p}} \Big(\int_{\mathbb{R}^n} \Big(\frac{t|y|}{(|y|^2+t^2)^{\frac{n+3}{2}}} \Big)^{p'\theta} dy \Big)^{\frac{1}{p'}}.$$

Now

$$\int_{\mathbb{R}^n} \Big(\frac{t|y|}{(|y|^2+t^2)^{\frac{n+3}{2}}} \Big)^{p'\theta} dy = t^{n-(n+1)p'\theta} \int_{\mathbb{R}^n} \Big(\frac{|y|}{(1+|y|^2)^{\frac{n+3}{2}}} \Big)^{p'\theta} dy \tag{2.24}$$

$$= c(n,p) t^{n-(n+1)p'\theta},$$

because $(n+2)p'\theta > n$. This and (2.23) imply

$$\int_{\mathbb{R}^n} dx \int_0^\infty |\nabla' u(x,t)|^p dt \tag{2.25}$$

$$\le c(n,p) \int_{\mathbb{R}^n} dy \int_{\mathbb{R}^n} dx \int_0^\infty t^{\frac{np}{p'}-(n+1)p\theta} \Big(\frac{t|y|}{(|y|^2+t^2)^{\frac{n+3}{2}}} \Big)^{p(1-\theta)} |g(x-y)-g(x)|^p dt$$

$$= c(n,p) \int_{\mathbb{R}^n} \frac{\omega_g(-y)^p}{|y|^{n+p-1}} \Big(\int_0^\infty t^{\frac{np}{p'}-(n+1)p\theta} \Big(\frac{t}{(1+t^2)^{\frac{n+3}{2}}} \Big)^{p(1-\theta)} dt \Big) dy,$$

from the choice of θ, we observe that

$$\int_0^\infty t^{\frac{np}{p'}-(n+1)p\theta} \Big(\frac{t}{(1+t^2)^{\frac{n+3}{2}}} \Big)^{p(1-\theta)} dt < \infty, \tag{2.26}$$

From (2.3), (2.25) and (2.26) we get

$$\int_{\mathbb{R}^n} dx \int_0^\infty |\nabla' u(x,t)|^p dt \le c(n,p) \int_{\mathbb{R}^n} \frac{\omega_g(-y)^p}{|y|^{n+p-1}} dy \tag{2.27}$$

$$= c(n,p) \int_{\mathbb{R}^n} \frac{\omega_g(y)^p}{|y|^{n+p-1}} dy = c(n,p) [g]^p_{1-\frac{1}{p},p,\mathbb{R}^n}.$$

(2.19) and (2.27) together imply the proposition. **Q.E.D.**

Remark 2.1 *Suppose we replace the $\mathbb{R}^n$ by S^n, and $\mathbb{R}^{n+1}_+$ by B^{n+1}_1, then we may prove similar results by the Poisson integral formula for balls. The case $p = 2$ was done in [Ri], where the expansion in terms of spherical harmonic functions was used. This kind of method works well only for $p = 2$ case. For reasons we refer to chapter 12 of [Z].*

3 Finding the connecting current

In this section, we give the way to find the connecting current for $\mathrm{Sing}(g)$ in the Theorem 1.1.

Proof of Theorem 1.1 Suppose we are given g as in the theorem, let u be the Poisson integral of g, then $|u(x,t)| \leq 1$ and we know from Proposition 2.1 and 2.2 that

$$(3.1)\quad \int_{\mathbb{R}^{m+1}_+} |\nabla u|^n dH^{m+1} \leq c(m,n)[g]^n_{1-\frac{1}{n},n,\mathbb{R}^m}, \quad \int_{\mathbb{R}^m} |u(x,t)-g(x)|^n dH^m(x) = o(t^{n-1}).$$

From the co-area formula we know

$$(3.2)\quad \int_{B^n_1} H^{m-n+1}(u^{-1}(a))dH^n(a) = \int_{\mathbb{R}^{m+1}_+} |\mathrm{Jac}(u)|dH^{m+1}$$

$$\leq c(m,n)\int_{\mathbb{R}^{m+1}_+} |\nabla u|^n dH^{m+1} \leq c(m,n)[g]^n_{1-\frac{1}{n},n,\mathbb{R}^m} < \infty.$$

Hence for a.e. $a \in B^n_1$, $H^{m-n+1}(u^{-1}(a)) < \infty$. By Sard theorem we may also assume such a is a regular value of u in $\mathbb{R}^{m+1}_+$.

Claim: For generic a and b in B^n_1, for any $\tilde{\tau}$, a smooth $(m-n)$ form on $\mathbb{R}^{m+1}$ with compact support, we have $\int_{u^{-1}(a)} d\tilde{\tau} = \int_{u^{-1}(b)} d\tilde{\tau}$.

Proof of the claim. For any $\theta \in (0,1)$, define $E_{t,\theta} = \{(x,t)\backslash x \in \mathbb{R}^m, |u(x,t)| < \theta\}$. Choose any $e \in S^{n-1}$, denote π_e as the orthogonal projection $\pi_e(\xi) = \xi - (\xi\cdot e)e$, for any $\xi \in \mathbb{R}^n$. Define $u_e(x,t) = \pi_e(u(x,t))$, then by co-area formula we have

$$(3.3)\quad \int_{B^{n-1}_\theta} H^{m-n+1}\Big(u^{-1}(\pi_e^{-1}(b)\cap B^n_\theta)\cap(\mathbb{R}^m\times\{t\})\Big)dH^{n-1}(b)$$

$$= \int_{\mathbb{R}^m\times\{t\}} |\mathrm{Jac}(u_e|_{\mathbb{R}^m\times\{t\}})|\chi_{E_{t,\theta}} dH^m \leq c(m,n)\int_{\mathbb{R}^m\times\{t\}} \chi_{E_{t,\theta}}|\nabla u|^{n-1}dH^m$$

$$\leq c(m,n)\Big(\int_{\mathbb{R}^m\times\{t\}} |\nabla u|^n dH^m\Big)^{\frac{n-1}{n}} H^m(E_{t,\theta})^{\frac{1}{n}}.$$

For any $r > 0$, we have

$$\frac{1}{r}\int_r^{2r} dt \int_{\mathbb{R}^m} t|\nabla u(x,t)|^n dx \leq 2\int_r^{2r} dt \int_{\mathbb{R}^m} |\nabla u(x,t)|^n dx \tag{3.4}$$

$$\leq 2\int_{\mathbb{R}_+^{m+1}} |\nabla u|^n dH^{m+1} < \infty.$$

Define

$$E_r = \{t\backslash r < t < 2r \text{ such that } \int_{\mathbb{R}^m} t|\nabla u(x,t)|^n dx > 4\Big(\int_{\mathbb{R}_+^{m+1}} |\nabla u|^n dH^{m+1} + 1\Big)\}, \tag{3.5}$$

then we have $|E_r| < \frac{r}{2}$. Define

$$E = \bigcup_{j\in\mathbb{Z}} \Big((2^j, 2^{j+1})\backslash E_{2^j}\Big). \tag{3.6}$$

For any $t \in E$, we have

$$\int_{\mathbb{R}^m} |\nabla u(x,t)|^n dx \leq \frac{4}{t}\Big(\int_{\mathbb{R}_+^{m+1}} |\nabla u|^n dH^{m+1} + 1\Big). \tag{3.7}$$

Since $|g(x)| = 1$, we have

$$H^m(E_{t,\theta}) \leq \frac{1}{(1-\theta)^n}\int_{\mathbb{R}^m} |u(x,t) - g(x)|^n dH^m(x). \tag{3.8}$$

(3.1) and (3.8) imply

$$H^m(E_{t,\theta}) = o(t^{n-1}) \text{ as } t \to 0+0. \tag{3.9}$$

For $t \in E$, from (3.3), (3.7) and (3.9) we get

$$\int_{B_\theta^{n-1}} H^{m-n+1}\Big(u^{-1}(\pi_e^{-1}(b) \cap B_\theta^n) \cap (\mathbb{R}^m \times \{t\})\Big) dH^{n-1}(b) \tag{3.10}$$

$$\leq c(m,n)\Big(\frac{4(\int_{\mathbb{R}_+^{m+1}} |\nabla u|^n dH^{m+1} + 1)}{t}\Big)^{\frac{n-1}{n}} o(t^{\frac{n-1}{n}}) = o(1).$$

Choose a sequence $t_j \in E$, $t_j \to 0$ such that $H^{m-n+1}\Big(u^{-1}(\pi_e^{-1}(b) \cap B_\theta^n) \cap (\mathbb{R}^m \times \{t_j\})\Big) \to 0$, H^{n-1} a.e. $b \in B_\theta^{n-1}$. Note $\pi_e^{-1}(b)$ is a line. If we have a_1, a_2 such that $|a_1|, |a_2| < \theta$ and

$\pi_e(a_1) = \pi_e(a_2) = b$ for a generic b, denote $[a_1, a_2] = \{(1-\lambda)a_1 + \lambda a_2 \backslash 0 \le \lambda \le 1\}$. From Stokes formula we have

$$0 = \int_{\partial\left(\left(u|_{\mathbb{R}^m\times[t_j,\infty)}\right)^{-1}\left([a_1,a_2]\right)\right)} d\tilde{\tau} \tag{3.11}$$

$$= \int_{u^{-1}(a_2)\cap\left(\mathbb{R}^m\times(t_j,\infty)\right)} d\tilde{\tau} - \int_{u^{-1}(a_1)\cap\left(\mathbb{R}^m\times(t_j,\infty)\right)} d\tilde{\tau} + \int_{u^{-1}\left([a_1,a_2]\right)\cap\left(\mathbb{R}^m\times\{t_j\}\right)} d\tilde{\tau}.$$

but

$$|\int_{u^{-1}\left([a_1,a_2]\right)\cap\left(\mathbb{R}^m\times\{t_j\}\right)} d\tilde{\tau}| \le c(\tilde{\tau})H^{m-n+1}\left(u^{-1}(\pi_e^{-1}(b)\cap B_\theta^n)\cap(\mathbb{R}^m \times \{t_j\})\right) \to 0 \tag{3.12}$$

as $j \to \infty$. Take a limit on both sides of (3.11) we get $\int_{u^{-1}(a_1)} d\tilde{\tau} = \int_{u^{-1}(a_2)} d\tilde{\tau}$. By choosing e as the n coordinate directions also noticing that θ could be arbitrary close to 1, we have $\int_{u^{-1}(a_1)} d\tilde{\tau} = \int_{u^{-1}(a_2)} d\tilde{\tau}$ for generic $a_1, a_2 \in B_1^n$. This proves the claim.

Now let us continue the proof of the theorem, choose a generic a_0, put $J = u^{-1}(a_0)$. From (3.2) we may assume $H^{m-n+1}(u^{-1}(a_0)) \le c(m,n)[g]^n_{1-\frac{1}{n},n,\mathbb{R}^m}$. For a generic a, we know $\int_{u^{-1}(a)} d\tilde{\tau} = \int_J d\tilde{\tau}$. For any smooth $(m-n)$ form τ on $\mathbb{R}^m$ with compact support, choose a smooth $(m-n)$ form $\tilde{\tau}$ on $\mathbb{R}^{m+1}$ with compact support such that $\tilde{\tau}|_{\mathbb{R}^m} = \tau$, then by (1.3) and the co-area formula, we have

$$\langle \mathrm{Sing}(g), \tau\rangle = \frac{1}{\omega_n}\int_{\mathbb{R}^{m+1}_+} d\tilde{\tau} \wedge u^*(dy^1 \wedge \cdots \wedge dy^n) \tag{3.13}$$

$$= \frac{1}{\omega_n}\int_{B_1^n} dH^n(a)\int_{u^{-1}(a)} d\tilde{\tau} = \frac{1}{\omega_n}\int_{B_1^n}\left(\int_J d\tilde{\tau}\right)dH^n(a)$$

$$= \int_J d\tilde{\tau} = \langle \partial J, \tau\rangle.$$

This proves the theorem. **Q.E.D.**

Proof of Corollary 1.1 From theorem 1.1 we know there exists an integer multiplicity rectifiable current J of finite mass as the connecting current, now suppose $\mathrm{Sing}(g)$ is a measure, it follows from the boundary rectifiability theorem in section 30 of [S] that $\mathrm{Sing}(g)$ is also an integer multiplicity current. **Q.E.D.**

4 Second approach to Theorem 1.1

We note that the proof in the former section is elementary in the sense that it only uses co-area formula. In this section we will give another approach which uses the degree theory developed in [BN1,2]. The proof is adapted from a proof given in [Ri] for the case $m = n = 2$.

Proof of Theorem 1.1 Suppose we are given g as in the theorem, let u be the Poisson integral of g, then $|u(x,t)| \le 1$ and we know from Proposition 2.2 that

$$\int_{\mathbb{R}_+^{m+1}} |\nabla u|^n dH^{m+1} \le c(m,n)[g]^n_{1-\frac{1}{n},n,\mathbb{R}^m}. \tag{4.1}$$

From the co-area formula we know

$$\int_{B_1^n} H^{m-n+1}(u^{-1}(a))dH^n(a) = \int_{\mathbb{R}_+^{m+1}} |\mathrm{Jac}(u)|dH^{m+1} \tag{4.2}$$

$$\le c(m,n)\int_{\mathbb{R}_+^{m+1}} |\nabla u|^n dH^{m+1} \le c(m,n)[g]^n_{1-\frac{1}{n},n\mathbb{R}^m} < \infty.$$

Choose a generic a_0, for which we may assume $H^{m-n+1}(u^{-1}(a_0)) \le c(m,n)[g]^n_{1-\frac{1}{n},n,\mathbb{R}^m}$ by (4.2) and it is a regular value of u in $\mathbb{R}_+^{m+1}$. Denote $J = u^{-1}(a_0)$. Given any τ, a smooth $(m-n)$ form with compact support on $\mathbb{R}^m$, choose $\tilde{\tau}$, a smooth $(m-n)$ form with compact support in $\mathbb{R}^{m+1}$ such that $\tilde{\tau}|_{\mathbb{R}^m} = \tau$, we want to show

$$\frac{1}{\omega_n}\int_{\mathbb{R}_+^{m+1}} d\tilde{\tau} \wedge u^*(dy^1 \wedge \cdots \wedge dy^n) = \int_J d\tilde{\tau}. \tag{4.3}$$

Because both sides are linear in $\tilde{\tau}$, we may assume $\tilde{\tau} = f^1 df^2 \wedge \cdots \wedge df^{m-n+1}$, where $f^1, \ldots, f^{m-n+1}$ are smooth functions on $\mathbb{R}^{m+1}$ with compact support. Define

$$F = (f^1, \cdots, f^{m-n+1})$$

as a map from $\overline{\mathbb{R}_+^{m+1}}$ to $\mathbb{R}^{m-n+1}$. Then

$$\frac{1}{\omega_n}\int_{\mathbb{R}_+^{m+1}} d\tilde{\tau} \wedge u^*(dy^1 \wedge \cdots \wedge dy^n) \tag{4.4}$$

$$= (-1)^{mn}\frac{1}{\omega_n}\int_{\mathbb{R}_+^{m+1}} u^*(dy^1 \wedge \cdots \wedge dy^n) \wedge df^1 \wedge \cdots \wedge df^{m-n+1}$$

$$= (-1)^{mn} \frac{1}{\omega_n} \int_{\mathbb{R}^{m-n+1}} da \int_{F^{-1}(a)} u^*(dy^1 \wedge \cdots \wedge dy^n).$$

We claim that for a.e. $a \in \mathbb{R}^{m-n+1}$,

$$\frac{1}{\omega_n} \int_{F^{-1}(a)} u^*(dy^1 \wedge \cdots \wedge dy^n) = F^{-1}(a) \wedge J, \tag{4.5}$$

where $F^{-1}(a) \wedge J$ is the algebraic intersection number between $F^{-1}(a)$ and J. For a.e. a, using co-area formula , $u \in W^{1,n}(F^{-1}(a), \mathbb{R}^n)$, thus $g \in W^{1-\frac{1}{n},n}(\partial F^{-1}(a), \mathbb{R}^n)$. Because F is compactly supported, we know for $a \neq 0$, $F^{-1}(a)$ is compact. From [BN1,2] we know

$$\deg_{\partial F^{-1}(a)} g = \frac{1}{\omega_n} \int_{F^{-1}(a)} u^*(dy^1 \wedge \cdots \wedge dy^n). \tag{4.6}$$

On the other hand

$$\deg_{\partial F^{-1}(a)} g = F^{-1}(a) \wedge u^{-1}(a_0) = F^{-1}(a) \wedge J. \tag{4.7}$$

So we get (4.5). From (4.4) and (4.5) we have

$$\frac{1}{\omega_n} \int_{\mathbb{R}^{m+1}_+} d\tilde{\tau} \wedge u^*(dy^1 \wedge \cdots \wedge dy^n) = \int_{\mathbb{R}^{m-n+1}} J \wedge F^{-1}(a) da \tag{4.8}$$

$$= \int_J df^1 \wedge \cdots \wedge df^{m-n+1} = \int_J d\tilde{\tau}.$$

This implies (4.3). Now in view of (1.3), we know

$$\langle \mathrm{Sing}(g), \tau \rangle = \langle \partial J, \tau \rangle.$$

Q.E.D.

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