EXISTENCE OF FADDEEV KNOTS IN GENERAL HOPF DIMENSIONS

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Abstract. In this paper, we present an existence theory for absolute minimizers of the Faddeev knot energies in the general Hopf dimensions. These minimizers are topologically classified by the Hopf-Whitehead invariant, Q, represented as an integral of the Chern-Simons type. Our method involves an energy decomposition relation and a fractionally powered universal topological growth law. We prove that there is an infinite subset $\mathbb{S}$ of the set of all integers such that for each $N \in \mathbb{S}$ there exists an energy minimizer in the topological sector $Q = N$. In the compact setting, we show that there exists an absolute energy minimizer in the topological sector $Q = N$ for any given integer N that may be realized as a Hopf-Whitehead number. We also obtain a precise energy-splitting relation and an existence result for the Skyrme model.

1. Introduction

In knot theory, an interesting problem concerns the existence of "ideal knots", which promises to provide a natural link between the geometric and topological contents of knotted structures. This problem has its origin in theoretical physics in which one wants to ask the existence and predict the properties of knots "based on a first principle approach" [N]. In other words, one is interested in determining the detailed physical characteristics of a knot such as its energy (mass), geometric conformation, and topological identification, via conditions expressed in terms of temperature, viscosity, electromagnetic, nuclear, and possibly gravitational, interactions, which is also known as an Hamiltonian approach to realizing knots as field-theoretical stable solitons. Based on high-power computer simulations, Faddeev and Niemi [FN1] carried out such a study on the existence of knots in the Faddeev quantum field theory model [F1]. Later, Faddeev addressed the existence problem and noted the mathematical challenges it gives rise to [F2]. The purpose of the present work is to develop a systematic existence theory of these Faddeev knots in their most general settings.

Recall that for the classical Faddeev model [BS1, BS2, F1, F2, FN1, FN2, Su] formulated over the standard $(3+1)$-dimensional Minkowski space of signature $(+---)$, the Lagrangian action density in normalized form reads

$$\mathcal{L} = \partial_\mu u \cdot \partial^\mu u - \frac{1}{2} F_{\mu\nu}(u) F^{\mu\nu}(u), \tag{1.1}$$

where the field $u = (u_1, u_2, u_3)$ assumes its values in the unit 2-sphere and

$$F_{\mu\nu}(u) = u \cdot (\partial_\mu u \wedge \partial_\nu u) \tag{1.2}$$

is the induced "electromagnetic" field. Since u is parallel to $\partial_\mu u \wedge \partial_\nu u$, it is seen that $F_{\mu\nu}(u)F^{\mu\nu}(u) = (\partial_\mu u \wedge \partial_\nu u) \cdot (\partial^\mu u \wedge \partial^\nu u)$, which may be identified with the well-known Skyrme term [E1, E2, MRS, S1, S2, S3, S4, ZB] when one embeds S^2 into $S^3 \approx SU(2)$. Hence, the Faddeev model may be viewed as a refined Skyrme model governing the interaction of baryons and mesons and the solution configurations of the former are the solution configurations of the latter with a restrained range [C].

We will be interested in the static field limit of the Faddeev model for which the total energy is given by

$$E(u) = \int_{\mathbb{R}^3} \left\{ \sum_{j=1}^{3} |\partial_j u|^2 + \frac{1}{2} \sum_{j,k=1}^{3} |F_{jk}(u)|^2 \right\} \mathrm{d}x. \tag{1.3}$$

Finite-energy condition implies that u approaches a constant vector u_∞ at spatial infinity (of $\mathbb{R}^3$). Hence we may compactify $\mathbb{R}^3$ into S^3 and view the fields as maps from S^3 to S^2. As a consequence, we see that each finite-energy field configuration u is associated with an integer, $Q(u)$, in $\pi_3(S^2) = \mathbb{Z}$ (the set of all integers). In fact, such an integer $Q(u)$ is known as the Hopf invariant which has the following integral characterization: The differential form $F = F_{jk}(u)\mathrm{d}x^j \wedge \mathrm{d}x^k$ $(j,k = 1,2,3)$ is closed in $\mathbb{R}^3$. Thus, there is a one form, $A = A_j \mathrm{d}x^j$ so that $F = \mathrm{d}A$. Then the Hopf charge $Q(u)$ of the map u may be evaluated by the integral

$$Q(u) = \frac{1}{16\pi^2} \int_{\mathbb{R}^3} A \wedge F, \tag{1.4}$$

due to J. H. C. Whitehead [Wh]. The integral (1.4) is in fact a special form of the Chern–Simons invariant [CS1, CS2] whose extended form in $(4n-1)$ dimensions (cf. (2.2) below) is also referred to as the Hopf–Whitehead invariant.

The Faddeev knots, or rather, knotted soliton configurations representing concentrated energy along knotted or linked curves, are realized as the solutions to the minimization problem [F2], also known as the Faddeev knot problem, given as

$$E_N \equiv \inf\{E(u) \,|\, E(u) < \infty,\ Q(u) = N\}, \qquad N \in \mathbb{Z}. \tag{1.5}$$

In [LY1, LY4], it is shown that E_N is attainable at $N = \pm 1$ and that there is an infinite subset of $\mathbb{Z}$, say $\mathbb{S}$, such that E_N is attainable for any $N \in \mathbb{S}$. The purpose of the present work is to extend this existence theory for the Faddeev knot problem to arbitrary settings beyond 3 dimensions.

Our motivation of engaging in a study of the Faddeev knot problem beyond 3 dimensions comes from several considerations: (i) Theoretical physics, especially quantum field theory, not only thrives in higher dimensions but although requires higher dimensions [GSW, P, Z]. (ii) The 3-dimensional Faddeev model may be viewed naturally as a special case of an elegant class of knot energies stratified by the Hopf invariant in general dimensions (see our formulation below). (iii) Progress in general dimensions helps us achieve an elevated level of understanding [LY3, LY5] of the intriguing relations between knot energy and knot topology and the mathematical mechanism for the formation of knotted structures. (iv) Knot theory in higher dimensions [H, K, R] is an actively pursued subject, and hence, it will be important to carry out a study of "ideal" knots for the Faddeev model in higher dimensions.

Note that minimization of knot energies subject to knot invariants based on diagrammatic considerations has been studied considerably in literature. For example, knot energies designed for measuring knotted/tangled space curves include the Gromov distortion energy [G1, G2], the Möbius energy [BFHW, FHW, O1, O2], and the ropelength energy [B, CKS1, CKS2, GM, Na]. See [JvR] for a rather comprehensive survey of these and other knot energies and related interesting works. See also [KBMSDS, Kf, M, S, SKK].

Although there are various available formulations when one tries to generalize the Faddeev energy (1.3), the core consideration is still to maintain an appropriate conformal structure for the energy functional which works to prevent the energy to collapse to zero. The simplest energy is the conformally invariant n-harmonic map energy, where n is the dimension of the domain space, which is also known as the Nicole model [Ni] when specialized to govern maps from $\mathbb{R}^3$ into S^2. Another type of energy functionals is of the Skyrme type [MRS, S1, S2, S3, S4, ZB] whose energy densities contain terms with opposite scaling properties and jointly prevent energy collapse. In fact, these terms interact to reach a suitable balance to ensure solitons of minimum energy to exist. The Faddeev model (1.3) belongs to this latter category for which the solitons of minimum energy are realized as knotted energy concentration configurations [BS1, BS2, FN1, FN2, Su]. In this paper, our main interest is to develop an existence theory for the energy minimizers of these two types of knotted soliton energies.

Specifically, we will study both the Nicole–Faddeev–Skyrme (NFS) type and Faddeev type knot energy (see (2.4), (2.5) and (2.6) for definitions). The two energy functionals have very different analytical properties. In particular, the conformally invariant term

$$\int_{\mathbb{R}^{4n-1}} |\nabla u|^{4n-1}\,\mathrm{d}x \tag{1.6}$$

in the NFS model enables us to carry out a straightforward argument which shows that the Hopf–Whitehead invariant $Q\,(u)$ (see (2.3)) must be an integer for any map u with finite NFS energy. More importantly, it allows us to get an annulus lemma (Lemma 3.1) which permits us to freely cut and paste maps under appropriate energy control. In this way, as in [LY2], the minimization problem fits well in the classical framework of the concentration-compactness principle [E1, E2, L1, L2]. Along this line, we shall arrive at the main result, Theorem 7.1, which guarantees the existence of extremal maps for an infinite set of integer values of the Hopf–Whitehead invariant. The situation is different for the Faddeev energy (see (2.6)). In this case, it seems difficult to know whether a map with finite energy can be approximated by smooth maps with similar energy control. In particular, it is not clear anymore why the Hopf–Whitehead invariant (see (2.3)), which is given by an integral expression, should always be an integer. Based on some recent observations of Hardt–Riviere [HR] in the study of the behavior of weak limits of smooth maps between manifolds in the Sobolev spaces, and some earlier approach of Esteban–Muller–Sverak [Sv, EM], we are able to show that the Hopf–Whitehead invariant of a map with finite Faddeev energy must be an integer (see Theorem 10.1). Such a statement is not only useful for a reasonable formulation of the Faddeev model but also plays a crucial role in understanding the behavior of minimizing sequence and the existence of extremal maps. One of the main difficulties in understanding the Faddeev model is that it is still not known whether an annulus lemma similar to Lemma 3.1 exists or not. In particular, we are not able to freely cut and paste maps with finite energy and it is not clear whether the minimizing problem would break into a finite region one and another at the infinity. That is, in this situation, the minimizing problem does not fit in the framework of the classical concentration-compactness principle anymore. This difficulty will be bypassed by a decomposition lemma (Lemma 12.1) for an arbitrary map with finite Faddeev energy (in the same spirit as in [LY1] for maps from $\mathbb{R}^3$ to S^2). Roughly speaking, the lemma says we may break the domain spaces into infinitely many blocks, each of which can be designated with some “degree”. By collecting those nonzero “degree” blocks suitably we may have a reasonable understanding of the minimizing sequence of maps for the Faddeev energy (Theorem 13.1). Based on this and the sublinear growth law for the Faddeev energy, we will obtain several existence results of extremal maps for the Faddeev energy (see Section 13.1). We point out that the method to bypass the breakdown of the concentration-compactness principle is along the same line as [LY1]. However, due to the fact that we do not have the tool of lifting through the classical Hopf map $S^3 \to S^2$ in higher dimensions, we have to resort to different approaches to deal with the nonlocally defined Hopf–Whitehead invariant. When reduced to the Faddeev model from $\mathbb{R}^3$ to S^2, this method gives a different route towards the main results in [LY1]. Moreover, by establishing the subaddivity of the Faddeev energy spectrum (see Corollary 13.3), we are able to strengthen the Substantial Inequality in [LY1] to an equality. That is, we are actually able to establish an additivity property for the Faddeev knot energy spectrum. We will also use the same approach to improve the Substantial Inequality for the Skyrme model to an equality (see Theorem 14.3).

Here is a sketch of the plan for the rest of the paper.

The first part, consisting of Sections 2–7, is about the NFS model. In Section 2, we introduce the generalized knot energies of the Nicole type [AS, ASVW, Ni, We], the NFS type extending the two-dimensional Skyrme model [Co, dW, GP, KPZ, LY2, PMTZ, PSZ1, PSZ2, PZ, SB, Wei], and the Faddeev type [F1, F2], all in light of the integral representation of the Hopf invariant in the general $(4n-1)$ dimensions (referred to as the Hopf dimensions). We will also obtain some growth estimates of the knot energies with respect to the Hopf number in view of the earlier work [LY3, LY5]. In Section 3, we establish a technical (annulus) lemma for the NFS model which allows truncation of a finite-energy map and plays a crucial role in proving the integer-valuedness of the Hopf–Whitehead integral and the validity of an energy-splitting relation called the "Substantial Inequality" [LY4]. We shall see that the conformal structure of the leading term in the energy density is essential. In Section 4, we show that the Hopf–Whitehead integral takes integer value for a finite-energy map in the NFS model. In Section 5, we consider the minimization process in view of the concentration-compactness principle of Lions [L1, L2] and we rule out the "vanishing" alternative for the nontrivial situation. We also show that the "compactness" alternative is needed for the solvability of the Faddeev knot problem stated in Section 2 for the NFS energy. In Section 6, we show that the "dichotomy" alternative implies the energy splitting relation or the Substantial Inequality. These results, combined with the energy growth law stated in Section 2, lead to the existence of the NFS energy minimizers stratified by infinitely many Hopf charges, as recognized in [LY1]. We state these results as the first existence theorem in Section 7. We then establish a simple but general existence theorem for both the generalized NFS model and the generalized Faddeev model in the compact case. For the Nicole model over $\mathbb{R}^3$ or S^3, we prove the existence of a finite-energy critical point among the topological class whose Hopf number is arbitrarily given.

The second part, consisting of Sections 8–13, is about the Faddeev model. In Section 8, we briefly describe the formulation of Faddeev model. In Section 9, various basic tools necessary for the study of Faddeev model are discussed. Section 10 is devoted to showing that for a map with finite Faddeev energy, the Hopf–Whitehead invariant is well defined and takes only integer values. We also derive a similar result for maps with mixed differentiability (see Section 10.1). Such kind of results are needed in proving the crucial decomposition lemma (Lemma 12.1). In Section 11, we describe some basic rules concerning the Hopf-Whitehead invariant for maps with finite Faddeev energy and the sublinear energy growth rate. Note that such kind of sublinear growth is a special case of results derived in [LY5]. The arguments are presented here to facilitate the discussions in Section 11, Section 12 and Section 13. In Section 12, we prove a crucial technical fact: the validity of a certain decomposition lemma for a map with finite Faddeev energy. The proof of this lemma shares the same spirit as that in [LY1] but is technically different due to the lack of lifting arguments. In Section 13, we prove the main result of the second part, namely, Theorem 13.1, which describes the behavior of a minimizing sequence of maps. Based on this description and the sublinear growth law, we discuss some facts about the existence of minimizers in Section 13.1.

In Section 14, we apply our approach in the second part to the standard Skyrme model to derive the subadditivity of the Skyrme energy spectrum and strengthen the substantial inequality to an equality.

Finally, we conclude with Section 15.

2. Knot Energies in General Hopf Dimensions

Recall that the integral representation of the Hopf invariant by Whitehead [Wh] of the classical fibration $S^3 \to S^2$ can be extended to the general case of the fibration $S^{4n-1} \to S^{2n}$. More precisely, let $u : S^{4n-1} \to S^{2n}$ $(n \geq 1)$ be a differentiable map. Then there is an integer representation of u in the homotopy group $\pi_{4n-1}(S^{2n})$, say $Q(u)$, called the generalized Hopf index of u, which has a similar integral representation as (1.4) as follows. Let $\omega_{S^{2n}}$ be a volume element of S^{2n} so that

$$|S^{2n}| \equiv \int_{S^{2n}} \omega_{S^{2n}} \tag{2.1}$$

is the total volume of S^{2n} and u^* the pullback map $\Lambda(S^{2n}) \to \Lambda(S^{4n-1})$ (a homomorphism between the rings of differential forms). Since u^* commutes with d, we see that $\mathrm{d}u^*(\omega_{S^{2n}}) = 0$; since the de-Rham cohomology $H^{2n}(S^{4n-1}, \mathbb{R})$ is trivial, there is a $(2n-1)$-form v on S^{4n-1} so that $\mathrm{d}v = u^*(\omega_{S^{2n}})$ (sometimes we also write $u^*(\omega_{S^{2n}})$ simply as $u^*\omega_{S^{2n}}$ when there is no risk of confusion). Of course, the normalized volume form $\tilde{\omega}_{S^{2n}} = |S^{2n}|^{-1}\omega_{S^{2n}}$ gives the unit volume and $\tilde{v} = |S^{2n}|^{-1}v$ satisfies $\mathrm{d}\tilde{v} = u^*(\tilde{\omega}_{S^{2n}})$. Since $\tilde{\omega}_{S^{2n}}$ can be viewed also as an orientation class, $Q(u)$ may be represented as [GHV, Hu]

$$Q(u) = \int_{S^{4n-1}} \tilde{v} \wedge u^*(\tilde{\omega}_{S^{2n}}) = \frac{1}{|S^{2n}|^2} \int_{S^{4n-1}} v \wedge u^*(\omega_{S^{2n}}). \tag{2.2}$$

The conformal invariance of (2.2) enables us to come up with the Hopf invariant, or the Hopf–Whitehead invariant, $Q(u)$, for a map u from $\mathbb{R}^{4n-1}$ to S^{2n} which approaches a fixed direction at infinity, as

$$Q(u) = \frac{1}{|S^{2n}|^2} \int_{\mathbb{R}^{4n-1}} v \wedge u^*(\omega_{S^{2n}}), \quad \mathrm{d}v = u^*(\omega_{S^{2n}}). \tag{2.3}$$

With the above preparation, we introduce the generalized Faddeev knot energies, subclassified as the Nicole, NFS, Faddeev energies over $\mathbb{R}^{4n-1}$, respectively, as

$$E_{\text{Nicole}}(u) = \int_{\mathbb{R}^{4n-1}} |\nabla u|^{4n-1}, \tag{2.4}$$

$$E_{\text{NFS}}(u) = \int_{\mathbb{R}^{4n-1}} \{|\nabla u|^{4n-1} + |u^*(\omega_{S^{2n}})|^2 + |\mathbf{n} - u|^2\}, \tag{2.5}$$

$$E_{\text{Faddeev}}(u) = \int_{\mathbb{R}^{4n-1}} \left\{|\nabla u|^{4n-2} + \frac{1}{2}|u^*(\omega_{S^{2n}})|^2\right\}, \tag{2.6}$$

where and in the sequel, we omit the Lebesgue volume element $\mathrm{d}x$ in various integrals whenever there is no risk of confusion, we use the notation $|\nabla u|$, $|\mathrm{d}u|$, and $|Du|$ interchangeably wherever appropriate, and we use $\mathbf{n}$ to denote a fixed unit vector in $\mathbb{R}^{2n+1}$ or a point on S^{2n}. Besides, we use c_0 to denote the best constant in the Sobolev inequality

$$c_0\|f\|_q \leq \|\nabla f\|_2 \tag{2.7}$$

over $\mathbb{R}^{4n-1}$ with q satisfying $1/q = 1/2 - 1/(4n-1) = (4n-3)/2(4n-1)$, given by the expression

$$c_0 = ([4n-1][4n-3])^{\frac{1}{2}} \left(\omega_{4n-1} \frac{\Gamma(2n-\frac{1}{2})\Gamma(2n+\frac{1}{2})}{\Gamma(4n-1)}\right)^{\frac{1}{(4n-1)}}, \tag{2.8}$$

with ω_m being the volume of the unit ball in $\mathbb{R}^m$.

Theorem 2.1.. *Let E be the energy functional defined by one of the energy functionals given by the expressions (2.4), (2.5), and (2.6). Then there is a universal constant $C = C(n) > 0$ such that*

$$E(u) \geq C|Q(u)|^{\frac{4n-1}{4n}}. \tag{2.9}$$

In the case when E is given by (2.6), the constant C has the explicit form

$$C(n) = 2^n (c_0|S^{2n}|^2)^{\frac{4n-1}{4n}} n^{\frac{2n-1}{2}}. \tag{2.10}$$

Proof. Recall the Sobolev inequality over $\mathbb{R}^{4n-1}$ of the form

$$C(n,p)\|f\|_q \leq \|\nabla f\|_p, \quad 1 < p < 4n-1, \quad q = \frac{(4n-1)p}{4n-1-p}. \tag{2.11}$$

From the pointwise bound

$$|u^*(\omega_{S^{2n}})| \leq C_1|\nabla u|^{2n}, \tag{2.12}$$

and assuming $\mathrm{d}v = u^*(\omega_{S^{2n}})$ and $\delta v = 0$, where δ is the codifferential of d which is often denoted by d^* as well, we have

$$\int_{\mathbb{R}^{4n-1}} |\nabla v|^{\frac{4n-1}{2n}} \leq C_2 \int_{\mathbb{R}^{4n-1}} |u^*(\omega_{S^{2n}})|^{\frac{4n-1}{2n}} \leq C_3 \int_{\mathbb{R}^{4n-1}} |\nabla u|^{4n-1}, \tag{2.13}$$

where we have used an L^p-version of the Gaffney type inequality [ISS, Sc] for differential forms (we thank Tom Otway for pointing out these references).

Choose $p = (4n-1)/2n$ so that $q = (4n-1)/(2n-1)$ in (2.11). The conjugate exponent q' with respect to q is $q' = q/(q-1) = (4n-1)/2n$. Thus the Hölder inequality and

(2.13) lead us to

$$|S^{2n}|^2|Q(u)| \leq \|v\|_q \|u^*(\omega_{S^{2n}})\|_{q'}$$
$$\leq C\|\nabla v\|_{(4n-1)/2n}\|u^*(\omega_{S^{2n}})\|_{(4n-1)/2n} \leq C_1\left(\int_{\mathbb{R}^{4n-1}}|\nabla u|^{4n-1}\right)^{\frac{4n}{4n-1}}, \tag{2.14}$$

which establishes (2.9) for the energy functional given by (2.4) or (2.5).

Consider now the energy functional

$$E_p(u) = \int_{\mathbb{R}^{4n-1}}\left\{|\nabla u|^p + \frac{1}{2}|u^*(\omega_{S^{2n}})|^2\right\}. \tag{2.15}$$

In [LY5], we have shown that, when the exponent p in (2.15) lies in the interval

$$1 < p < \frac{4n(4n-1)}{4n+1}, \tag{2.16}$$

there holds the universal fractionally-powered topological lower bound

$$E_p(u) \geq C(n,p)|Q(u)|^{\frac{4n-1}{4n}}, \tag{2.17}$$

where the positive constant $C(n,p)$ may be explicitly expressed as

$$C(n,p) =$$
$$(c_0|S^{2n}|^2)^{\frac{4n-1}{4n}}(2n)^{\frac{p}{2(4n-p)}}(4n-p)\left(\frac{4n}{(4n-1)(8n-p)-p(4n+1)}\right)^{\frac{(4n-1)(8n-p)-p(4n+1)}{8n(4n-p)}}. \tag{2.18}$$

It is seen that our stated lower bound for the energy defined in (2.6) corresponds to $p = 4n-2$ so that $C(n, 4n-2)$ is given by (2.10) as claimed. □

For the earlier work in the classical situation, $n = 1$, see [KR, Sh, VK].

Note that the energy

$$E_{\mathrm{AFZ}}(u) = \int_{\mathbb{R}^{4n-1}}|u^*(\omega_{S^{2n}})|^{\frac{4n-1}{2n}} \tag{2.19}$$

is also of interest and referred to as the AFZ model [AFZ] when $n = 1$. Combining (2.13) and (2.14), we have

$$C|Q(u)| \leq \|u^*(\omega_{S^{2n}})\|^2_{(4n-1)/2n}, \tag{2.20}$$

which implies that the energy E_{AFZ} defined in (2.19) satisfies the general fractionally-powered topological lower bound (2.9) as well.

We next show that the lower bound (2.9) is sharp.

Theorem 2.2.. *Let E be defined by one of the expressions stated in (2.4), (2.5), (2.6), and (2.19). Then for any given integer N which may be realized as the value of the Hopf–Whitehead invariant, i.e., $Q(u) = N$ for some differentiable map $u : \mathbb{R}^{4n-1} \to S^{2n}$, and for the positive number E_N defined as*

$$E_N = \inf\{E(u)|E(u) < \infty, Q(u) = N\}, \tag{2.21}$$

we have the universal topological upper bound

$$E_N \leq C|N|^{\frac{4n-1}{4n}}, \tag{2.22}$$

where $C > 0$ is a constant independent of N.

Proof. In [LY5], we have proved the theorem for the general energy functional

$$E(u) = \int_{\mathbb{R}^{4n-1}} \mathcal{H}(\nabla u)\,\mathrm{d}x,$$

where the energy density function $\mathcal{H}$ is assumed to be continuous with respect to its arguments and satisfies the natural condition $\mathcal{H}(\mathbf{0}) = 0$. Hence the theorem is valid for the energy functionals (2.4) and (2.6). For the energy functional (2.5), there is an extra potential term $|u-\mathbf{n}|^2$. However, this term does not cause problem in our proof because the crucial step is to work on a ball in $\mathbb{R}^{4n-1}$ of radius $|N|^{\frac{1}{4n}}$ and $u = \mathbf{n}$ outside the ball. Therefore, the potential term upon integration contributes a quantity proportional to the volume of the ball, which is of the form $C|N|^{\frac{4n-1}{4n}}$. □

In the following first few sections, we will concentrate on the energy functional (2.5).

3. Technical Lemma

Let B be a subdomain in $\mathbb{R}^{4n-1}$ and consider the knot energy (2.5) restricted to B,

$$E(u; B) = \int_B \{|\nabla u|^{4n-1} + |u^*(\omega_{S^{2n}})|^2 + |u-\mathbf{n}|^2\}. \tag{3.1}$$

We use B_R to denote the ball in $\mathbb{R}^{4n-1}$ centered at the origin and of radius $R > 0$. The following technical lemma plays an important part in our investigation of the first part of this paper.

Lemma 3.1.. *For any small $\varepsilon > 0$ and $R \geq 1$, let $u : \overline{B_{2R} \setminus B_R} \to S^{2n}$ satisfy $E(u; B_{2R} \setminus B_R) < \varepsilon$. Then there is a map $\tilde{u} : \overline{B_{2R} \setminus B_R} \to S^{2n}$ such that (i) $\tilde{u} = u$ on ∂B_R, (ii) $\tilde{u} = \mathbf{n}$ on ∂B_{2R}, (iii) $E(\tilde{u}; B_{2R} \setminus B_R) < C\varepsilon$, where $C > 0$ is an absolute constant independent of $R, \varepsilon,$ and u. The same statement is also valid when $\tilde{u}$ is modified to satisfy $\tilde{u} = \mathbf{n}$ on ∂B_R and $\tilde{u} = u$ on ∂B_{2R}.*

To obtain a proof, it will be convenient to work on a standard small domain. First, for the map stated in the lemma, define

$$u^R(y) = u(Ry) \quad \text{for } x = Ry \in B_{2R} \setminus B_R. \tag{3.2}$$

Hence $y \in B_2 \setminus B_1$ and

$$\begin{aligned} \varepsilon &> E(u; B_{2R} \setminus B_R) \\ &= \int_{B_2 \setminus B_1} \{|\nabla_y u^R(y)|^{4n-1} + |(u^R)^*(\omega_{S^{2n}})(y)|^2 R^{-1} + R^{4n-1}|u^R(y) - \mathbf{n}|^2\}\,\mathrm{d}y. \end{aligned} \tag{3.3}$$

Consequently, we have

$$\varepsilon > \int_1^{3/2} \mathrm{d}r \int_{\partial B_r} \mathrm{d}S_r \{|\nabla u^R|^{4n-1} + |(u^R)^*(\omega_{S^{2n}})|^2 R^{-1} + R^{4n-1}|u^R - \mathbf{n}|^2\}. \tag{3.4}$$

Hence, there is an $r \in (1, 3/2)$ such that

$$\int_{\partial B_r} \{|\nabla u^R|^{4n-1} + |(u^R)^*(\omega_{S^{2n}})|^2 R^{-1} + R^{4n-1}|u^R - \mathbf{n}|^2\} \, \mathrm{d}S_r < 2\varepsilon. \tag{3.5}$$

In what follows, we fix such an r determined by (3.5).

Consider a map $v^R : \mathbb{R}^{4n-1} \to \mathbb{R}^{2n}$ defined by

$$\Delta v^R = 0 \quad \text{in } B_2 \setminus B_r, \tag{3.6}$$

$$v^R = u^R \quad \text{on } \partial B_r, \quad v^R = \mathbf{n} \quad \text{on } \partial B_2. \tag{3.7}$$

Then, for $p = (4n-1)^2/(4n-2)$, we have, in view of (3.6) and (3.7), the bound

$$\|\nabla v^R\|_{L^p(B_2 \setminus B_r)} \leq C \|\nabla u^R\|_{L^{4n-1}(\partial B_r)}, \tag{3.8}$$

which in terms of (3.5) leads to

$$\int_{B_2 \setminus B_r} |\nabla v^R|^{\frac{(4n-1)^2}{(4n-2)}} \leq C_1 \varepsilon^{\frac{4n-1}{4n-2}}. \tag{3.9}$$

Since $(4n-1)^2 > 4n(4n-2)$, we have $p > 4n$. So the Hölder inequality with conjugate exponents s and t gives us

$$\int_{B_2 \setminus B_r} |\nabla v^R|^{4n} \leq |B_2 \setminus B_r|^{\frac{1}{t}} \left(\int_{B_2 \setminus B_r} |\nabla v^R|^p \right)^{\frac{1}{s}}, \tag{3.10}$$

where $4ns = p = (4n-1)^2/(4n-2)$ and $t = s/(1-s)$. Therefore, we have, in view of (3.9) and (3.10),

$$\int_{B_2 \setminus B_r} |\nabla v^R|^{4n} \leq C_2 \varepsilon^{\frac{4n}{4n-1}}. \tag{3.11}$$

Recall that, since $R \geq 1$, we also have $\int_{\partial B_r} |u^R - \mathbf{n}|^2 \, \mathrm{d}S_r < 2\varepsilon$. Hence, for any $q > 2$, we have $\int_{\partial B_r} |u^R - \mathbf{n}|^q \, \mathrm{d}S_r \leq C \int_{\partial B_r} |u^R - \mathbf{n}|^2 \, \mathrm{d}S_r \leq C_1 \varepsilon$. Since the ball is in $\mathbb{R}^{4n-1}$, we see that for $q = 4n(4n-2)/(4n-1)$ (of course, $q > 2$), we have

$$\|v^R - \mathbf{n}\|_{L^{4n}(B_2 \setminus B_r)} \leq C \|u^R - \mathbf{n}\|_{L^q(\partial B_r)} \leq C_1 \varepsilon^{\frac{1}{q}}. \tag{3.12}$$

Therefore, we have seen that $(v^R - \mathbf{n})$ has small $W^{1,4n}(B_2 \setminus B_r)$-norm. Using the embedding $W^{1,4n}(B_2 \setminus B_r) \to C(\overline{B_2 \setminus B_r})$ (noting that $\dim(B_2 \setminus B_r) = 4n - 1 < 4n$), we see that $(v^R - \mathbf{n})$ has small $C(\overline{B_2 \setminus B_r})$-norm. As a consequence, we may assume

$$\mathbf{n} \cdot v^R > \frac{1}{2} \quad \text{on } \overline{B_2 \setminus B_r}. \tag{3.13}$$

Since v^R is harmonic, $|v^R - \mathbf{n}|^2$ is subharmonic, $\Delta |v^R - \mathbf{n}|^2 \geq 0$, on $B_2 \setminus B_r$. Hence

$$\int_{B_2 \setminus B_r} |v^R - \mathbf{n}|^2 \leq C \int_{\partial B_r} |v^R - \mathbf{n}|^2 \, \mathrm{d}S_r \leq \frac{2\varepsilon C}{R^{4n-1}}. \tag{3.14}$$

To get a map from $B_2 \setminus B_r$, we need to normalize v^R, which is ensured by (3.13). Thus, we set

$$w^R = \frac{v^R}{|v^R|} \quad \text{on } B_2 \setminus B_r. \tag{3.15}$$

Then $w^R \in S^{2n}$. We can check that $|w^R - \mathbf{n}| < 4|v^R - \mathbf{n}|$ and $|\partial_j w^R| < 4|\partial_j v^R|$ in view of (3.13). Therefore we have

$$\int_{B_2\setminus B_r} R^{4n-1}|w^R - \mathbf{n}|^2 \leq 8C\varepsilon, \tag{3.16}$$

$$\int_{B_2\setminus B_r} R^{-1}|(w^R)^*(\omega_{S^{2n}})|^2 \leq C\int_{B_2\setminus B_r}|\nabla v^R|^{4n} \leq C_1\varepsilon^{\frac{4n}{4n-1}}, \tag{3.17}$$

$$\begin{aligned}\int_{B_2\setminus B_r}|\nabla w^R|^{4n-1} &\leq C_2\int_{B_2\setminus B_r}|\nabla v^R|^{4n-1}\\ &\leq C_2|B_2\setminus B_r|^{\frac{1}{t}}\left(\int_{B_2\setminus B_r}|\nabla v^R|^{4n}\right)^{\frac{1}{s}},\end{aligned} \tag{3.18}$$

where $t = s/(s-1)$ and $s = 4n/(4n-1)$. The bounds (3.11) and (3.18) may be combined to yield

$$\int_{B_2\setminus B_r}|\nabla w^R|^{4n-1} \leq C_3\varepsilon. \tag{3.19}$$

Thus, we can summarize (3.16), (3.17), and (3.19) and write down the estimate

$$\int_{B_2\setminus B_r}\{|\nabla w^R|^{4n-1} + R^{-1}|(w^R)^*(\omega_{S^{2n}})|^2 + R^{4n-1}|w^R - \mathbf{n}|^2\} < C\varepsilon. \tag{3.20}$$

On ∂B_2, $w^R = \mathbf{n}$; on ∂B_r, $w^R = u^R/|u^R| = u^R$.

Define

$$\tilde{u}(x) = w^R\left(\frac{1}{R}x\right) \quad \text{for } x \in B_{2R}\setminus B_{rR}; \quad \tilde{u}(x) = u(x) \quad \text{for } x \in B_{rR}. \tag{3.21}$$

We see that the statements of the lemma in the first case are all established.

The proof can be adapted to the case of the interchanged boundary conditions $\tilde{u} = u$ on B_{2R} and $\tilde{u} = \mathbf{n}$ on B_R. Hence, all the statements of the lemma in the second case are also established.

4. Integer-Valuedness of the Hopf–Whitehead Integral

As the first application of the technical lemma established in the previous section, we prove

Theorem 4.1.. *If $u : \mathbb{R}^{4n-1} \to S^{2n}$ is of finite energy, $E(u) < \infty$, where the energy E is as given in (2.5), then the Hopf–Whitehead integral (2.3) with $\delta v = 0$ is an integer.*

Let the pair u, v be given as in the theorem and $\{\varepsilon_j\}$ be a sequence of positive numbers so that $\varepsilon_j \to 0$ as $j \to \infty$ and $\{R_j\}$ be a corresponding sequence so that $R_j \to \infty$ as $j \to \infty$ and $E(u; \mathbb{R}^{4n-1}\setminus B_{R_j}) < \varepsilon_j$, $j = 1, 2, \cdots$. Let $\{u_j\}$ be a sequence of modified maps from $\mathbb{R}^{4n-1}$ to S^{2n} produced by the technical lemma so that $u_j = u$ in B_{R_j} and $u_j = \mathbf{n}$ on $\mathbb{R}^{4n-1}\setminus B_{2R_j}$. Then

$$Q(u_j) = \frac{1}{|S^{2n}|^2}\int_{\mathbb{R}^{4n-1}} v_j \wedge u_j^*(\omega_{S^{2n}}) \tag{4.1}$$

is a sequence of integers. We prove that $Q(u_j) \to Q(u)$ as $j \to \infty$.

We know that $\{|u_j^*(\omega_{S^{2n}})|\}$ is bounded in $L^2(\mathbb{R}^{4n-1})$ and $L^{\frac{4n-1}{2n}}(\mathbb{R}^{4n-1})$ due to the structure of the knot energy (2.5), the definition of u_j, and the relation (2.12). By interpolation, we see that the sequence is bounded in $L^p(\mathbb{R}^{4n-1})$ for all $p \in [\frac{4n-1}{2n}, 2]$. From the relations $\mathrm{d}v_j = u_j^*(\omega_{S^{2n}})$ and $\delta v_j = 0$, we see that $\{|\nabla v_j|\}$ is bounded in $L^p(\mathbb{R}^{4n-1})$ for all $p \in [\frac{4n-1}{2n}, 2]$ as well. Using the Sobolev inequality

$$C(m,p)\|f\|_q \le \|\nabla f\|_p \tag{4.2}$$

in $\mathbb{R}^m$ with $q = mp/(m-p)$ and $1 < p < m$, we get the boundedness of $\{v_j\}$ in $L^q(\mathbb{R}^{4n-1})$ for $q = (4n-1)p/(4n-1-p)$ with $\frac{4n-1}{2n} \le p \le 2$, which gives the range for q,

$$q(n) \equiv \frac{4n-1}{2n-1} \le q \le \frac{2(4n-1)}{4n-3}. \tag{4.3}$$

To proceed, we consider the estimate

$$\begin{aligned}
&|S^{2n}|^2|Q(u) - Q(u_j)| \\
&= \left| \int_{\mathbb{R}^{4n-1}} (v \wedge u^*(\omega_{S^{2n}}) - v_j \wedge u_j^*(\omega_{S^{2n}})) \right| \\
&\le \left| \int_{\mathbb{R}^{4n-1}} (v \wedge u^*(\omega_{S^{2n}}) - v \wedge u_j^*(\omega_{S^{2n}})) \right| + \left| \int_{\mathbb{R}^{4n-1}} (v \wedge u_j^*(\omega_{S^{2n}}) - v_j \wedge u_j^*(\omega_{S^{2n}})) \right| \\
&\equiv I_j^{(1)} + I_j^{(2)}.
\end{aligned} \tag{4.4}$$

To show that $I_j^{(1)} \to 0$ as $j \to \infty$, we look at the bottom numbers (for example) for which

$$u_j^*(\omega_{S^{2n}}) \to u^*(\omega_{S^{2n}}) \quad \text{weakly in } L^p(\mathbb{R}^{4n-1}) \tag{4.5}$$

for $p = \frac{4n-1}{2n}$ so that the conjugate of p is $p' = \frac{p}{p-1} = \frac{4n-1}{2n-1} = q(n)$, as defined in (4.3). Hence the claim $I_j^{(1)} \to 0$ immediately follows from (4.5).

On the other hand, since $q(n) > 2$, we see that $\{v_j\}$ is bounded in $W^{1,2}(B)$ for any bounded domain B in $\mathbb{R}^{4n-1}$. Using the compact embedding $W^{1,2}(B) \to L^2(B)$ and a subsequence argument, we may assume that $\{v_j\}$ is strongly convergent in $L^2(B)$ for any bounded domain B. Thus, we have

$$\begin{aligned}
I_j^{(2)} &\le \|v - v_j\|_{L^2(B)} E(u_j)^{\frac{1}{2}} + (\|v\|_{\frac{4n-1}{2n-1}} + \|v_j\|_{\frac{4n-1}{2n-1}}) \left(\int_{\mathbb{R}^{4n-1}\setminus B} |u_j^*(\omega_{S^{2n}})|^{\frac{4n-1}{2n}} \right)^{\frac{2n}{4n-1}} \\
&\le C_1\|v - v_j\|_{L^2(B)} + C_2 E(u_j; \mathbb{R}^{4n-1} \setminus B)^{\frac{2n}{4n-1}}.
\end{aligned} \tag{4.6}$$

It is not hard to see that the quantity $E(u_j; \mathbb{R}^{4n-1} \setminus B)$ may be made uniformly small. Indeed, for any $\varepsilon > 0$, we can choose B sufficiently large so that $E(u; \mathbb{R}^{4n-1} \setminus B) < \varepsilon$. Let j be large enough so that $B_{R_j} \supset B$. Then

$$\begin{aligned}
E(u_j; \mathbb{R}^{4n-1} \setminus B) &\le E(u; \mathbb{R}^{4n-1} \setminus B) + E(u_j; B_{2R_j} \setminus B_{R_j}) \\
&\le \varepsilon + C\varepsilon_j,
\end{aligned} \tag{4.7}$$

in view of Lemma 3.1. Using (4.7) in (4.6), we see that $I_j^{(2)} \to 0$ as $j \to \infty$.

Consequently, we have established $Q(u_j) \to Q(u)$ as $j \to \infty$. In particular, $Q(u)$ must be an integer because $Q(u_j)$'s are all integers.

5. Minimization for the Nicole–Faddeev–Skyrme Model

Consider the minimization problem (2.21) where the energy functional E is defined by (2.5). Let $\{u_j\}$ be a minimizing sequence of (2.21) and set

$$f_j(x) = (|\nabla u_j|^{4n-1} + |u_j^*(\omega_{S^{2n}})|^2 + |\mathbf{n} - u_j|^2)(x). \tag{5.1}$$

Then we have

$$f_j \in L(\mathbb{R}^{4n-1}), \quad \|f_j\|_1 \geq C|N|^{\frac{4n-1}{4n}}, \tag{5.2}$$

and $\|f_j\|_1 \leq E_N + 1$ (say) for all j.

Use $B(y, R)$ to denote the ball in $\mathbb{R}^{4n-1}$ centered at y and of radius $R > 0$. According to the concentration-compactness principle of P. L. Lions [L1, L2], one of the following three alternatives holds for the sequence $\{f_j\}$:

(a) Compactness: There is a sequence $\{y_j\}$ in $\mathbb{R}^{4n-1}$ such that for any $\varepsilon > 0$, there is an $R > 0$ such that

$$\sup_j \int_{\mathbb{R}^{4n-1}\setminus B(y_j,R)} f_j(x)\,\mathrm{d}x < \varepsilon. \tag{5.3}$$

(b) Vanishing: For any $R > 0$,

$$\lim_{j\to\infty} \left(\sup_{y\in\mathbb{R}^{4n-1}} \int_{B(y,R)} f_j(x)\,\mathrm{d}x \right) = 0. \tag{5.4}$$

(c) Dichotomy: There is a sequence $\{y_j\} \subset \mathbb{R}^{4n-1}$ and a positive number $t \in (0, 1)$ such that for any $\varepsilon > 0$ there is an $R > 0$ and a sequence of positive numbers $\{R_j\}$ satisfying $\lim_{j\to\infty} R_j = \infty$ so that

$$\left| \int_{B(y_j,R)} f_j(x)\,\mathrm{d}x - t\|f_j\|_1 \right| < \varepsilon, \tag{5.5}$$

$$\left| \int_{\mathbb{R}^{4n-1}\setminus B(y_j,R_j)} f_j(x)\,\mathrm{d}x - (1-t)\|f_j\|_1 \right| < \varepsilon. \tag{5.6}$$

We have the following.

Lemma 5.1.. *The alternative (b) (or vanishing) stated in (5.4) does not happen for the minimization problem when $N \neq 0$.*

Proof. Let B be a bounded domain in $\mathbb{R}^m$ and recall the continuous embedding $W^{1,p}(B) \to L^{\frac{mp}{m-p}}(B)$ for $p < m$. We need a special case of this at $p = 1$:

$$W^{1,1}(B) \to L^{\frac{m}{m-1}}(B) \quad (m > 1). \tag{5.7}$$

Hence, for any function w, we have

$$\begin{aligned}
\left(\int_B |w|^{k\frac{m}{m-1}}\right)^{\frac{m-1}{m}} &\leq C_B\left(\int_B (|w|^k + |w|^{k-1}|\nabla w|)\right)\\
&\leq C_B\left(\int_B |w|^k + \left[\int_B |w|^{(k-1)\frac{m}{m-1}}\right]^{\frac{m-1}{m}}\left[\int_B |\nabla w|^m\right]^{\frac{1}{m}}\right)\\
&\leq C\left(\int_B |w|^k + \int_B |w|^{(k-1)\frac{m}{m-1}} + \int_B |\nabla w|^m\right)\\
&\quad (\text{if } |w| \text{ is bounded, } k \geq 2,\ (k-1)\tfrac{m}{m-1} \geq 2, \text{ then})\\
&\leq C\left(\int_B |w|^2 + \int_B |\nabla w|^m\right). \qquad (5.8)
\end{aligned}$$

Now taking $m = 4n-1$ so that $\frac{m}{m-1} = \frac{4n-1}{4n-2} > 1$, $k = 4$, $w = u_j - \mathbf{n}$, and $B = B(y_j, R)$, we have from (5.8) the inequality

$$\int_{B(y_j,R)} |u_j - \mathbf{n}|^{\frac{2(4n-1)}{2n-1}} \leq C\left(\int_{B(y_j,R)} |u_j - \mathbf{n}|^2 + \int_{B(y_j,R)} |\nabla u_j|^{4n-1}\right)^{1+\frac{1}{4n-2}}. \qquad (5.9)$$

We now decompose $\mathbb{R}^{4n-1}$ into the union of a countable family of balls,

$$\mathbb{R}^{4n-1} = \cup_{i=1}^{\infty} B(y_i, R), \qquad (5.10)$$

so that each point in $\mathbb{R}^{4n-1}$ lies in at most m such balls. Then define the quantity

$$a_j = \sup_i \left(\int_{B(y_i,R)} |u_j - \mathbf{n}|^2 + \int_{B(y_i,R)} |\nabla u_j|^{4n-1}\right). \qquad (5.11)$$

Thus the alternative (b) (vanishing) implies $a_j \to 0$ as $j \to \infty$. Therefore

$$\begin{aligned}
\int_{\mathbb{R}^{4n-1}} |u_j - \mathbf{n}|^{\frac{2(4n-1)}{2n-1}} &\leq \sum_{i=1}^{\infty} \int_{B(y_i,R)} |u_j - \mathbf{n}|^{\frac{2(4n-1)}{2n-1}}\\
&\leq a_j^{\frac{1}{4n-2}} C \sum_{i=1}^{\infty}\left(\int_{B(y_i,R)} |u_j - \mathbf{n}|^2 + \int_{B(y_i,R)} |\nabla u_j|^{4n-1}\right)\\
&\leq m a_j^{\frac{1}{4n-1}} C\left(\int_{\mathbb{R}^{4n-1}} (|u_j - \mathbf{n}|^2 + |\nabla u_j|^{4n-1})\right)\\
&\leq m a_j^{\frac{1}{4n-1}} C E(u_j) \to 0 \quad \text{as } j \to \infty. \qquad (5.12)
\end{aligned}$$

Define the set $A_j = \{x \in \mathbb{R}^{4n-1} \,|\, |u_j(x) - \mathbf{n}| \geq 1\}$ (say). Then (5.12) implies

$$\lim_{j\to\infty} |A_j| = 0, \qquad (5.13)$$

where $|A_j|$ denotes the Lebesgue measure of A_j. Since $Q(u_j) = N \neq 0$, we see that $u_j(\mathbb{R}^{4n-1})$ covers S^{2n} (except possibly skipping $\mathbf{n}$). The definition of A_j says $u_j(A_j)$ contains the half-sphere below the equator of S^{2n}. Consequently,

$$\int_{A_j} |u_j^*(\omega_{S^{2n}})| \, \mathrm{d}x \geq |u_j(A_j)| \geq \frac{1}{2}|S^{2n}|, \qquad (5.14)$$

where $|S^{2n}|$ is the total volume of S^{2n}. However, the Schwartz inequality and (5.13) give us

$$\int_{A_j} |u_j^*(\omega_{S^{2n}})|\,\mathrm{d}x \le |A_j|^{\frac{1}{2}}\left(\int_{\mathbb{R}^{4n-1}} |u_j^*(\omega_{S^{2n}})|^2\right)^{\frac{1}{2}}$$
$$\le |A_j|^{\frac{1}{2}}(E_N+1)^{\frac{1}{2}} \to 0, \tag{5.15}$$

as $j\to\infty$, which is a contradiction to (5.14). □

Suppose that (a) holds. Using the notation of (a), we can translate the minimizing sequence $\{u_j\}$ to

$$\{u_j(\cdot - y_j)\} = \{\tilde{u}_j(\cdot)\} \tag{5.16}$$

so that $\{\tilde{u}_j\}$ is also a minimizing sequence of the same Hopf charge. Passing to a subsequence if necessary, we may assume without loss of generality that $\{\tilde{u}_j\}$ weakly converges in a well-understood sense over $\mathbb{R}^{4n-1}$ to its weak limit, say u. Of course,

$$E(u) \le \liminf_{j\to\infty}\{E(u_j)\} = E_N. \tag{5.17}$$

Lemma 5.2.. *The alternative (a) (or compactness) stated in (5.3) implies the preservation of the Hopf charge in the limit described in (5.17). In other words, $Q(u) = N$ so that u gives rise to a solution of the direct minimization problem (2.21).*

Proof. Let ε and R be the pair stated in the alternative (a). Then

$$\sup_j \int_{\mathbb{R}^{4n-1}\setminus B_R} \{|\nabla \tilde{u}_j|^{4n-1} + |\tilde{u}_j^*(\omega_{S^{2n}})|^2 + |\tilde{u}_j - \mathbf{n}|^2\} < \varepsilon. \tag{5.18}$$

Besides, for the weak limit u of the sequence $\{\tilde{u}_j\}$, we have

$$\int_{\mathbb{R}^{4n-1}\setminus B_R} \{|\nabla u|^{4n-1} + |u^*(\omega_{S^{2n}})|^2 + |u - \mathbf{n}|^2\} \le \varepsilon \tag{5.19}$$

and

$$|S^{2n}|^2|Q(u) - Q(\tilde{u}_j)| \le I_j + J + K_j, \tag{5.20}$$

where

$$I_j = \left|\int_{B_R} v\wedge u^*(\omega_{S^{2n}}) - \int_{B_R} \tilde{v}_j \wedge \tilde{u}_j^*(\omega_{S^{2n}})\right|,$$
$$J = \left|\int_{\mathbb{R}^{4n-1}\setminus B_R} v\wedge u^*(\omega_{S^{2n}})\right|,$$
$$K_j = \left|\int_{\mathbb{R}^{4n-1}\setminus B_R} \tilde{v}_j\wedge \tilde{u}_j^*(\omega_{S^{2n}})\right|. \tag{5.21}$$

It is not hard to see that the quantities J and K_j are small with a magnitude of some power of ε. In fact, (2.5) and (2.12) indicate that $|\tilde{u}_j^*(\omega_{S^{2n}})|$ is uniformly bounded in $L^p(\mathbb{R}^{4n-1})$ for $p\in[\frac{4n-1}{2n}, 2]$. Then the relation $\mathrm{d}\tilde{v}_j = \tilde{u}_j^*(\omega_{S^{2n}})$, $\delta\tilde{v}_j = 0$, and the Sobolev inequality (4.2) imply that $\tilde{v}_j$ is uniformly bounded in $L^q(\mathbb{R}^{4n-1})$ for $q\in[\frac{4n-1}{2n-1}, \frac{2(4n-1)}{4n-3}]$ (see (4.3)). Using (2.12) again, we have

$$K_j \le \|\tilde{v}_j\|_{L^{\frac{4n-1}{2n-1}}(\mathbb{R}^{4n-1}\setminus B_R)} \|\tilde{u}_j^*(\omega_{S^{2n}})\|_{L^{\frac{4n-1}{2n}}(\mathbb{R}^{4n-1}\setminus B_R)}$$
$$\le CE(\tilde{u}_j;\mathbb{R}^{4n-1}\setminus B_R)^{\frac{2n}{4n-1}} \le C\varepsilon^{\frac{2n}{4n-1}}. \tag{5.22}$$

By the same method, we can show that the quantity J obeys a similar bound as well.

For I_j, we observe that since $\tilde{u}_j^*(\omega_{S^{2n}})$ converges to $u^*(\omega_{S^{2n}})$ weakly in $L^2(B_R)$ and $\tilde{v}_j$ converges to v strongly in $L^2(B_R)$, we have $I_j \to 0$ as $j \to \infty$.

Summarizing the above results, we conclude that $Q(\tilde{u}_j) \to Q(u)$ as $j \to \infty$. □

In the next section, we will characterize the alternative (c) (dichotomy).

6. Dichotomy and Energy Splitting in Minimization

Use the notation of the previous section and suppose that (c) (or dichotomy) happens. Then, after possible translations, we may assume that there is a number $t \in (0,1)$ such that for any $\varepsilon > 0$ there is an $R > 0$ and a sequence of positive numbers $\{R_j\}$ satisfying $\lim_{j\to\infty} R_j = \infty$ so that

$$\left| \int_{B_R} f_j(x)\,\mathrm{d}x - tE(u_j) \right| < \varepsilon, \tag{6.1}$$

$$\left| \int_{\mathbb{R}^{4n-1}\setminus B_{R_j}} f_j(x)\,\mathrm{d}x - (1-t)E(u_j) \right| < \varepsilon. \tag{6.2}$$

For convenience, we assume $R_j > 2R$ for all j. Therefore, from the decomposition

$$E(u_j) = \int_{B_R} f_j(x)\,\mathrm{d}x + \int_{\mathbb{R}^{4n-1}\setminus B_{R_j}} f_j(x)\,\mathrm{d}x + E(u_j; B_{R_j} \setminus B_R), \tag{6.3}$$

and (6.1), (6.2), we have

$$\begin{aligned} E(u_j; B_{2R} \setminus B_R) &\le E(u_j; B_{R_j} \setminus B_R) < 2\varepsilon, \\ E(u_j; B_{R_j} \setminus B_{R_j/2}) &\le E(u_j; B_{R_j} \setminus B_R) < 2\varepsilon. \end{aligned} \tag{6.4}$$

Using Lemma 3.1, we can find maps $u_j^{(1)}$ and $u_j^{(2)}$ from $\mathbb{R}^{4n-1}$ to S^{2n} such that $u_j^{(1)} = u_j$ in B_R, $u_j^{(1)} = \mathbf{n}$ in $\mathbb{R}^{4n-1} \setminus B_{2R}$, and $E(u_j^{(1)}; B_{2R} \setminus B_R) < C\varepsilon$; $u_j^{(2)} = u_j$ in $\mathbb{R}^{4n-1} \setminus B_{R_j}$, $u_j^{(2)} = \mathbf{n}$ in $B_{R_j/2}$, and $E(u_j^{(2)}; B_{R_j} \setminus B_{R_j/2}) < C\varepsilon$. Here $C > 0$ is an irrelevant constant.

Use the notation $F(u) = v \wedge u^*(\omega_{S^{2n}})$. Since $F(u)$ depends on u nonlocally, we need to exert some care when we make argument involving truncation.

In view of the fact that u_j and $u_j^{(1)}$ coincide on B_R and u_j and $u_j^{(2)}$ coincide on $\mathbb{R}^{4n-1} \setminus B_{R_j}$, we have

$$\begin{aligned} &\int_{\mathbb{R}^{4n-1}} |u_j^*(\omega_{S^{2n}}) - (u_j^{(1)})^*(\omega_{S^{2n}}) - (u_j^{(2)})^*(\omega_{S^{2n}})|^{\frac{4n-1}{2n}} \\ &\le C(E(u_j; B_{R_j} \setminus B_R) + E(u_j^{(1)}; B_{2R} \setminus B_R) + E(u_j^{(2)}; B_{R_j} \setminus B_{R_j/2})) \\ &\le C\varepsilon. \end{aligned} \tag{6.5}$$

Consequently, using the relations $\mathrm{d}v_j = u_j^*(\omega_{S^{2n}})$, $\delta v_j = 0$, $\mathrm{d}v_j^{(i)} = (u_j^{(i)})^*(\omega_{S^{2n}})$, $\delta v_j^{(i)} = 0$, $i = 1, 2$, we have in view of (6.5) and (4.2) with $p = (4n-1)/2n$ and $q = (4n-1)/(2n-1)$ that

$$\begin{aligned} \|v_j - v_j^{(1)} - v_j^{(2)}\|_{\frac{4n-1}{2n-1}} &\le C\|u_j^*(\omega_{S^{2n}}) - (u_j^{(1)})^*(\omega_{S^{2n}}) - (u_j^{(2)})^*(\omega_{S^{2n}})\|_{\frac{4n-1}{2n}} \\ &\le C_1 \varepsilon^{\frac{2n}{4n-1}}. \end{aligned} \tag{6.6}$$

Since the numbers p, q above are also conjugate exponents, we obtain from (6.6) the bound

$$\begin{aligned}
&\int_{B_R\cup\{\mathbb{R}^{4n-1}\setminus B_{R_j}\}} |F(u_j)-F(u_j^{(1)})-F(u_j^{(2)})| \\
&= \int_{B_R\cup\{\mathbb{R}^{4n-1}\setminus B_{R_j}\}} |(v_j - v_j^{(1)} - v_j^{(2)})\wedge u_j^*(\omega_{S^{2n}})| \\
&\leq \|v_j - v_j^{(1)} - v_j^{(2)}\|_{\frac{4n-1}{2n-1}} \|u_j^*(\omega_{S^{2n}})\|_{\frac{4n-1}{2n}} \\
&\leq C\varepsilon^{\frac{2n}{4n-1}}. \qquad (6.7)
\end{aligned}$$

Applying (6.7), we have

$$\begin{aligned}
&|S^{2n}|^2 |Q(u_j) - (Q(u_j^{(1)}) + Q(u_j^{(2)}))| \\
&\leq \int_{B_R\cup\{\mathbb{R}^{4n-1}\setminus B_{R_j}\}} |F(u_j)-F(u_j^{(1)})-F(u_j^{(2)})| \\
&\quad + \int_{B_{R_j}\setminus B_R} |F(u_j)| + \int_{B_{2R}\setminus B_R} |F(u_j^{(1)})| + \int_{B_{R_j}\setminus B_{R_j/2}} |F(u_j^{(2)})| \\
&\leq C_1\varepsilon^{\frac{2n}{4n-1}} \\
&\quad + C_2(E(u_j;B_{R_j}\setminus B_R)^{\frac{2n}{4n-1}} + E(u_j^{(1)};B_{2R}\setminus B_R)^{\frac{2n}{4n-1}} + E(u_j^{(2)};B_{R_j}\setminus B_{R_j/2})^{\frac{2n}{4n-1}}) \\
&\leq C\varepsilon^{\frac{2n}{4n-1}}. \qquad (6.8)
\end{aligned}$$

Since $\varepsilon > 0$ can be arbitrarily small and $Q(u_j), Q(u_j^{(1)}), Q(u_j^{(2)})$ are integers, the uniform bound (6.8) enables us to assume that

$$N \equiv Q(u_j) = Q(u_j^{(1)}) + Q(u_j^{(2)}), \quad \forall j. \qquad (6.9)$$

On the other hand, since (2.9) implies that

$$\begin{aligned}
|Q(u_j^{(1)})|^{\frac{4n-1}{4n}} &\leq CE(u_j^{(1)}) = C(E(u_j;B_R) + E(u_j^{(1)};B_{2R}\setminus B_R)) \\
&\leq CE(u_j) + C_1\varepsilon, \qquad (6.10)
\end{aligned}$$

we see that $\{Q(u_j^{(1)})\}$ is bounded.

We claim that $Q(u_j^{(1)}) \neq 0$ for j sufficiently large. Indeed, if $Q(u_j^{(1)}) = 0$ for infinitely many j's, then, by going to a subsequence when necessary, we may assume that $Q(u_j^{(1)}) = 0$ for all j. Thus we see that $Q(u_j^{(2)}) = N$ in (6.9) for all j and

$$E(u_j^{(2)}) \leq E(u_j;\mathbb{R}^{4n-1}\setminus B_{R_j}) + C\varepsilon = \int_{\mathbb{R}^{4n-1}\setminus B_{R_j}} f_j(x)\,\mathrm{d}x + C\varepsilon. \qquad (6.11)$$

As a consequence, we have in view of (6.11) and (6.2) that

$$\begin{aligned}
E_N &\leq \limsup_{j\to\infty} E(u_j^{(2)}) \leq (1-t)\lim_{j\to\infty} E(u_j) + \varepsilon + C\varepsilon \\
&\leq (1-t)E_N + C_1\varepsilon. \qquad (6.12)
\end{aligned}$$

Since $0 < t < 1$ and ε is arbitrarily small, we obtain $E_N = 0$, which contradicts the topological lower bound $E_N \geq C|N|^{\frac{4n-1}{4n}}$ $(N \neq 0)$ stated in (2.9).

Similarly, we may assume that $Q(u_j^{(2)}) \neq 0$ for j sufficiently large. Of course, $\{Q(u_j^{(2)})\}$ is bounded as well.

Hence, extracting a subsequence if necessary, we may assume that there are integers $N_1 \neq 0$ and $N_2 \neq 0$ such that

$$Q(u_j^{(1)}) = N_1, \quad Q(u_j^{(2)}) = N_2, \quad \forall j. \tag{6.13}$$

Furthermore, for the respective energy infima at the Hopf charges N_1, N_2, N, we have

$$\begin{aligned} &E_{N_1} + E_{N_2} \leq E(u_j^{(1)}) + E(u_j^{(2)}) \\ &= E(u_j; B_R) + E(u_j; \mathbb{R}^{4n-1} \setminus B_{R_j}) + E(u_j^{(1)}; B_{2R} \setminus B_R) + E(u_j^{(2)}; B_{R_j} \setminus B_{R_j/2}) \\ &\leq E(u_j) + 2C\varepsilon. \end{aligned} \tag{6.14}$$

Since $\varepsilon > 0$ may be arbitrarily small, we can take the limit $j \to \infty$ in (6.14) to arrive at

$$E_{N_1} + E_{N_2} \leq E_N, \quad N = N_1 + N_2. \tag{6.15}$$

We can now establish the following energy-splitting lemma.

Lemma 6.1.. *If the alternative (c) (or dichotomy) stated in (5.5) and (5.6) happens at the Hopf charge $N \neq 0$, then there are nonzero integers $N_1, N_2, \cdots, N_k$ such that*

$$E_N \geq E_{N_1} + E_{N_2} + \cdots + N_k, \quad N = N_1 + N_2 + \cdots + N_k, \tag{6.16}$$

and that the alternative (a) (or compactness) stated in (5.3) takes place at each of these integers $N_1, N_2, \cdots, N_k$.

Proof. If the alternative (c) happens at $N \neq 0$, we have the splitting (6.15). We may repeat this procedure at all the sublevels wherever the alternative (c) happen. Since (2.9) and (2.10) imply that there is a universal constant $C > 0$ such that $E_\ell \geq C$ for any $\ell \neq 0$. Hence the above splitting procedure ends after a finitely many steps at (6.16) for which the alternative (c) cannot happen anymore at $N_1, N_2, \cdots, N_k$. Since the alternative (b) never happens because $N_s \neq 0$ ($s = 1, 2, \cdots, k$) in view of Lemma 5.1, we see that (a) takes place at each of these integer levels. □

The energy splitting inequality, (6.16), is referred to as the "Substantial Inequality" in [LY4] which is crucial for obtaining existence theorems in a noncompact situation.

7. Existence Theorems

We say that an integer $N \neq 0$ satisfies the condition (S) if the nontrivial splitting as described in Lemma 6.1 cannot happen at N. Define

$$\mathbb{S} = \{N \in \mathbb{Z} \,|\, N \text{ satisfies condition (S)}\}. \tag{7.1}$$

It is clear that, for any $N \in \mathbb{S}$, the minimization problem (2.21) has a solution. As a consequence of our study in the previous sections, we arrive at

Theorem 7.1.. *Consider the minimization problem (2.21) in which the energy functional is of the NFS type given in (2.5). Then there is an infinite subset of $\mathbb{Z}$, say $\mathbb{S}$, such that, for any $N \in \mathbb{S}$, the problem (2.21) has a solution. In particular, the minimum-mass or minimum-energy Hopf charge N_0 defined by*

$$N_0 \textit{ is such that } E_{N_0} = \min\{E_N \,|\, N \neq 0\} \tag{7.2}$$

is an element in $\mathbb{S}$*. Furthermore, for any nonzero* $N \in \mathbb{Z}$*, we can find* $N_1, \cdots, N_k \in \mathbb{S}$ *such that the substantial inequality (6.16) is strengthened to the equalities*

$$E_N = E_{N_1} + E_{N_2} + \cdots + N_k, \quad N = N_1 + N_2 + \cdots + N_k, \tag{7.3}$$

which simply express energy and charge conservation laws of the model in regards of energy splitting.

Proof. Use the Technical Lemma (Lemma 3.1) as in [LY1] to get (7.3). The rest may also follow the argument given in [LY1]. □

Next, we show that, in the compact situation, the minimization problem (2.21) has a solution for any integer N. For this purpose, let $E(u)$ denote the energy functional of the NFS type or the Faddeev type given as in (2.5) or (2.6) evaluated over S^{4n-1} for a map u from S^{4n-1} into S^{2n}. Namely,

$$E_{\text{NFS}}(u) = \int_{S^{4n-1}} \{|\mathrm{d}u|^{4n-1} + |u^*(\omega_{S^{2n}})|^2 + |\mathbf{n} - u|^2\}\, \mathrm{d}S, \tag{7.4}$$

$$E_{\text{Faddeev}}(u) = \int_{S^{4n-1}} \left\{|\mathrm{d}u|^{4n-2} + \frac{1}{2}|u^*(\omega_{S^{2n}})|^2\right\} \mathrm{d}S. \tag{7.5}$$

The Hopf invariant $Q(u)$ of u is given in (2.2). We have

Theorem 7.2.. *For any nonzero integer* N *which may be realized as a Hopf number, i.e., there exists a map* $u : S^{4n-1} \to S^{2n}$ *such that* $Q(u) = N$*, the minimization problem* $E_N = \inf\{E(u)\,|\,E(u) < \infty, Q(u) = N\}$ *over* S^{4n-1} *has a solution when* E *is given either by (7.4) or (7.5).*

Proof. Let $\{u_j\}$ be a minimizing sequence of the stated topologically constrained minimization problem and v_j be the “potential” $(2n-1)$-form satisfying

$$\mathrm{d}v_j = u_j^*(\omega_{S^{2n}}), \quad \delta v_j = 0, \quad j = 1, 2, \cdots. \tag{7.6}$$

Passing to a subsequence if necessary, we may assume that there is a finite-energy map u (say) such that $u_j \to u$, $\mathrm{d}u_j \to u$, and $u_j^*(\omega_{S^{2n}}) \to u^*(\omega_{S^{2n}})$ weakly in obvious function spaces, respectively, as $j \to \infty$, which lead us to the correct comparison $E(u) \leq E_N$ by the weakly lower semi-continuity of the given energy functional. To see that $Q(u) = N$, we recall that the sequence $\{v_j\}$ may be chosen [Mo] such that it is bounded in $W^{1,2}(S^{4n-1})$ by the $L^2(S^{4n-1})$ bound of $\{u_j^*(\omega_{S^{2n}})\}$. Hence $v_j \to$ some $v \in W^{1,2}(S^{4n-1})$ weakly as $j \to \infty$. Therefore $v_j \to v$ strongly in $L^2(S^{4n-1})$ as $j \to \infty$. Of course, $\mathrm{d}v = u^*(\omega_{S^{2n}})$ and $\delta v = 0$. Consequently, we immediately obtain

$$Q(u) = \frac{1}{|S^{2n}|^2} \int_{S^{4n-1}} v \wedge u^*(\omega_{S^{2n}}) = \frac{1}{|S^{2n}|^2} \lim_{j\to\infty} \int_{S^{4n-1}} v_j \wedge u_j^*(\omega_{S^{2n}}) = N, \tag{7.7}$$

and the proof is complete. □

Note that the existence of global minimizers for the compact version of the Nicole energy (2.4),

$$E(u) = \int_{S^{4n-1}} |\mathrm{d}u|^{4n-1}\, \mathrm{d}S, \tag{7.8}$$

was studied by Riviere [Ri] for $n = 1$. See also [L] and [DK]. In particular, he showed that there exist infinitely many homotopy classes from S^3 into S^2 having energy minimizers.

We now address the general problem of the existence of critical points of (7.8) at the bottom dimension $n = 1$ whose conformal structure prompts us to simply consider it over $\mathbb{R}^3$. Thus we are led to the Nicole model. Specifically, for a map $u : \mathbb{R}^3 \to S^2$, the Nicole energy [Ni] is given by

$$E(u) = \int_{\mathbb{R}^3} |\nabla u|^3. \tag{7.9}$$

For convenience, we may use the stereographic projection of $S^2 \to \mathbb{C}$ from the south pole to represent $u = (u_1, u_2, u_3)$ by a complex-valued function $U = U_1 + \mathrm{i}U_2$ as follows,

$$U_1 = \frac{u_1}{1+u_3}, \quad U_2 = \frac{u_2}{1+u_3}, \tag{7.10}$$

where $u_3 = \pm\sqrt{1-u_1^2-u_2^2}$ for u belonging to the upper or lower hemisphere, $S^2_\pm$. Following [AFZ] (see also [ASVW, HS]), we use the toroidal coordinates (η, ξ, φ) to represent a point $x = (x^1, x^2, x^3)$ in $\mathbb{R}^3$ by

$$x^1 = q^{-1}\sinh\eta\cos\varphi, \quad x^2 = q^{-1}\sinh\eta\sin\varphi, \quad x^3 = q^{-1}\sin\xi, \tag{7.11}$$

where $q = \cosh\eta - \cos\xi$ and $0 < \eta < \infty, 0 \le \xi, \varphi \le 2\pi$. The AFZ ansatz [AFZ, ASVW, HS] reads

$$U(\eta, \xi, \varphi) = f(\eta)\,\mathrm{e}^{\mathrm{i}m\varphi + \mathrm{i}n\xi}, \quad m, n \in \mathbb{Z}, \tag{7.12}$$

where the undetermined function f satisfies the "normalized" boundary condition

$$f(0) = \lim_{\eta\to 0} f(\eta) = 0, \quad f(\infty) = \lim_{\eta\to\infty} f(\eta) = \infty, \tag{7.13}$$

so that the Hopf map is given by the choice $f(\eta) = \sinh\eta$ with $m = n = 1$, or

$$U(\eta, \xi, \varphi) = \sinh\eta\;\mathrm{e}^{\mathrm{i}\xi + \mathrm{i}\varphi}. \tag{7.14}$$

After some calculation, it can be shown [AFZ, HS] that the Hopf invariant of u designated by (7.10)–(7.13) is given as

$$Q(u) = mn. \tag{7.15}$$

Besides, with the new variable

$$t = \sinh\eta, \tag{7.16}$$

the function f becomes a function of t, which is still denoted by $f(t)$ for simplicity, so that the Nicole energy (7.9) takes the form [ASVW]

$$E(f) = 32\pi^2 \int_0^\infty \left\{ t(1+t^2)\left(\frac{f_t^2}{(1+f^2)^2} + \frac{1}{1+t^2}\left[\frac{m^2}{t^2} + n^2\right]\frac{f^2}{(1+f^2)^2}\right)^{\frac{3}{2}}\right\} \mathrm{d}t, \tag{7.17}$$

and the boundary condition (7.13) is reinterpreted in terms of t given in (7.16). The Euler–Lagrange equation of (7.17) is [ASVW]

$$\begin{aligned} &t^2(1+t^2)(1+f^2)(2t^2[1+t^2]f_t^2 + [m^2+n^2t^2]f^2)f_{tt} - 4t^4(1+t^2)^2 f f_t^4 \\ &+ t^3(1+3t^2)(1+t^2)(1+f^2)f_t^3 - 2t^2(1+t^2)(m^2+n^2t^2)f^3 f_t^2 \\ &+ t^3(m^2+n^2[1+2t^2])(1+f^2)f^2 f_t - (m^2+n^2t^2)^2 f^3(1-f^2) = 0. \end{aligned} \tag{7.18}$$

It is important to note that the advantage of using the AFZ ansatz (7.12) is that it is a compatible ansatz [ASVW], meaning that (7.18) gives rise to the critical points of the original Nicole energy (7.9). More precisely, the critical points of (7.17) subject to the boundary condition $f(t) \to 0$ as $t \to 0$, $f(t) \to \infty$ as $t \to \infty$, give rise to the critical points of the Hopf number (7.15) for the Nicole energy through (7.10)–(7.12) and (7.16). Although (7.18) looks complicated, it has a nontrivial solution $f(t) = t$ when $m = n = 1$, which implies that the Hopf map is an explicit critical point [ASVW]. Our purpose below is to show that, for any m, n, the equation (7.18) has a finite-energy solution satisfying the stated boundary condition at $t = 0$ and $t = \infty$. In fact, such a solution also minimizes the energy (7.17).

To proceed, we introduce another new variable

$$g = \arctan f. \tag{7.19}$$

Then the boundary condition for f becomes

$$g(0) = 0, \quad g(\infty) = \frac{\pi}{2}, \tag{7.20}$$

and the energy (7.17) is converted into the simplified form given as

$$I(g) = \int_0^\infty \left\{ t(1+t^2)\left(g_t^2 + \frac{1}{1+t^2}\left[\frac{m^2}{t^2} + n^2\right]\frac{\tan^2 g}{(1+\tan^2 g)^2}\right)^{\frac{3}{2}} \right\} \mathrm{d}t, \tag{7.21}$$

where we have suppressed an irrelevant constant factor. It is seen that the Hopf map, defined by $g(t) = \arctan t$, is of finite energy for any integers m, n.

We now define the admissible space as

$$\begin{aligned} \mathcal{A} = \{ g(t)\,|\, g(t) &\text{ is absolutely continuous over the interval } (0,\infty), \\ &\text{satisfies the boundary condition (7.20), and } I(g) < \infty \}, \end{aligned} \tag{7.22}$$

and consider the associated minimization problem

$$I_0 \equiv \inf\{ I(g) \,|\, g \in \mathcal{A}\}. \tag{7.23}$$

Let $\{g_j\}$ be a minimizing sequence of (7.23). We may assume that $I(g_j) \le I_0 + 1$ (say) for all $j = 1, 2, \cdots$. We will show that $\{g_j\}$ contains a subsequence which converges in a well-defined way to an element in $\mathcal{A}$, g_0 (say), and $I(g_0) = I_0$.

In fact, collectively writing

$$P(g) = \frac{\tan^2 g}{(1+\tan^2 g)^2}, \tag{7.24}$$

we see that $P(\cdot)$ is a periodic even function of period π, whose singularities at odd-integer multiples of $\pi/2$ are removed if we understand $P(\frac{\pi}{2}) = \lim_{g\to\frac{\pi}{2}} P(g) = 0$, etc. In the sequel, we always observe such a convention for $P(\cdot)$. Therefore, for any $g \in \mathcal{A}$, the modified function

$$\tilde{g}(t) = \begin{cases} |g(t)|, & \text{if } |g(t)| < \frac{\pi}{2}, \\ \frac{\pi}{2}, & \text{if } |g(t)| \ge \frac{\pi}{2}, \end{cases} \tag{7.25}$$

lies in $\mathcal{A}$ and satisfies $0 \le \tilde{g} \le \frac{\pi}{2}$ and $I(\tilde{g}) \le I(g)$. Hence, with suitable modifications if necessary, we may assume that our minimizing sequence $\{g_j\}$ satisfies the same boundedness condition $0 \le g_j \le \frac{\pi}{2}$, $j = 1, 2, \cdots$.

On the other hand, near $t = 0$ and $t = \infty$, we have, respectively,

$$\begin{aligned} 0 \le g_j(t) &\le \left(\int_0^t (s^{-\frac{1}{3}})^{\frac{3}{2}}\,\mathrm{d}s \right)^{\frac{2}{3}} \left(\int_0^t s\left|\frac{\mathrm{d}g_j}{\mathrm{d}s}\right|^3 \mathrm{d}s\right)^{\frac{1}{3}} \\ &\le 2^{\frac{2}{3}} t^{\frac{1}{3}} (I(g_j))^{\frac{1}{3}}, \end{aligned} \tag{7.26}$$

and

$$\left|\frac{\pi}{2}-g_j(t)\right| \le \left(\int_t^\infty s^{-\frac{3}{2}}\,\mathrm{d}s\right)^{\frac{2}{3}}\left(\int_t^\infty s^3\left|\frac{\mathrm{d}g_j}{\mathrm{d}s}\right|^3\mathrm{d}s\right)^{\frac{1}{3}}$$
$$\le 2^{\frac{2}{3}}t^{-\frac{1}{3}}(I(g_j))^{\frac{1}{3}}, \tag{7.27}$$

which indicates in particular that $\{g_j\}$ satisfies the boundary condition (7.20) uniformly.

The structure of the energy I given in (7.21) shows that for any numbers $0<a<b<\infty$, the sequence $\{g_j\}$ is bounded in $W^{1,3}(a,b)$. Using a diagonal subsequence argument, we may assume without loss of generality that $\{g_j\}$ is weakly convergent in $W^{1,3}(a,b)$ for any $0<a<b<\infty$. We use g_0 to denote the so-obtained weak limit of $\{g_j\}$ over the entire interval $(0,\infty)$. We need to prove that $g_0\in\mathcal{A}$ and $I(g_0)=I_0$.

For convenience, we set

$$J(g,h;a,b)=\int_a^b\left\{t(1+t^2)\left(g_t^2+\frac{1}{1+t^2}\left[\frac{m^2}{t^2}+n^2\right]P(h)\right)^{\frac{3}{2}}\right\}\mathrm{d}t, \tag{7.28}$$

where g,h are absolulutely continuous over $(0,\infty)$ and $P(\cdot)$ is defined by (7.24). We note that

$$P'(h)=\frac{2\tan h(1-\tan^2 h)}{(1+\tan^2 h)^2},\quad h\neq \text{odd-integer multiple of }\frac{\pi}{2};$$
$$P'(h)=0,\quad h=\text{odd-integer multiple of }\frac{\pi}{2}. \tag{7.29}$$

Hence, P' is bounded. Besides, we may check that $J(\cdot,h;a,b)$ is convex for fixed h,a,b. Therefore, we have

$$\lim_{j\to\infty}\left(J(g_j,g_j;a,b)-J(g_j;g_0;a,b)\right)=0, \tag{7.30}$$

and the weakly lower semicontinuity of $J(\cdot,g_0;a,b)$ implies that

$$J(g_0,g_0;a,b)\le\liminf_{j\to\infty}J(g_j,g_0;a,b). \tag{7.31}$$

Consequently, we get

$$\begin{aligned}I_0&=\lim_{j\to\infty}I(g_j)\\&\ge\liminf_{j\to\infty}J(g_j,g_j;a,b)\\&=\lim_{j\to\infty}\left(J(g_j,g_j;a,b)-J(g_j;g_0;a,b)\right)+\liminf_{j\to\infty}J(g_j,g_0;a,b)\\&\ge J(g_0,g_0;a,b).\end{aligned} \tag{7.32}$$

Letting $a\to 0$ and $b\to\infty$ in (7.32), we see that $I(g_0)=J(g_0,g_0;0,\infty)\le I_0$ as claimed. The fact that g_0 satisfies the boundary condition (7.20) follows from the uniform bounds (7.26) and (7.27). Thus, $g_0\in\mathcal{A}$.

The Euler–Lagrange equation of (7.21) is

$$\left\{t(1+t^2)\left(g_t^2+\frac{1}{1+t^2}\left(\frac{m^2}{t^2}+n^2\right)P(g)\right)^{\frac{1}{2}}g_t\right\}_t$$
$$=\frac{t}{2}\left(g_t^2+\frac{1}{1+t^2}\left(\frac{m^2}{t^2}+n^2\right)P(g)\right)^{\frac{1}{2}}\left(\frac{m^2}{t^2}+n^2\right)P'(g). \tag{7.33}$$

With the help of this equation, we may show that g_0 satisfies

$$0 < g_0(t) < \frac{\pi}{2}, \quad 0 < t < \infty. \tag{7.34}$$

In fact, if there is a point $t_0 > 0$ such that $g_0(t_0) = 0$ or $g(t_0) = \pi/2$, then the property $0 \le g_0(t) \le \pi/2$ implies that $g_0'(t_0) = 0$. In view of the uniqueness theorem for the initial value problem of an ordinary differential equation, we infer that $g_0(t) \equiv 0$ or $g_0(t) \equiv \pi/2$ since $g = 0$ and $g = \pi/2$ are two trivial solutions of (7.33). This conclusion contradicts the boundary condition (7.20) enjoyed by the function g_0 obtained earlier.

The property (7.34) ensures the invertibility of the transformation (7.19) so that we obtain a critical point for the original energy (7.17).

We may summarize our study above in the form of the following existence theorem.

Theorem 7.3.. *For any $N \in \mathbb{Z}$, the Nicole energy (7.9) has a finite-energy critical point u in the topological class $Q = N$. More precisely, for any $m, n \in \mathbb{Z}$, the energy functional (7.9) has a finite-energy critical point u represented in terms of the toroidal coordinates through the expressions (7.10)–(7.13) so that its Hopf invariant satisfies $Q = mn$, its associated configuration function f defined in (7.12) is positive-valued with range equal to the full interval $(0, \infty)$ and minimizes the reduced one-dimensional energy (7.17) in the variable $t = \sinh \eta$.*

As mentioned already, since (7.9) is conformally invariant, it covers the spherical energy (7.8) when $n = 1$. Therefore, Theorem 7.3 establishes the existence of a critical point of the energy (7.8) at $n = 1$ among the topological class $Q = N$ for each $N \in \mathbb{Z}$.

8. Generalized Faddeev Knot Energy

In the subsequent sections, we shall study the topologically constrained minimization problem of the generalized Faddeev knot energy in arbitrary $(4n-1)$ dimensions. The generalization we will be focused on is defined by the energy

$$E(u) = \int_{\mathbb{R}^{4n-1}} \{|\mathrm{d}u|^{4n-2} + |u^*\omega_{S^{2n}}|^2\}, \tag{8.1}$$

where, for convenience, we have absorbed the unimportant coefficient $\frac{1}{2}$ in (2.6) to unity. One may argue that a more natural generalization of the Faddeev knot energy should take the original "quadratic" form so that

$$E(u) = \int_{\mathbb{R}^{4n-1}} \{|\mathrm{d}u|^2 + |u^*\omega_{S^{2n}}|^2\}. \tag{8.2}$$

However, at this moment, the energy (8.2) seems to be too hard to approach. Indeed for $n \ge 2$ and a map $u : \mathbb{R}^{4n-1} \to S^{2n}$ with $\int_{\mathbb{R}^{4n-1}} \{|\mathrm{d}u|^2 + |u^*\omega_{S^{2n}}|^2\} < \infty$, it is not necessary that $u^*\omega_{S^{2n}}$ is a closed form. On the other hand, it is worth mentioning that (8.1) may be viewed as a "natural" extension of the Faddeev energy as well because (i) both energy density terms are quadratic when $n = 1$, and (ii) with respect to the rescaling of coordinates, $x \mapsto \lambda x$ $(\lambda > 0)$, the two energy terms respond with λ^{-1} and λ, respectively, as in the classical Faddeev model.

As mentioned in the introduction, one of the main difficulties in understanding the Faddeev model is that it is still not known whether an annulus lemma similar to Lemma 3.1 exists or not. In particular we are not able to freely cut and paste maps with finite energy and it is not clear the minimizing problem would break into a finite region one and a problem at the infinity, that is, the minimizing problem does not fit in the frame of classical concentration compactness principle anymore. This difficulty will be bypassed by a decomposition lemma (Lemma 12.1) for an arbitrary map with finite Faddeev energy (in the same spirit as in [LY1] for maps from $\mathbb{R}^3$ to S^2).

9. Some General Facts and Useful Properties and Relations

In this section we collect some basic facts which will be used frequently later.

We will use the algebraic notations in [F, Chapter 1]. Assume n is an integer and $1 \le k \le n$. Then we denote

$$\Lambda(n,k) = \{\lambda = (\lambda_1, \cdots, \lambda_k) \,|\, \lambda_i\text{'s are integers and } 1 \le \lambda_1 < \cdots < \lambda_k \le n\}. \tag{9.1}$$

If $\lambda = (\lambda_1, \cdots, \lambda_k) \in \Lambda(n,k)$, $k < n$, then $\overline{\lambda} \in \Lambda(n, n-k)$ is obtained from the complement $\{1, \cdots, n\} \setminus \{\lambda_1, \cdots, \lambda_k\}$. If $x_1, \cdots, x_n$ are the coordinates on $\mathbb{R}^n$, then we write

$$\mathrm{d}x_\lambda = \mathrm{d}x_{\lambda_1} \wedge \mathrm{d}x_{\lambda_2} \wedge \cdots \wedge \mathrm{d}x_{\lambda_k}.$$

If, for every $\lambda \in \Lambda(n,k)$, ω_λ is a distribution on an open subset of $\mathbb{R}^n$, then we call

$$\omega = \sum_{\lambda \in \Lambda(n,k)} \omega_\lambda \mathrm{d}x_\lambda$$

a (k-form) distribution.

Occasionally, we will need to verify some weak differential identities. It is convenient to have the following basic rule.

Lemma 9.1.. *Assume that $1 \le p_1, p_2, p_3 \le \infty$, Ω is an open subset of $\mathbb{R}^n$, $\alpha \in L^{p_1}_{loc}(\Omega)$ is a k-form, $\beta \in L^{p_1'}_{loc}(\Omega)$ is another form such that $d\alpha \in L^{p_2}_{loc}(\Omega)$, $\beta \in L^{p_2'}_{loc}(\Omega)$, $\alpha \in L^{p_3}_{loc}(\Omega)$ and $d\beta \in L^{p_3'}_{loc}(\Omega)$. Then, in sense of distribution, we have*

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta.$$

Here $p_1' = \frac{p_1}{p_1 - 1}$ is the conjugate power of p_1. Similar for p_2' and p_3'.

Proof. First assume $1 \le p_1, p_2, p_3 < \infty$. By mollifying arguments we may find a sequence of smooth k-forms $\alpha_i \in C^\infty(\Omega)$ such that $\alpha_i \to \alpha$ in $L^{p_1}_{loc}(\Omega)$ and $L^{p_3}_{loc}(\Omega)$, $\mathrm{d}\alpha_i \to \mathrm{d}\alpha$ in $L^{p_2}_{loc}(\Omega)$. Taking a limit in the equation

$$\mathrm{d}(\alpha_i \wedge \beta) = \mathrm{d}\alpha_i \wedge \beta + (-1)^k \alpha_i \wedge \mathrm{d}\beta,$$

the lemma follows. For the remaining cases, without loss of generality, we assume $p_1 = \infty$, $1 \le p_2, p_3 < \infty$. Then we may find a sequence of smooth k-forms $\alpha_i \in C^\infty(\Omega)$ such that $\alpha_i \overset{*}{\rightharpoonup} \alpha$ in $L^\infty_{loc}(\Omega)$, $\alpha_i \to \alpha$ in $L^{p_3}_{loc}(\Omega)$ and $\mathrm{d}\alpha_i \to \mathrm{d}\alpha$ in $L^{p_2}_{loc}(\Omega)$. Then $\alpha_i \wedge \beta \to \alpha \wedge \beta$ in sense of distribution and $\mathrm{d}\alpha_i \wedge \beta \to \mathrm{d}\alpha \wedge \beta$, $\alpha_i \wedge \mathrm{d}\beta \to \alpha \wedge \mathrm{d}\beta$ in $L^1_{loc}(\Omega)$. The same limit process as above implies the lemma. □

For a smooth map u, the exterior differential d commutes with the pullback operator u^*. It remains true under suitable integrability condition on the derivatives of u when it is only weakly differentiable.

Lemma 9.2.. *Assume that $\Omega \subset \mathbb{R}^n$ is open, α is a smooth k-form on $\mathbb{R}^l$ with compact support, $u \in W^{1,k+1}_{loc}(\Omega, \mathbb{R}^l)$. Then in sense of distribution*

$$du^*\alpha = u^* d\alpha.$$

Proof. we may find $u_i \in C^{\infty}\left(\Omega, \mathbb{R}^l\right)$ such that $u_i \to u$ in $W^{1,k+1}_{loc}\left(\Omega, \mathbb{R}^l\right)$ and $u_i \to u$ a.e. It follows that $u_i^*\alpha \to u^*\alpha$ in $L^{\frac{k+1}{k}}_{loc}(\Omega)$ and $u_i^* \mathrm{d}\alpha \to u^* \mathrm{d}\alpha$ in $L^1_{loc}(\Omega)$. Taking limit in $\mathrm{d}u_i^*\alpha = u_i^*\mathrm{d}\alpha$, we arrive at the conclusion. □

The conclusion of the above lemma can be strengthened when we know that the map is bi-Lipschitz.

Lemma 9.3.. *Assume that Ω_1, Ω_2 are open subsets in $\mathbb{R}^n$, $\phi : \Omega_1 \to \Omega_2$ is a bi-Lipschitz map, and $\alpha \in L^1_{loc}(\Omega_2)$ is a k-form such that $d\alpha \in L^1_{loc}(\Omega_2)$. Then $d\phi^*\alpha = \phi^* d\alpha$.*

Proof. We may find a sequence of smooth k-forms $\alpha_i \in C_c^{\infty}(\Omega_2)$ such that $\alpha_i \to \alpha$ in $L^1_{loc}(\Omega_2)$ and $\mathrm{d}\alpha_i \to \mathrm{d}\alpha$ in $L^1_{loc}(\Omega_2)$. Hence $\phi^*\alpha_i \to \phi^*\alpha$ in $L^1_{loc}(\Omega_1)$ and $\phi^*\mathrm{d}\alpha_i \to \phi^*\mathrm{d}\alpha$ in $L^1_{loc}(\Omega_1)$. It follows from Lemma 9.2 that $\mathrm{d}\phi^*\alpha_i = \phi^*\mathrm{d}\alpha_i$. Letting $i \to \infty$, we obtain $\mathrm{d}\phi^*\alpha = \phi^*\mathrm{d}\alpha$. □

Later on we will need to verify weak differential identities for maps with mixed differentiability on different domains. For that purpose we state the following smoothing lemma.

Lemma 9.4.. *Let $\Omega = B_1^{n-1} \times (-1,1)$, $f : B_1^{n-1} \to (-1,1)$ be a continuous function. Assume that $1 \le p_1, q_1 < \infty$, $1 \le p_2, q_2 < \infty$, $\alpha \in L^{p_1}_{loc}(\Omega)$ is a k-form such that $d\alpha \in L^{p_2}_{loc}(\Omega)$. For $x \in \Omega$, we write $x = (x', x_n)$, $x' \in \mathbb{R}^{n-1}$. Denote $\Omega_- = \{x \in \Omega : x_n < f(x')\}$. If*

$$\alpha|_{\Omega_-} \in L^{q_1}_{loc}(\{x \in \Omega \,|\, x_n \le f(x')\})$$

and

$$d\alpha|_{B_1^-} \in L^{q_2}_{loc}(\{x \in \Omega : x_n \le f(x')\}),$$

then there exists a sequence of smooth k-forms α_i on Ω such that

$$\begin{aligned} \alpha_i &\to \alpha \text{ in } L^{p_1}_{loc}(\Omega), \\ d\alpha_i &\to d\alpha \text{ in } L^{p_2}_{loc}(\Omega), \\ \alpha_i|_{\Omega_-} &\to \alpha|_{\Omega_-} \text{ in } L^{p_2}_{loc}(\{x \in \Omega \,|\, x_n \le f(x')\}), \\ d\alpha_i|_{\Omega_-} &\to d\alpha|_{\Omega_-} \text{ in } L^{q_2}_{loc}(\{x \in \Omega \,|\, x_n \le f(x')\}). \end{aligned}$$

If any one of the p_1, p_2, q_1, q_2 is infinite, then the conclusion remains true if we replace the strong convergence by the weak $*$ convergence in L^{∞}_{loc}.

Proof. For $\delta > 0$ small we denote $i_\delta : x \mapsto (x', x_n - \delta)$, then $i_\delta^*\alpha$ is defined on $B_1^{n-1} \times (-1+\delta, 1)$. We may choose $0 < \varepsilon < \delta$ small enough such that for $y' \in B^{n-1}_{1-2\delta}$, we have $f(x') + \delta > f(y') + \varepsilon$ for all $x' \in B_\varepsilon^{n-1}(y')$. For $\rho \in C^{\infty}(\mathbb{R}^n, \mathbb{R})$, $\rho(x) = 0$ for $x \in \mathbb{R}^n \backslash B_1$ and $\int_{\mathbb{R}^n} \rho(x)\,\mathrm{d}x = 1$, write $\rho_\varepsilon(x) = \frac{1}{\varepsilon^n}\rho\left(\frac{x}{\varepsilon}\right)$. Let $\beta_\delta = \rho_\varepsilon * i_\delta^*\alpha$ be defined on $B^{n-1}_{1-3\delta} \times (-1+3\delta, 1-3\delta)$. Choose a $\phi_\delta \in C_c^{\infty}\left(B^{n-1}_{1-3\delta} \times (-1+3\delta, 1-3\delta)\right)$ with $\phi_\delta = 1$ on $B^{n-1}_{1-4\delta} \times (-1+4\delta, 1-4\delta)$. Then $\alpha_\delta = \phi_\delta \cdot \beta_\delta$ satisfies all the requirements as $\delta \to 0^+$. □

Based on the above smoothing lemma, we can derive another differential identity.

Lemma 9.5.. *Assume that Ω is an open subset in $\mathbb{R}^n$, $\Sigma \subset \Omega$ is a continuous hypersurface which separates Ω into Ω_1 and Ω_2 i.e. $\Omega \backslash \Sigma = \Omega_1 \cup \Omega_2$, $1 \le p_1, p_2, p_3, q_1, q_2, q_3 \le \infty$, $\alpha \in L^{p_1}_{loc}(\Omega)$ is a k-form with $d\alpha \in L^{p_2}_{loc}(\Omega)$, $\alpha \in L^{p_3}_{loc}(\Omega)$ and*

$$\alpha|_{\Omega_2} \in L^{q_1}_{loc}(\Omega_2 \cup \Sigma), \quad d\alpha|_{\Omega_2} \in L^{q_2}_{loc}(\Omega_2 \cup \Sigma), \quad \alpha|_{\Omega_2} \in L^{q_3}_{loc}(\Omega_2 \cup \Sigma).$$

Let $\beta \in L^1_{loc}(\Omega)$ be another form with $d\beta \in L^1_{loc}(\Omega)$ and

$$\beta|_{\Omega_1} \in L^{p_1'}_{loc}(\Omega_1 \cup \Sigma), \quad \beta|_{\Omega_1} \in L^{p_2'}_{loc}(\Omega_1 \cup \Sigma), \quad d\beta|_{\Omega_1} \in L^{p_3'}_{loc}(\Omega_1 \cup \Sigma).$$
$$\beta|_{\Omega_2} \in L^{q_1'}_{loc}(\Omega_2 \cup \Sigma), \quad \beta|_{\Omega_2} \in L^{q_2'}_{loc}(\Omega_2 \cup \Sigma), \quad d\beta|_{\Omega_2} \in L^{q_3'}_{loc}(\Omega_2 \cup \Sigma).$$

Then, in sense of distribution,

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta.$$

Proof. Without loss of generality, we may assume that $1 \le p_1, p_2, p_3, q_1, q_2, q_3 < \infty$. By localization and rotation, we may assume that Ω is a cylinder and Σ is the graph of a continuous function. It follows from Lemma 9.3 that we can find a sequence of smooth k-forms α_i on Ω such that

$$\alpha_i \to \alpha \text{ in } L^{p_1}_{loc}(\Omega), \quad \mathrm{d}\alpha_i \to \mathrm{d}\alpha \text{ in } L^{p_2}_{loc}(\Omega), \quad \alpha_i \to \alpha \text{ in } L^{p_3}_{loc}(\Omega),$$
$$\alpha_i \to \alpha \text{ in } L^{q_1}_{loc}(\Omega_2 \cup \Sigma), \quad \mathrm{d}\alpha_i \to \mathrm{d}\alpha \text{ in } L^{q_2}_{loc}(\Omega_2 \cup \Sigma), \quad \alpha_i \to \alpha \text{ in } L^{q_3}_{loc}(\Omega_2 \cup \Sigma).$$

Taking limit in $\mathrm{d}(\alpha_i \wedge \beta) = \mathrm{d}\alpha_i \wedge \beta + (-1)^k \alpha_i \wedge \mathrm{d}\beta$, we obtain the conclusion. □

For later purposes, we review a little bit of the Hodge theory on domains ([T, Section 9 of Chapter 5]). Let $\Omega \subset \mathbb{R}^n$ be a bounded open subset with smooth boundary $\Sigma = \partial\Omega$, ν be the outer normal direction and $i : \Sigma \to \overline{\Omega}$ be the natural put-in map. For $1 < p < \infty$,

$$W^{1,p}_R(\Omega) = \left\{\alpha \in W^{1,p}(\Omega) \mid \alpha \text{ is a form with } i^*\alpha = 0\right\},$$
$$\mathcal{H}_R\left(\overline{\Omega}\right) = \left\{\alpha \in C^\infty\left(\overline{\Omega}\right) \mid \alpha \text{ is a form with } \mathrm{d}\alpha = 0, \mathrm{d}^*\alpha = 0, i^*\alpha = 0\right\}.$$

Here the subscript R refers to the imposed relative boundary condition: $i^*\alpha = 0$. That is, the tangential part of α on the boundary Σ is zero. Then we have

$$L^p(\Omega) = \mathrm{d}W^{1,p}_R(\Omega) \oplus \mathrm{d}^*W^{1,p}_R(\Omega) \oplus \mathcal{H}_R\left(\overline{\Omega}\right)$$

and $\mathcal{H}_R\left(\overline{\Omega}\right) \cong H^*\left(\overline{\Omega}, \partial\Omega, \mathbb{R}\right)$, the real singular cohomology group. More precisely, if $\omega \in L^p(\Omega)$, then

$$\omega = \mathrm{d}\alpha + \mathrm{d}^*\beta + \gamma,$$

with $\alpha, \beta \in W^{1,p}_R(\Omega)$, $\gamma \in \mathcal{H}_R\left(\overline{\Omega}\right)$ and $\|\alpha\|_{W^{1,p}(\Omega)}, \|\beta\|_{W^{1,p}(\Omega)} \le c(p, \Omega)\|\omega\|_{L^p(\Omega)}$. If we know $\int_\Omega \langle \omega, \mathrm{d}^*\varphi\rangle\,\mathrm{d}x = 0$ for every smooth form φ on $\overline{\Omega}$ with $i^*\varphi = 0$, then $\omega = \mathrm{d}\alpha + \gamma$ for $\alpha \in W^{1,p}_R(\Omega)$, $\gamma \in \mathcal{H}_R\left(\overline{\Omega}\right)$. Indeed it follows from integration by parts formula that

$$\int_\Omega \langle \omega, \mathrm{d}^*\varphi\rangle\,\mathrm{d}x = \int_\Omega \langle \mathrm{d}^*\beta, \mathrm{d}\varphi\rangle\,\mathrm{d}x = 0.$$

Hence $\int_\Omega \langle \mathrm{d}^*\beta, \mathrm{d}\varphi\rangle\,\mathrm{d}x = 0$ for every $\varphi \in W^{1,p'}_R(\Omega)$, $p' = \frac{p}{p-1}$. For every $\tau \in L^{p'}(\Omega)$, $\tau = \mathrm{d}\alpha_1 + \mathrm{d}\beta_1 + \gamma_1$ for $\alpha_1, \beta_1 \in W^{1,p'}_R(\Omega)$ and $\gamma_1 \in \mathcal{H}_R\left(\overline{\Omega}\right)$, hence

$$\int_\Omega \langle \mathrm{d}^*\beta, \tau\rangle\,\mathrm{d}x = \int_\Omega \langle \mathrm{d}^*\beta, \mathrm{d}\beta_1\rangle\,\mathrm{d}x = 0.$$

This implies $\mathrm{d}^*\beta = 0$.

One of the ingredients in proving the crucial decomposition lemma (Lemma 12.1) is the construction of suitable functions on annulus which connects the original map to constant maps. The next two lemmas are about the existence of such auxiliary functions. First, we derive some basic inequalities for the harmonic extension of a function on the boundary of a domain.

Lemma 9.6.. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open subset with smooth boundary Σ, $1 < p < \infty$, $f \in W^{1,p}(\Sigma)$, and u the harmonic extension of f to Ω. Then*

$$\|u\|_{W^{1,\frac{np}{n-1}}(\Omega)} \leq c(p,\Omega)\|f\|_{W^{1,p}(\Sigma)}.$$

Proof. We need the following basic fact (compare with [HWY1, Proposition 2.1]): Assume that $\rho \in C_c^\infty(\mathbb{R}^{n-1})$, g is a function on $\mathbb{R}^{n-1}$, and

$$(Tg)(x) = \int_{\mathbb{R}^{n-1}} \rho(\xi) g(x' - x_n \xi) \, \mathrm{d}\xi$$

for $x \in \mathbb{R}^n_+$, $x = (x', x_n)$, $1 < p < \infty$. Then

$$\|Tg\|_{L^{\frac{np}{n-1}}(\mathbb{R}^n_+)} \leq c(n,p,\rho)\|g\|_{L^p(\mathbb{R}^{n-1})},$$
$$\|\nabla Tg\|_{L^{\frac{np}{n-1}}(\mathbb{R}^n_+)} \leq c(n,p,\rho)\|\nabla g\|_{L^p(\mathbb{R}^{n-1})}.$$

To prove the two inequalities, we claim that

$$\|Tg\|_{L_W^{\frac{n}{n-1}}(\mathbb{R}^n_+)} \leq c(n,\rho)\|g\|_{L^1(\mathbb{R}^{n-1})}.$$

If the claim is true, then the first inequality follows from the Marcinkiewicz interpolation theorem (see [SW, p197]) and the basic fact that $\|Tg\|_{L^\infty(\mathbb{R}^n_+)} \leq \|\rho\|_{L^1(\mathbb{R}^{n-1})}\|g\|_{L^\infty(\mathbb{R}^{n-1})}$. To prove the claim, assume that $\|g\|_{L^1(\mathbb{R}^{n-1})} = 1$. Then $|Tg(x)| \leq \frac{c(n,\rho)}{x_n^{n-1}}$ and

$$\int_{x\in\mathbb{R}^n_+, 0<x_n<a} |Tg(x)| \, \mathrm{d}x \leq c(n,\rho)\, a$$

for $a > 0$. Hence, for $t > 0$,

$$\begin{aligned} ||Tg| > t| &= \left|\left\{x \in \mathbb{R}^n_+ : 0 < x_n < c(n,\rho)\, t^{-\frac{1}{n-1}}, |Tg(x)| > t\right\}\right| \\ &\leq \frac{1}{t}\int_{0<x_n<c(n)t^{-\frac{1}{n-1}}, x'\in\mathbb{R}^{n-1}} |Tg|(x) \, \mathrm{d}x \leq c(n,\rho)\, t^{-\frac{n}{n-1}}. \end{aligned}$$

Thus the claim follows. Next we observe that, for $1 \leq i \leq n-1$,

$$\partial_i (Tg)(x) = \int_{\mathbb{R}^{n-1}} \rho(\xi)\, \partial_i g(x' - x_n\xi) \, \mathrm{d}\xi$$

and

$$\partial_n (Tg)(x) = -\sum_{j=1}^{n-1} \int_{\mathbb{R}^{n-1}} \rho(\xi)\, \xi_j \partial_j g(x' - x_n\xi) \, \mathrm{d}\xi.$$

Hence it follows that, for $1 < p < \infty$, $\|\nabla Tg\|_{L^{\frac{np}{n-1}}(\mathbb{R}^n_+)} \leq c(n,p,\rho)\|\nabla g\|_{L^p(\mathbb{R}^{n-1})}$.

By decomposition of unity, flattening the boundary and applying the above fact, we may find some $v \in W^{1,\frac{np}{n-1}}(\Omega)$ with $|v\|_{W^{1,\frac{np}{n-1}}(\Omega)} \leq c(p,\Omega)\|f\|_{W^{1,p}(\Sigma)}$ and $v|_\Sigma = f$. Then $\Delta(u - v) = -\Delta v$ and $(u-v)|_{\partial\Omega} = 0$. It follows from elliptic estimate that

$$\|u - v\|_{W^{1,\frac{np}{n-1}}(\Omega)} \leq c(p,\Omega)\|v\|_{W^{1,\frac{np}{n-1}}(\Omega)}.$$

Hence

$$\|u\|_{W^{1,\frac{np}{n-1}}(\Omega)} \leq c(p,\Omega)\|v\|_{W^{1,\frac{np}{n-1}}(\Omega)} \leq c(p,\Omega)\|f\|_{W^{1,p}(\Sigma)}.$$

$\square$

The next lemma gives us the existence of suitable auxiliary functions with energy control.

Lemma 9.7.. *Assume that* $n \geq 3$*,* $f : \partial B_1^n \to S^{l-1} \subset \mathbb{R}^l$ *such that* $\int_{\partial B_1} |df|^{n-1} dS \leq \varepsilon(l,n)$ *small,then there exists a* $u \in W^{1,n}\left(B_2 \backslash B_1, S^{l-1}\right)$ *such that* $u|_{\partial B_1} = f$*,* $u|_{\partial B_2} = \mathrm{const}$ *and*

$$\|\nabla u\|_{L^n(B_2\backslash B_1)} \leq c(l,n)\|df\|_{L^{n-1}(\partial B_1)}.$$

Proof. Set $f_{\partial B_1} = \frac{1}{|\partial B_1|}\int_{\partial B_1} f \mathrm{d}S$. By the Poincaré inequality, we have

$$\int_{\partial B_1} |f - f_{\partial B_1}| \,\mathrm{d}S \leq c(l,n)\|\mathrm{d}f\|_{L^{n-1}(\partial B_1)} \leq c(l,n)\,\varepsilon^{\frac{1}{n-1}}.$$

Hence $||f_{\partial B_1}| - 1| \leq c(l,n)\,\varepsilon^{\frac{1}{n-1}}$. We can solve the Dirichlet problem

$$\begin{cases} \Delta v = 0 \quad \text{on } B_2\backslash B_1, \\ v|_{\partial B_1} = f, \\ v|_{\partial B_2} = f_{\partial B_1}. \end{cases}$$

Then $\Delta(v - f_{\partial B_1}) = 0$ on $B_2\backslash B_1$, $(v - f_{\partial B_1})|_{\partial B_1} = f$, $(v - f_{\partial B_1})|_{\partial B_2} = 0$. It follows from Lemma 9.6 that

$$\|v - f_{\partial B_1}\|_{W^{1,n}(B_2\backslash B_1)} \leq c(l,n)\|f - f_{\partial B_1}\|_{W^{1,n-1}(\partial B_1)} \leq c(l,n)\,\varepsilon^{\frac{1}{n-1}}.$$

It follows that, for $\delta > 0$ small,

$$\|v - f_{\partial B_1}\|_{L^\infty(B_2\backslash B_{1+\delta})} \leq c(n,l,\delta)\,\varepsilon^{\frac{1}{n-1}}.$$

For $x \in B_{\frac{3}{2}}\backslash B_1$, $\xi \in \partial B_1 \cup \partial B_2$, we let $P(x,\xi)$ be the Poisson kernel. For $\xi \in \partial B_2$, define $f(\xi) = f_{\partial B_1}$. Then $v(x) = \int_{\partial B_1 \cup \partial B_2} P(x,\xi) f(\xi)\,\mathrm{d}S(\xi)$. Set $\xi_0 = \frac{x}{|x|}$, $r = |x| - 1$. Then classical estimate for the Poisson kernel gives (see [HWY2, lemma 2.2 and section 5])

$$0 \leq P(x,\xi) \leq \frac{c(n)\,r}{\left(r^2 + |\xi - \xi_0|^2\right)^{\frac{n}{2}}}.$$

For $k \geq 1$ with $kr \leq \frac{1}{2}$, we write

$$f_{kr,\xi_0} = \frac{1}{|\partial B_1 \cap B_{kr}(\xi_0)|}\int_{\partial B_1 \cap B_{kr}(\xi_0)} f \mathrm{d}S.$$

Using the Poincaré inequality, we see that

$$\begin{aligned} \frac{1}{|\partial B_1 \cap B_{kr}(\xi_0)|}\int_{\partial B_1 \cap B_{kr}(\xi_0)} |f - f_{kr,\xi_0}|\,\mathrm{d}S &\leq c(l,n)\|\mathrm{d}f\|_{L^{n-1}(\partial B_1 \cap B_{kr}(\xi_0))} \\ &\leq c(l,n)\,\varepsilon^{\frac{1}{n-1}}. \end{aligned}$$

Hence $||f_{kr,\xi_0}|-1| \le c(l,n)\,\varepsilon^{\frac{1}{n-1}}$. On the other hand,

$$\begin{aligned}
|v(x) - f_{kr,\xi_0}| &= \left|\int_{\partial B_1\cup\partial B_2} P(x,\xi)\,(f(\xi) - f_{kr,\xi_0})\,\mathrm{d}S(\xi)\right| \\
&\le \int_{(\partial B_1\setminus B_{kr}(\xi_0))\cup\partial B_2} P(x,\xi)\,|f(\xi) - f_{kr,\xi_0}|\,\mathrm{d}S(\xi) \\
&\quad + \int_{\partial B_1\cap B_{kr}(\xi_0)} P(x,\xi)\,|f(\xi) - f_{kr,\xi_0}|\,\mathrm{d}S(\xi) \\
&\le c(l,n)\left(r + \frac{1}{k}\right) + \frac{c(n)}{r^{n-1}}\int_{\partial B_1\cap B_{kr}(\xi_0)} |f(\xi) - f_{kr,\xi_0}|\,\mathrm{d}S(\xi) \\
&\le \frac{c(l,n)}{k} + c(l,n)\,k^{n-1}\,|\mathrm{d}f|_{L^{n-1}(\partial B_1\cap B_{kr}(\xi_0))} \\
&\le c(l,n)\left(\frac{1}{k} + k^{n-1}\varepsilon^{\frac{1}{n-1}}\right).
\end{aligned}$$

Hence

$$||v(x)| - 1| \le c(l,n)\left(\frac{1}{k} + k^{n-1}\varepsilon^{\frac{1}{n-1}}\right).$$

By fixing k large, r small, and then ε small, we have $\|\,|v|-1\|_{L^\infty(B_2\setminus B_1)} \le \frac{1}{2}$. Let $u(x) = \frac{v(x)}{|v(x)|}$. Then u satisfies all the requirements of the lemma. □

To prove that the Hopf–Whitehead invariant $Q(u)$ must be an integer for any map u with finite Faddeev energy, we need to show that the invariant of a suitable weakly differentiable map must be an integer. For this purpose, we recall some ideas from [Sv, EM].

Proposition 9.8.. ([Sv, Section 2]) *Assume that M^n and N^n are both smoothly oriented Riemannian manifolds, $u \in W^{1,1}_{loc}(M^n, N^n)$ such that and $J_u = |\det du| \in L^1(M^n)$. Then there exists a measure zero subset E of M^n such that the function*

$$d(u,y) = \sum_{x\in u^{-1}(y)\setminus E} \operatorname{sgn}(\det du(x))$$

is integrable on N^n and for every $f \in L^\infty(N^n)$,

$$\int_{M^n} u^*(f\omega_{N^n}) = \int_{M^n} f(u(x))\det du(x)\,d\mu_{M^n}(x) = \int_{N^n} f(y)\cdot d(u,y)\,d\mu_{N^n}(y).$$

Here ω_{N^n} is the volume form on N^n, μ_{M^n} is the measure on M^n associated with the Riemannian metric.

Proposition 9.8 follows from the Lusin type theorems and the usual coarea formula for Lipschitz functions. The idea of [Sv, EM] to show that $d(u,y)$ is independent of y is to show $\int_{M^n} u^*(f\omega_{N^n}) = 0$ whenever $\int_{N^n} f\omega_{N^n} = 0$. To achieve that, the following basic fact is useful.

Lemma 9.9.. *Assume that $n \ge 2$, $1 \le p \le \frac{n}{n-1}$, or $n = 1$ but $1 \le p < \infty$, and $\alpha \in L^p(\mathbb{R}^n)$ is a $(n-1)$-form with $d\alpha \in L^1(\mathbb{R}^n)$. Then $\int_{\mathbb{R}^n} d\alpha = 0$.*

Proof. By a mollifying function argument, we may assume that $\alpha \in C^{\infty}(\mathbb{R}^n)$. Fix some $\phi \in C_c^{\infty}(\mathbb{R}^n)$ such that $\phi|_{B_{1/2}} = 1$ and $\phi|_{\mathbb{R}^n \backslash B_1} = 0$. For $R > 0$, we write $\phi_R(x) = \phi\left(\frac{x}{R}\right)$. Then

$$0 = \int_{\mathbb{R}^n} \mathrm{d}(\phi_R \alpha) = \int_{\mathbb{R}^n} \mathrm{d}\phi_R \wedge \alpha + \int_{\mathbb{R}^n} \phi_R \mathrm{d}\alpha.$$

Note that

$$\left| \int_{\mathbb{R}^n} \mathrm{d}\phi_R \wedge \alpha \right| \leq \frac{c(n)}{R} \int_{B_R \backslash B_{R/2}} |\alpha| \leq c(n,p) \left(\int_{B_R \backslash B_{R/2}} |\alpha|^p \right)^{\frac{1}{p}} R^{n-1-\frac{n}{p}} \to 0$$

as $R \to \infty$. Hence, by letting $R \to \infty$ in the first equation, we get $\int_{\mathbb{R}^n} \mathrm{d}\alpha = 0$. □

In Lemma 9.9, the requirement $p \leq \frac{n}{n-1}$ is crucial. Indeed, for $n \geq 2$, let Γ be the fundamental solution of the Laplacian, $\phi \in C_c^{\infty}(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} \phi(x)\,\mathrm{d}x = 1$, and let

$$\alpha = (-1)^{n+1} * \mathrm{d}(\phi * \Gamma).$$

Then for any $q > \frac{n}{n-1}$, $\alpha \in L^q(\mathbb{R}^n)$ and $\mathrm{d}\alpha = \phi \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n$. Hence $\int_{\mathbb{R}^n} \mathrm{d}\alpha = 1$.

10. The Hopf–Whitehead Invariant: Integer-Valuedness

In this section, we will prove that for a map with finite Faddeev energy, the Hopf–Whitehead invariant $Q(u)$ is always an integer. This fact is not only needed for us to come up with a reasonable mathematical formulation for the Faddeev model but also plays a crucial role in understanding the minimizing sequences for the minimization problems.

Theorem 10.1.. *Assume that $u \in W^{1,1}_{loc}(\mathbb{R}^{4n-1}, S^{2n})$ such that*

$$\int_{\mathbb{R}^{4n-1}} |du|^{4n-2} + |u^* \omega_{S^{2n}}|^2 < \infty,$$

where $\omega_{S^{2n}}$ is the volume form on S^{2n}. Then $du^* \omega_{S^{2n}} = 0$. Let

$$\Gamma(x) = \frac{1}{(4n-3)|S^{4n-2}||x|^{4n-3}}, \quad \tau = d^*(\Gamma * u^* \omega_{S^{2n}}),$$

where d^* is the L^2-dual of d, $|S^{4n-2}|$ is the area of S^{4n-2}. Then $\tau \in L^2(\mathbb{R}^{4n-1})$, $d\tau = u^* \omega_{S^{2n}}$, $d^* \tau = 0$, and the Hopf–Whitehead invariant

$$Q(u) = \frac{1}{|S^{2n}|^2} \int_{\mathbb{R}^{4n-1}} u^* \omega_{S^{2n}} \wedge \tau$$

is well defined and equal to an integer.

To prove Theorem 10.1, we first show that $\mathrm{d}u^* \omega_{S^{2n}} = 0$.

Claim 10.2.. *For any smooth $2n$-form α on S^{2n}, we have $du^* \alpha = u^* d\alpha = 0$.*

Proof. By linearity we may assume $\alpha = f_0 \mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_{2n}$, where $f_0, \cdots, f_{2n} \in C_c^{\infty}(\mathbb{R}^{2n+1}, \mathbb{R})$. Because $u \in W^{1,4n-2}(\mathbb{R}^{4n-1}) \subset W^{1,2n}_{loc}(\mathbb{R}^{4n-1})$, it follows from Lemma 9.2 that

$$\mathrm{d}u^*(f_1 \mathrm{d}f_2 \wedge \cdots \wedge \mathrm{d}f_{2n}) = u^*(\mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_{2n}).$$

Hence

$$\mathrm{d}u^*(\mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_{2n}) = 0.$$

For any integer k, we write

$$\Lambda_k\left(\mathrm{d}u\right)=\overbrace{\mathrm{d}u\wedge\cdots\wedge\mathrm{d}u}^{k\text{ times}}.$$

Then $\left|u^*\omega_{S^{2n}}\right|=\left|\Lambda_{2n}\left(\mathrm{d}u\right)\right|$. It follows that $\Lambda_{2n}\left(\mathrm{d}u\right)\in L^2\left(\mathbb{R}^{4n-1}\right)$. Hence

$$u^*\left(\mathrm{d}f_1\wedge\cdots\wedge\mathrm{d}f_{2n}\right)\in L^2\left(\mathbb{R}^{4n-1}\right).$$

On the other hand, because $f_0\circ u\in L^\infty\left(\mathbb{R}^{4n-1}\right)$, $\mathrm{d}\left(f_0\circ u\right)\in L^{4n-2}(\mathbb{R}^{4n-1})\subset L^2_{loc}(\mathbb{R}^{4n-1})$, it follows from Lemma 9.1 that

$$\begin{aligned}\mathrm{d}u^*\alpha&=\mathrm{d}\left(f_0\circ u\cdot u^*\left(\mathrm{d}f_1\wedge\cdots\wedge\mathrm{d}f_{2n}\right)\right)\\&=\mathrm{d}\left(f_0\circ u\right)\wedge u^*\left(\mathrm{d}f_1\wedge\cdots\wedge\mathrm{d}f_{2n}\right)\\&=u^*\mathrm{d}\alpha=0.\end{aligned}$$

□

Note that $u^*\omega_{S^{2n}}\in L^{\frac{2n-1}{n}}\cap L^2$ where and in the sequel, we often omit the domain space when there is no risk of confusion. Hence, if we let $\eta=\Gamma*u^*\omega_{S^{2n}}$, then

$$\mathrm{d}\eta=0,\quad \mathrm{d}\mathrm{d}^*\eta=\Delta\eta=u^*\omega_{S^{2n}}.$$

Here Γ is the fundamental solution of the Laplacian operator on $\mathbb{R}^{4n-1}$, $*$ means we convolute each component of $u^*\omega_{S^{2n}}$ with Γ and in $\Delta\eta$, the Δ is equal to $\mathrm{d}\mathrm{d}^*+\mathrm{d}^*\mathrm{d}$ (the Hodge Laplacian, it is the negative of the standard Laplacian when acting on functions). Let $\tau=\mathrm{d}^*\eta$. Then $\mathrm{d}\tau=u^*\omega_{S^{2n}}$. It follows from the usual singular integral estimate that ([St])

$$\begin{aligned}&\tau\in L^{\frac{8n^2-6n+1}{4n^2-3n+1}}\cap L^{\frac{2(4n-1)}{4n-3}},\quad D\tau\in L^{\frac{2n-1}{n}}\cap L^2\quad\text{when } n\geq 2;\\&\tau\in L^{\frac{3}{2}+\varepsilon}\cap L^6,\quad D\tau\in L^{1+\varepsilon}\cap L^2\quad\text{when } n=1.\end{aligned}$$

Here ε is an arbitrarily small positive number. In particular, we always have $\tau\in L^2\left(\mathbb{R}^{4n-1}\right)$ and

$$Q\left(u\right)=\frac{1}{\left|S^{2n}\right|^2}\int_{\mathbb{R}^{4n-1}}u^*\omega_{S^{2n}}\wedge\tau$$

is well defined. To show it is an integer, we will first use an idea from [HR, Section II.4] which would imply that $Q\left(u\right)$ is equal to the usual Hopf–Whitehead invariant of another weakly differentiable map. Then we will apply ideas from [Sv, EM] to show that the invariant is an integer.

Claim 10.3.. *Let* $U:\mathbb{R}^{4n-1}\times\mathbb{R}^{4n-1}\to S^{2n}\times S^{2n}\times S^{4n-2}$ *be given by*

$$U\left(x,y\right)=\left(u\left(x\right),u\left(y\right),\frac{x-y}{\left|x-y\right|}\right).$$

Then $U^*\omega_{S^{2n}\times S^{2n}\times S^{4n-2}}\in L^1$ *and*

$$Q\left(u\right)=-\frac{1}{\left|S^{2n}\right|^2\left|S^{4n-2}\right|}\int_{\mathbb{R}^{4n-1}\times\mathbb{R}^{4n-1}}U^*\omega_{S^{2n}\times S^{2n}\times S^{4n-2}}.$$

Roughly speaking, the claim says the Hopf invariant of u is equal to the degree of U. This is a special case of a more general formula for rational homotopy in [HR, section II.4]. Since we will need the proof later on and for completeness, we present the argument in this special case.

Proof. Let $J_u = |u^*\omega_{S^{2n}}|$ be the Jacobian of u, then

$$J_U(x,y) \le c(n)\frac{J_u(x)\,J_u(y)}{|x-y|^{4n-2}}.$$

Because $J_u \in L^{\frac{2n-1}{n}} \cap L^2$, we see $J_u \in L^{\frac{4n-1}{2n}}(\mathbb{R}^{4n-1})$. It follows from the classical Hardy–Littlewood–Sobolev inequality ([St]) that $J_U \in L^1(\mathbb{R}^{8n-2})$, that is $U^*\omega_{S^{2n}\times S^{2n}\times S^{4n-2}} \in L^1$. □

To continue, note that for $x, y \in \mathbb{R}^m$, the map $\frac{x-y}{|x-y|} : \mathbb{R}^m \times \mathbb{R}^m \to S^{m-1}$ satisfies

$$\begin{aligned}&\left(\frac{x-y}{|x-y|}\right)^* \omega_{S^{m-1}}\\ &= \frac{1}{|x-y|^m}\sum_{k=0}^{m}\sum_{\lambda\in\Lambda(m,k)}\sum_{i=0}^{k}(-1)^{m-k}\operatorname{sgn}\left(\lambda,\overline{\lambda}\right)(x_{\lambda_i}-y_{\lambda_i})\,(\mathrm{d}x_\lambda)\Big\lfloor\partial_{x_{\lambda_i}} \wedge \mathrm{d}y_{\overline{\lambda}}.\end{aligned}$$

Indeed, under the spherical coordinate, the metric and volume forms of $\mathbb{R}^m$ and S^{m-1} are given by

$$\begin{aligned} g_{\mathbb{R}^m} &= \mathrm{d}r\otimes\mathrm{d}r + r^2\sum_{1\le i,j\le m-1} b_{ij}(\theta)\,\mathrm{d}\theta_i\otimes\mathrm{d}\theta_j,\\ \omega_{S^{m-1}} &= \sqrt{B(\theta)}\mathrm{d}\theta_1\wedge\cdots\wedge\mathrm{d}\theta_{m-1},\end{aligned}$$

respectively, where $B(\theta) = \det(b_{ij}(\theta))$. Hence

$$\begin{aligned}\left(\frac{x}{|x|}\right)^*\omega_{S^{m-1}} &= \sqrt{B(\theta)}\mathrm{d}\theta_1\wedge\cdots\wedge\mathrm{d}\theta_{m-1}\\ &= \frac{1}{r^{m-1}}\left(r^{m-1}\sqrt{B(\theta)}\mathrm{d}r\wedge\mathrm{d}\theta_1\wedge\cdots\wedge\mathrm{d}\theta_{m-1}\right)\lfloor\partial_r\\ &= \frac{1}{|x|^m}(\mathrm{d}x_1\wedge\cdots\wedge\mathrm{d}x_m)\Bigg\lfloor\sum_{k=1}^{m}x_k\partial_{x_k}.\end{aligned}$$

It follows that

$$\begin{aligned}&\left(\frac{x-y}{|x-y|}\right)^*\omega_{S^{m-1}}\\ &= \frac{1}{|x-y|^m}\sum_{j=1}^{m}(-1)^{j-1}(x_j-y_j)(\mathrm{d}x_1-\mathrm{d}y_1)\wedge\cdots\wedge(\mathrm{d}x_{j-1}-\mathrm{d}y_{j-1})\\ &\wedge(\mathrm{d}x_{j+1}-\mathrm{d}y_{j+1})\wedge\cdots\wedge(\mathrm{d}x_m-\mathrm{d}y_m).\end{aligned}$$

Developing the product out we get the needed formula.

Proof continued. We may write

$$u^*\omega_{S^{2n}} = \sum_\lambda f_\lambda(x)\,\mathrm{d}x_\lambda.$$

Here λ runs over elements in $\Lambda(4n-1,2n)$, and the same for μ,ν we use below. Then

$$\begin{aligned}
&U^*\omega_{S^{2n}\times S^{2n}\times S^{4n-2}}\\
&=-\sum_\lambda f_\lambda(x)\,\mathrm{d}x_\lambda\wedge\sum_\mu f_\mu(y)\,\mathrm{d}y_\mu\wedge\frac{1}{|x-y|^{4n-1}}\sum_\nu\sum_{i=0}^{2n}\operatorname{sgn}(\nu,\overline{\nu})(x_{\nu_i}-y_{\nu_i})\cdot\\
&(\mathrm{d}x_\nu)\lfloor\partial_{x_{\nu_i}}\wedge\mathrm{d}y_{\overline{\nu}}\\
&=-\frac{1}{|x-y|^{4n-1}}\sum_\lambda\sum_\mu\sum_{i=0}^{2n}f_\lambda(x)f_\mu(y)\,\mathrm{d}x_\lambda\wedge\mathrm{d}y_\mu\wedge\operatorname{sgn}(\mu,\overline{\mu})(x_{\mu_i}-y_{\mu_i})\cdot\\
&(\mathrm{d}x_\mu)\lfloor\partial_{x_{\mu_i}}\wedge\mathrm{d}y_{\overline{\mu}}\\
&=-\frac{1}{|x-y|^{4n-1}}\sum_\lambda\sum_\mu\sum_{j=0}^{4n-1}f_\lambda(x)f_\mu(y)\,\mathrm{d}x_\lambda\wedge(x_j-y_j)(\mathrm{d}x_\mu)\lfloor\partial_{x_j}\wedge\mathrm{d}y_1\wedge\cdots\wedge\mathrm{d}y_n.
\end{aligned}$$

Hence

$$\begin{aligned}
&\int_{\mathbb{R}^{4n-1}\times\mathbb{R}^{4n-1}}U^*\omega_{S^{2n}\times S^{2n}\times S^{4n-2}}\\
&=-\sum_\lambda\sum_\mu\sum_{j=0}^{4n-1}\int_{\mathbb{R}^{4n-1}}\left(\int_{\mathbb{R}^{4n-1}}f_\lambda(x)f_\mu(y)\frac{x_j-y_j}{|x-y|^{4n-1}}\mathrm{d}y\right)\mathrm{d}x_\lambda\wedge(\mathrm{d}x_\mu)\lfloor\partial_{x_j}\\
&=\left|S^{4n-2}\right|\int_{\mathbb{R}^{4n-1}}\sum_\lambda\sum_\mu\sum_{j=0}^{4n-1}\int_{\mathbb{R}^{4n-1}}\partial_j(\Gamma*f_\mu)(x)f_\lambda(x)\,dx_\lambda\wedge(\mathrm{d}x_\mu)\lfloor\partial_{x_j}\\
&=-\left|S^{4n-2}\right|\int_{\mathbb{R}^{4n-1}}u^*\omega_{S^{2n}}\wedge\mathrm{d}^*\eta,
\end{aligned}$$

where

$$\eta=\sum_\lambda(\Gamma*f_\lambda)\,\mathrm{d}x_\lambda.$$

Hence

$$Q(u)=-\frac{1}{|S^{2n}|^2|S^{4n-2}|}\int_{\mathbb{R}^{4n-1}\times\mathbb{R}^{4n-1}}U^*\omega_{S^{2n}\times S^{2n}\times S^{4n-2}}.$$

□

It follows from Proposition 9.8 that there exists an integer-valued function $d_U\in L^1(S^{2n}\times S^{2n}\times S^{4n-2})$ such that for every $f\in L^\infty(S^{2n}\times S^{2n}\times S^{4n-2})$,

$$\begin{aligned}
&\int_{\mathbb{R}^{4n-1}\times\mathbb{R}^{4n-1}}f\left(u(x),u(y),\frac{x-y}{|x-y|}\right)(u^*\omega_{S^{2n}})(x)\wedge(u^*\omega_{S^{2n}})(y)\wedge\left(\frac{x-y}{|x-y|}\right)^*\omega_{S^{4n-2}}\\
&=\int_{S^{2n}\times S^{2n}\times S^{4n-2}}f(z)\,d_U(z)\,\mathrm{d}S(z')\,\mathrm{d}S(z'')\,\mathrm{d}S(z''').
\end{aligned}$$

Here $z=(z',z'',z''')$. Denote

$$C_1=\frac{1}{|S^{2n}|^2|S^{4n-2}|}\int_{S^{2n}\times S^{2n}\times S^{4n-2}}d_U(z)\,\mathrm{d}S(z')\,\mathrm{d}S(z'')\,\mathrm{d}S(z''').$$

Once we know $d_U \equiv C_1$, by choosing $f = 1$ in the above equation, it follows from Claim 10.3 that $H(u) = -C_1$ is an integer. To show $d_U \equiv C_1$, we only need to prove the following.

Claim 10.4.. *For every* $f \in L^\infty(S^{2n} \times S^{2n} \times S^{4n-2})$,

$$\begin{aligned}&\int_{S^{2n}\times S^{2n}\times S^{4n-2}} f(z)\, d_U(z)\, dS(z')\, dS(z'')\, dS(z''')\\&= C_1 \int_{S^{2n}\times S^{2n}\times S^{4n-2}} f(z)\, dS(z')\, dS(z'')\, dS(z''').\end{aligned}$$

By approximation we only need to verify the equality for

$$f(z) = f_1(z')\, f_2(z'')\, f_3(z'''),$$

$f_1, f_2 \in C^\infty(S^{2n})$, $f_3 \in C^\infty(S^{4n-2})$. To achieve this we only need to prove

(a) If $\int_{S^{4n-2}} f_3(z''')\,\mathrm{d}S(z''') = 0$, then

$$\int_{S^{2n}\times S^{2n}\times S^{4n-2}} f_1(z')\, f_2(z'')\, f_3(z''')\, d_U(z)\, \mathrm{d}S(z')\, \mathrm{d}S(z'')\, \mathrm{d}S(z''') = 0.$$

(b) If $\int_{S^{2n}} f_2(z'')\,\mathrm{d}S(z'') = 0$, then

$$\int_{S^{2n}\times S^{2n}\times S^{4n-2}} f_1(z')\, f_2(z'')\, d_U(z)\, \mathrm{d}S(z')\, \mathrm{d}S(z'')\, \mathrm{d}S(z''') = 0.$$

(c) If $\int_{S^{2n}} f_1(z')\,\mathrm{d}S(z') = 0$, then

$$\int_{S^{2n}\times S^{2n}\times S^{4n-2}} f_1(z')\, d_U(z)\, \mathrm{d}S(z')\, \mathrm{d}S(z'')\, \mathrm{d}S(z''') = 0.$$

Indeed, if (a)–(c) are true, then we have

$$\begin{aligned}&\int_{S^{2n}\times S^{2n}\times S^{4n-2}} f_1(z')\, f_2(z'')\, f_3(z''')\, d_U(z)\, \mathrm{d}S(z')\, \mathrm{d}S(z'')\, \mathrm{d}S(z''')\\
&= \frac{1}{|S^{4n-2}|}\int_{S^{4n-2}} f_3(z''')\,\mathrm{d}S(z''') \int_{S^{2n}\times S^{2n}\times S^{4n-2}} f_1(z')\, f_2(z'')\, d_U(z)\, \mathrm{d}S(z')\, \mathrm{d}S(z'')\, \mathrm{d}S(z''')\\
&= \frac{1}{|S^{2n}|}\frac{1}{|S^{4n-2}|}\int_{S^{2n}} f_2(z'')\,\mathrm{d}S(z'') \int_{S^{4n-2}} f_3(z''')\,\mathrm{d}S(z''')\,\cdot\\
&\int_{S^{2n}\times S^{2n}\times S^{4n-2}} f_1(z')\, d_U(z)\, \mathrm{d}S(z')\, \mathrm{d}S(z'')\, \mathrm{d}S(z''')\\
&= \frac{1}{|S^{2n}|^2}\frac{1}{|S^{4n-2}|}\int_{S^{2n}} f_1(z')\,\mathrm{d}S(z') \int_{S^{2n}} f_2(z'')\,\mathrm{d}S(z'') \int_{S^{4n-2}} f_3(z''')\,\mathrm{d}S(z''')\,\cdot\\
&\int_{S^{2n}\times S^{2n}\times S^{4n-2}} d_U(z)\, \mathrm{d}S(z')\, \mathrm{d}S(z'')\, \mathrm{d}S(z''')\\
&= C_1 \int_{S^{2n}\times S^{2n}\times S^{4n-2}} f_1(z')\, f_2(z'')\, f_3(z''')\, \mathrm{d}S(z')\, \mathrm{d}S(z'')\, \mathrm{d}S(z''').\end{aligned}$$

We start with (a). Since $\int_{S^{4n-2}} f_3(z''')\,\mathrm{d}S(z''') = 0$ we may find a smooth $(4n-3)$-form γ on S^{4n-2} such that $\mathrm{d}\gamma = f_3\omega_{S^{4n-2}}$. Note that

$$\begin{aligned}&\int_{S^{2n}\times S^{2n}\times S^{4n-2}} f_1(z') f_2(z'') f_3(z''') d_U(z)\,\mathrm{d}S(z')\,\mathrm{d}S(z'')\,\mathrm{d}S(z''')\\ &= \int_{\mathbb{R}^{4n-1}\times\mathbb{R}^{4n-1}} u^*(f_1\omega_{S^{2n}})(x)\wedge u^*(f_2\omega_{S^{2n}})(y)\wedge\left(\frac{x-y}{|x-y|}\right)^*(\mathrm{d}\gamma).\end{aligned}$$

Recall that $\Lambda_{2n}(\mathrm{d}u)\in L^{\frac{2n-1}{n}}\cap L^2\subset L^{\frac{4n-1}{2n}}$. Let $\theta=\frac{8n-2}{8n-3}$. Note that

$$\begin{aligned}&\left|u^*(f_1\omega_{S^{2n}})(x)\wedge u^*(f_2\omega_{S^{2n}})(y)\wedge\left(\frac{x-y}{|x-y|}\right)^*\gamma\right|\\ &\le c\frac{|\Lambda_{2n}(\mathrm{d}u)(x)|\,|\Lambda_{2n}(\mathrm{d}u)(y)|}{|x-y|^{4n-3}}.\end{aligned}$$

It follows from the fact $\Lambda_{2n}(\mathrm{d}u)\in L^{\frac{4n-1}{2n}}$, the Hardy–Littlewood–Sobolev inequality, and

$$\frac{2n\theta}{4n-1}+\frac{2n\theta}{4n-1}=1+\frac{4n-1-(4n-3)\theta}{4n-1}$$

that

$$\frac{|\Lambda_{2n}(\mathrm{d}u)(x)|^\theta\,|\Lambda_{2n}(\mathrm{d}u)(y)|^\theta}{|x-y|^{(4n-3)\theta}}\in L^1\left(\mathbb{R}^{4n-1}\times\mathbb{R}^{4n-1}\right).$$

Hence

$$u^*(f_1\omega_{S^{2n}})(x)\wedge u^*(f_2\omega_{S^{2n}})(y)\wedge\left(\frac{x-y}{|x-y|}\right)^*\gamma\in L^\theta\left(\mathbb{R}^{4n-1}\times\mathbb{R}^{4n-1}\right).$$

Claim 10.5..

$$\begin{aligned}&d\left[u^*(f_1\omega_{S^{2n}})(x)\wedge u^*(f_2\omega_{S^{2n}})(y)\wedge\left(\frac{x-y}{|x-y|}\right)^*\gamma\right]\\ &= u^*(f_1\omega_{S^{2n}})(x)\wedge u^*(f_2\omega_{S^{2n}})(y)\wedge\left(\frac{x-y}{|x-y|}\right)^*(d\gamma).\end{aligned}$$

Proof. Because $\frac{x-y}{|x-y|}\in W^{1,4n-2}_{loc}(\mathbb{R}^{4n-1}\times\mathbb{R}^{4n-1})$, it follows from Lemma 9.2 that

$$\mathrm{d}\left[\left(\frac{x-y}{|x-y|}\right)^*\gamma\right]=\left(\frac{x-y}{|x-y|}\right)^*(\mathrm{d}\gamma).$$

On the other hand, it follows from Claim 10.2 that

$$\mathrm{d}\left[u^*(f_1\omega_{S^{2n}})\right]=0.$$

By smoothing we may find a sequence of smooth $2n$-forms on $\mathbb{R}^{4n-1}$, namely α_i, such that

$$\alpha_i\to u^*(f_1\omega_{S^{2n}})\ \text{ in } L^{\frac{4n-1}{2n}}\left(\mathbb{R}^{4n-1}\right)$$

and $\mathrm{d}\alpha_i=0$. Similarly we may find a sequence of smooth $2n$-forms on $\mathbb{R}^{4n-1}$, namely β_i such that

$$\beta_i\to u^*(f_2\omega_{S^{2n}})\ \text{ in } L^{\frac{4n-1}{2n}}\left(\mathbb{R}^{4n-1}\right)$$

and $\mathrm{d}\beta_i=0$. It follows from Hardy–Littlewood–Sobolev inequality that

$$\alpha_i(x)\wedge\beta_i(y)\wedge\left(\frac{x-y}{|x-y|}\right)^*\gamma\to u^*(f_1\omega_{S^{2n}})(x)\wedge u^*(f_2\omega_{S^{2n}})(y)\wedge\left(\frac{x-y}{|x-y|}\right)^*\gamma$$

in $L^{\frac{8n-2}{8n-3}}\left(\mathbb{R}^{4n-1}\times\mathbb{R}^{4n-1}\right)$ as $i\to\infty$. Similarly

$$\alpha_i(x)\wedge\beta_i(y)\wedge\left(\frac{x-y}{|x-y|}\right)^*(\mathrm{d}\gamma)\to u^*\left(f_1\omega_{S^{2n}}\right)(x)\wedge u^*\left(f_2\omega_{S^{2n}}\right)(y)\wedge\left(\frac{x-y}{|x-y|}\right)^*(\mathrm{d}\gamma)$$

in $L^1\left(\mathbb{R}^{4n-1}\times\mathbb{R}^{4n-1}\right)$ as $i\to\infty$. Taking limit in the equality

$$\mathrm{d}\left[\alpha_i(x)\wedge\beta_i(y)\wedge\left(\frac{x-y}{|x-y|}\right)^*\gamma\right]=\alpha_i(x)\wedge\beta_i(y)\wedge\left(\frac{x-y}{|x-y|}\right)^*(\mathrm{d}\gamma),$$

we prove the claim. □

It follows from Claim 10.5, Lemma 9.9, and the fact $1<\frac{8n-2}{8n-3}<\frac{4n-1}{4n-2}$ that

$$\int_{\mathbb{R}^{4n-1}\times\mathbb{R}^{4n-1}}u^*\left(f_1\omega_{S^{2n}}\right)(x)\wedge u^*\left(f_2\omega_{S^{2n}}\right)(y)\wedge\left(\frac{x-y}{|x-y|}\right)^*(\mathrm{d}\gamma)=0.$$

Part (a) follows.

Next we check part (b). If $\int_{S^{2n}}f_2(z'')\,\mathrm{d}S(z'')=0$, then we may find a smooth $(2n-1)$-form γ on S^{2n} such that $\mathrm{d}\gamma=f_2\omega_{S^{2n}}$. We have

$$\begin{aligned}&\int_{S^{2n}\times S^{2n}\times S^{4n-2}}f_1(z')\,f_2(z'')\,d_U(z)\,\mathrm{d}S(z')\,\mathrm{d}S(z'')\,\mathrm{d}S(z''')\\&=\int_{\mathbb{R}^{4n-1}\times\mathbb{R}^{4n-1}}u^*\left(f_1\omega_{S^{2n}}\right)(x)\wedge u^*\left(f_2\omega_{S^{2n}}\right)(y)\wedge\left(\frac{x-y}{|x-y|}\right)^*\omega_{S^{4n-2}}\\&=-\left|S^{4n-2}\right|\int_{\mathbb{R}^{4n-1}}u^*\left(f_2\omega_{S^{2n}}\right)\wedge\tau_1.\end{aligned}$$

Here

$$\tau_1=\tau=\mathrm{d}^*\left(\Gamma*u^*\left(f_1\omega_{S^{2n}}\right)\right).$$

We have used the calculation in the proof of Claim 10.3 in the last step. By Claim 10.2, $\mathrm{d}u^*\left(f_1\omega_{S^{2n}}\right)=0$. This together with $u^*\left(f_1\omega_{S^{2n}}\right)\in L^{\frac{4n-1}{2n}}$ implies

$$\tau_1\in L^{\frac{4n-1}{2n-1}},\quad \mathrm{d}\tau_1=u^*\left(f_1\omega_{S^{2n}}\right).$$

Because $u\in W^{1,4n-2}\left(\mathbb{R}^{4n-1}\right)$, it follows from Lemma 9.2 that

$$u^*\left(f_2\omega_{S^{2n}}\right)=u^*(\mathrm{d}\gamma)=\mathrm{d}u^*\gamma.$$

Using $u^*\gamma\in L^2$, $\tau_1\in L^{\frac{4n-1}{2n-1}}$, $\mathrm{d}u^*\gamma=u^*\left(f_2\omega_{S^{2n}}\right)\in L^2$, $\mathrm{d}\tau_1=u^*\left(f_1\omega_{S^{2n}}\right)\in L^{\frac{4n-1}{2n}}\cap L^2$, it follows from Lemma 9.1 that

$$\begin{aligned}\mathrm{d}\left(u^*\gamma\wedge\tau_1\right)&=\mathrm{d}u^*\gamma\wedge\tau_1-u^*\gamma\wedge\mathrm{d}\tau_1\\&=\mathrm{d}u^*\gamma\wedge\tau_1-u^*\gamma\wedge u^*\left(f_1\omega_{S^{2n}}\right)\\&=\mathrm{d}u^*\gamma\wedge\tau_1\\&=u^*\left(f_2\omega_{S^{2n}}\right)\wedge\tau_1.\end{aligned}$$

Note that $u^*\gamma\wedge\tau_1\in L^{\frac{8n-2}{8n-3}}$ and $1<\frac{8n-2}{8n-3}<\frac{4n-1}{4n-2}$. Applying Lemma 9.9, we get

$$\int_{\mathbb{R}^{4n-1}}u^*\left(f_2\omega_{S^{2n}}\right)\wedge\tau_1=0.$$

Part (b) follows. Part (c) can be proved exactly in the same way as part (b). This finishes the proof of Claim 10.4 and hence Theorem 10.1.

It is worth pointing out that there is freedom in the choice of τ in Theorem 10.1. More precisely, we have

Proposition 10.6.. *Assume $u \in W^{1,1}_{loc}\left(\mathbb{R}^{4n-1}, S^{2n}\right)$ such that*

$$\int_{\mathbb{R}^{4n-1}} \left\{|du|^{4n-2} + |u^*\omega_{S^{2n}}|^2\right\} < \infty,$$

and that α is a smooth $2n$-form on S^{2n}. Then $du^*\alpha = 0$. If $2 \le p < \frac{(2n-1)(4n-1)}{n(4n-3)}$, $\beta \in L^p\left(\mathbb{R}^{4n-1}\right)$ is a $(2n-1)$-form such that $d\beta = u^*\alpha$, then

$$\int_{\mathbb{R}^{4n-1}} u^*\alpha \wedge \beta = Q(u)\left(\int_{S^{2n}} \alpha\right)^2.$$

Proof. Claim 10.2 implies that $\mathrm{d}u^*\alpha = 0$. Since $u^*\alpha \in L^{\frac{2n-1}{n}} \cap L^2$, it follows that $\mathrm{dd}^*\left(\Gamma * u^*\alpha\right) = u^*\alpha$ and $\mathrm{d}^*\left(\Gamma * u^*\alpha\right) \in L^p\left(\mathbb{R}^{4n-1}\right)$. Hence we may find $\beta \in L^p$ with $\mathrm{d}\beta = u^*\alpha$. Using $\frac{(2n-1)(4n-1)}{n(4n-3)} < \frac{2n-1}{n-1}$, we get $u^*\alpha \wedge \beta \in L^1\left(\mathbb{R}^{4n-1}\right)$. We claim that $\int_{\mathbb{R}^{4n-1}} u^*\alpha \wedge \beta$ does not depend on the choice of β. Indeed, if $\widetilde{\beta} \in L^p$ satisfies $\mathrm{d}\widetilde{\beta} = u^*\alpha$, then $\mathrm{d}\left(\beta - \widetilde{\beta}\right) = 0$. Hence $\beta - \widetilde{\beta} = \mathrm{d}\gamma$ for some $(2n-1)$-form $\gamma \in L^{p^*}\left(\mathbb{R}^{4n-1}\right)$, where $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{4n-1}$. Indeed we may choose $\gamma = \mathrm{d}^*\left(\Gamma * \left(\beta - \widetilde{\beta}\right)\right)$. Note that $u^*\alpha \wedge \gamma \in L^1$. It follows from Lemma 9.1 that

$$\mathrm{d}\left(u^*\alpha \wedge \gamma\right) = u^*\alpha \wedge \left(\beta - \widetilde{\beta}\right).$$

Using Lemma 9.9 we see

$$\int_{\mathbb{R}^{4n-1}} u^*\alpha \wedge \left(\beta - \widetilde{\beta}\right) = 0.$$

The claim follows. Next we look at the case $\int_{S^{2n}} \alpha = 0$. In this case we may find a smooth $(2n-1)$-form γ on S^{2n} such that $\alpha = \mathrm{d}\gamma$. It follows from Lemma 9.2 and the fact $u \in W^{1,4n-2}$ that $u^*\alpha = u^*\mathrm{d}\gamma = \mathrm{d}u^*\gamma$. Note that $u^*\gamma \in L^2$. Hence we may choose $\beta = u^*\gamma$. It follows that

$$\int_{\mathbb{R}^{4n-1}} u^*\alpha \wedge \beta = \int_{\mathbb{R}^{4n-1}} u^*\alpha \wedge u^*\gamma = 0 = Q(u)\left(\int_{S^{2n}} \alpha\right)^2.$$

Finally, if $\int_{S^{2n}} \alpha \neq 0$, by rescaling we may assume $\int_{S^{2n}} \alpha = |S^{2n}|$. Then $\alpha = \omega_{S^{2n}} + \mathrm{d}\gamma$ for some smooth $(2n-1)$-form γ. Hence

$$u^*\alpha = u^*\omega_{S^{2n}} + \mathrm{d}u^*\gamma = \mathrm{d}\tau + \mathrm{d}u^*\gamma$$

with $\tau = \mathrm{d}^*\left(\Gamma * u^*\omega_{S^{2n}}\right)$. Let $\beta = \tau + u^*\gamma$. Then $\beta \in L^2$ and $\mathrm{d}\beta = u^*\alpha$. Hence

$$\begin{aligned}\int_{\mathbb{R}^{4n-1}} u^*\alpha \wedge \beta &= \int_{\mathbb{R}^{4n-1}} u^*\alpha \wedge \tau + u^*\alpha \wedge u^*\gamma \\ &= \int_{\mathbb{R}^{4n-1}} u^*\omega_{S^{2n}} \wedge \tau + \int_{\mathbb{R}^{4n-1}} \mathrm{d}u^*\gamma \wedge \tau.\end{aligned}$$

Note that because $u^*\gamma \in L^2$, $\tau \in L^2$, $\mathrm{d}u^*\gamma = u^*\mathrm{d}\gamma \in L^2$, $\mathrm{d}\tau = u^*\omega_{S^{2n}} \in L^{\frac{4n-1}{2n}} \cap L^2$, we see that

$$\begin{aligned}\mathrm{d}\left(u^*\gamma \wedge \tau\right) &= \mathrm{d}u^*\gamma \wedge \tau - u^*\gamma \wedge \mathrm{d}\tau \\ &= \mathrm{d}u^*\gamma \wedge \tau - u^*\gamma \wedge u^*\omega_{S^{2n}} \\ &= \mathrm{d}u^*\gamma \wedge \tau.\end{aligned}$$

Hence $\int_{\mathbb{R}^{4n-1}} \mathrm{d}u^*\gamma \wedge \tau = \int_{\mathbb{R}^{4n-1}} \mathrm{d}\left(u^*\gamma \wedge \tau\right) = 0$. It follows that

$$\int_{\mathbb{R}^{4n-1}} u^*\alpha \wedge \beta = \int_{\mathbb{R}^{4n-1}} u^*\omega_{S^{2n}} \wedge \tau = \left|S^{2n}\right|^2 Q\left(u\right) = Q\left(u\right)\left(\int_{S^{2n}} \alpha\right)^2.$$

□

Using Proposition 10.6 we easily derive the following expected corollary.

Corollary 10.7. *For every $v \in C^\infty\left(S^{4n-1}, S^{2n}\right)$, let $u = v \circ \pi_{\mathbf{n}}^{-1}$, where $\pi_{\mathbf{n}} : S^{4n-1} \backslash \{\mathbf{n}\} \to \mathbb{R}^{4n-1}$ is the stereographic projection with respect to the north pole $\mathbf{n}$. Then $\int_{\mathbb{R}^{4n-1}} |du|^{4n-2} + |u^*\omega_{S^{2n}}|^2 < \infty$ and*

$$Q\left(u\right) = Q\left(v\right).$$

Here $Q\left(v\right)$ is defined as in [BT, p228] *as follows: If $v^*\omega_{S^{2n}} = d\eta$ for some smooth $(2n-1)$-form η on S^{4n-1}, then*

$$Q\left(v\right) = \frac{1}{\left|S^{2n}\right|^2} \int_{S^{4n-1}} v^*\omega_{S^{2n}} \wedge \eta.$$

Proof. Indeed since $\left|\nabla u\left(x\right)\right| \le \frac{c}{(|x|+1)^2}$, we see that $\int_{\mathbb{R}^{4n-1}} |\mathrm{d}u|^{4n-2} + |u^*\omega_{S^{2n}}|^2 < \infty$. On the other hand, $v^*\omega_{S^{2n}} = \mathrm{d}\eta$ implies

$$u^*\omega_{S^{2n}} = \left(\pi_{\mathbf{n}}^{-1}\right)^* v^*\omega_{S^{2n}} = \mathrm{d}\left(\pi_{\mathbf{n}}^{-1}\right)^* \eta = \mathrm{d}\widetilde{\tau}.$$

Here $\widetilde{\tau} = \left(\pi_{\mathbf{n}}^{-1}\right)^* \eta$. Then $|\widetilde{\tau}| \le \frac{c}{(|x|+1)^{4n-2}}$. It follows that $\widetilde{\tau} \in L^2\left(\mathbb{R}^{4n-1}\right)$. Using Proposition 10.6, we see that

$$Q\left(v\right) = \frac{1}{\left|S^{2n}\right|^2} \int_{S^{4n-1}} v^*\omega_{S^{2n}} \wedge \eta = \frac{1}{\left|S^{2n}\right|^2} \int_{\mathbb{R}^{4n-1}} u^*\omega_{S^{2n}} \wedge \widetilde{\tau} = Q\left(u\right).$$

□

When $n \ne 1, 2, 4$, $v \in C^\infty\left(S^{4n-1}, S^{2n}\right)$, classical algebraic topology tells us $Q\left(v\right)$ can only be an even integer (see [Hu, Corollary 3.6 on p214 and Theorem 4.3 on p215]). It is natural to make the following

Conjecture 1. *Under the assumption of Theorem 10.1, $Q\left(u\right)$ must be an even integer when $n \ne 1, 2, 4$.*

10.1. **Further Discussions on the Hopf–Whitehead Invariant.** In the proof of the crucial decomposition lemma (Lemma 12.1), we will see that some maps to be constructed have finite Faddeev energy on one piece of the domain and finite conformal dimensional energy on other piece of the domain. It is necessary to show such kind of maps still have integer Hopf invariant. Indeed we have the following analogue of Theorem 10.1.

Theorem 10.8.. *Assume that $u \in W^{1,1}_{loc}\left(\mathbb{R}^{4n-1}, S^{2n}\right)$ and that $\Omega \subset \mathbb{R}^{4n-1}$ is a bounded open subset with continuous boundary such that*

$$\int_\Omega |du|^{4n-2} + |u^*\omega_{S^{2n}}|^2 + \int_{\mathbb{R}^{4n-1}\backslash\Omega} |du|^{4n-1} < \infty,$$

where $\omega_{S^{2n}}$ is the volume form on S^{2n}. Then $du^*\omega_{S^{2n}} = 0$. Let

$$\Gamma(x) = \frac{1}{(4n-3)\left|S^{4n-2}\right| |x|^{4n-3}}, \quad \tau = d^*\left(\Gamma * u^*\omega_{S^{2n}}\right),$$

where d^* is the L^2-dual of d, $\left|S^{4n-2}\right|$ is the area of S^{4n-2}. Then $\tau \in L^{\frac{4n-1}{2n-1}}\left(\mathbb{R}^{4n-1}\right)$, $d\tau = u^*\omega_{S^{2n}}$, $d^*\tau = 0$. The generalized Hopf invariant

$$Q(u) = \frac{1}{|S^{2n}|^2} \int_{\mathbb{R}^{4n-1}} u^*\omega_{S^{2n}} \wedge \tau$$

is well defined and equal to an integer.

Again the first step is to show that $\mathrm{d}u^*\omega_{S^{2n}} = 0$.

Claim 10.9.. *For any smooth $2n$-form α on S^{2n}, we have $du^*\alpha = 0$.*

Proof. By linearity we may assume $\alpha = f_0 \mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_{2n}$, where $f_0, \cdots, f_{2n} \in C^\infty_c\left(\mathbb{R}^{2n+1}, \mathbb{R}\right)$. Because $u \in W^{1,2n}_{loc}$, it follows from Lemma 9.2 that

$$\mathrm{d}u^*\left(f_1 \mathrm{d}f_2 \wedge \cdots \wedge \mathrm{d}f_{2n}\right) = u^*\left(\mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_{2n}\right).$$

Hence

$$\mathrm{d}u^*\left(\mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_{2n}\right) = 0.$$

Note that $f_0 \circ u \in L^\infty\left(\mathbb{R}^{4n-1}\right)$, $\mathrm{d}\left(f_0 \circ u\right) \in L^{4n-2}_{loc}\left(\mathbb{R}^{4n-1}\right)$, $\mathrm{d}\left(f_0 \circ u\right) \in L^{4n-1}\left(\mathbb{R}^{4n-1}\backslash\Omega\right)$, $u^*\left(\mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_{2n}\right) \in L^2(\Omega)$, $u^*\left(\mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_{2n}\right) \in L^{\frac{4n-1}{2n}}\left(\mathbb{R}^{4n-1}\backslash\Omega\right)$, and

$$\mathrm{d}u^*\left(\mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_{2n}\right) \in L^\infty\left(\mathbb{R}^{4n-1}\right).$$

It follows from Lemma 9.5 that

$$\begin{aligned}
\mathrm{d}u^*\alpha &= \mathrm{d}\left(f_0 \circ u \cdot u^*\left(\mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_{2n}\right)\right) \\
&= \mathrm{d}\left(f_0 \circ u\right) \wedge u^*\left(\mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_{2n}\right) \\
&= u^*\mathrm{d}\alpha = 0.
\end{aligned}$$

□

To continue we observe that

$$\left(u^*\omega_{S^{2n}}\right)|_\Omega \in L^{\frac{2n-1}{n}}(\Omega) \cap L^2(\Omega), \quad \left(u^*\omega_{S^{2n}}\right)|_{\mathbb{R}^n\backslash\Omega} \in L^{\frac{4n-1}{2n}}\left(\mathbb{R}^n\backslash\Omega\right).$$

Hence $u^*\omega_{S^{2n}} \in L^{\frac{4n-1}{2n}}\left(\mathbb{R}^{4n-1}\right)$. Let $\tau = \mathrm{d}^*\left(\Gamma * u^*\omega_{S^{2n}}\right)$. Then

$$\tau \in L^{\frac{4n-1}{2n-1}}\left(\mathbb{R}^{4n-1}\right), \quad \mathrm{d}\tau = u^*\left(\omega_{S^{2n}}\right).$$

In particular,

$$Q(u) = \frac{1}{|S^{2n}|^2} \int_{\mathbb{R}^{4n-1}} u^*\omega_{S^{2n}} \wedge \tau$$

is well defined. Because $J_u = \left|u^*\omega_{S^{2n}}\right| \in L^{\frac{4n-1}{2n-1}}$, the proofs of Claim 10.3 and 10.4 remain valid with minor modifications (e.g., replacing Lemma 9.1 by Lemma 9.5 when necessary). Similar to Proposition 10.6, we have

Proposition 10.10.. *Assume that* $u \in W_{loc}^{1,1}\left(\mathbb{R}^{4n-1}, S^{2n}\right)$, $\Omega \subset \mathbb{R}^{4n-1}$ *is a bounded open subset with continuous boundary such that*

$$\int_{\Omega} |du|^{4n-2} + |u^{*}\omega_{S^{2n}}|^{2} + \int_{\mathbb{R}^{4n-1}\backslash\Omega} |du|^{4n-1} < \infty,$$

and that α *is a smooth* $2n$*-form on* S^{2n}*. Then* $du^{*}\alpha = 0$*. If* $\beta \in L^{\frac{4n-1}{2n-1}}\left(\mathbb{R}^{4n-1}\right)$ *such that* $d\beta = u^{*}\alpha$*, then for* $n \geq 2$ *we have*

$$\int_{\mathbb{R}^{4n-1}} u^{*}\alpha \wedge \beta = Q\left(u\right)\left(\int_{S^{2n}} \alpha\right)^{2}.$$

For $n = 1$*, the equality remains true if, in addition,* u *is constant near infinity.*

This follows from a similar argument as that in the proof of Proposition 10.6.

11. Energy Growth Estimate

In this section we will describe some basic rules concerning the Hopf invariant for maps with finite Faddeev energy and the sublinear energy growth law. Note that such kind of sublinear growth is a special case of results derived in [LY5]. We provide the arguments here to facilitate the further discussions in Section 12 and Section 13.

Recall for $u \in W_{loc}^{1,1}\left(\mathbb{R}^{4n-1}, S^{2n}\right)$, we denote

$$E\left(u\right) = \int_{\mathbb{R}^{4n-1}} \{|\mathrm{d}u|^{4n-2} + |u^{*}\omega_{S^{2n}}|^{2}\}.$$

Let

$$X = \left\{u \in W_{loc}^{1,1}\left(\mathbb{R}^{4n-1}, S^{2n}\right) \mid E\left(u\right) < \infty\right\}.$$

Lemma 11.1.. *For any* $u \in X$,

$$|Q\left(u\right)| \leq c\left(n\right) E\left(u\right)^{\frac{4n}{4n-1}}.$$

Proof. Indeed,

$$Q\left(u\right) = \frac{1}{|S^{2n}|^{2}} \int_{\mathbb{R}^{4n-1}} u^{*}\omega_{S^{2n}} \wedge \tau$$

with $\tau = \mathrm{d}^{*}\left(\Gamma * u^{*}\omega_{S^{2n}}\right)$. It follows that

$$\begin{aligned}
|Q\left(u\right)| &\leq c\left(n\right) \int_{\mathbb{R}^{4n-1}} |u^{*}\omega_{S^{2n}}| \cdot |\tau| \\
&\leq c\left(n\right) \|u^{*}\omega_{S^{2n}}\|_{L^{2}} \|\tau\|_{L^{2}} \\
&\leq c\left(n\right) \|u^{*}\omega_{S^{2n}}\|_{L^{2}} \|u^{*}\omega_{S^{2n}}\|_{L^{\frac{2(4n-1)}{4n+1}}} \\
&\leq c\left(n\right) \|u^{*}\omega_{S^{2n}}\|_{L^{2}} \|u^{*}\omega_{S^{2n}}\|_{L^{2}}^{\frac{1}{4n-1}} \|u^{*}\omega_{S^{2n}}\|_{L^{\frac{2n-1}{n}}}^{\frac{4n-2}{4n-1}} \\
&\leq c\left(n\right) \|u^{*}\omega_{S^{2n}}\|_{L^{2}}^{\frac{4n}{4n-1}} \|\nabla u\|_{L^{4n-2}}^{\frac{2n(4n-2)}{4n-1}} \\
&\leq c\left(n\right) E\left(u\right)^{\frac{4n}{4n-1}}.
\end{aligned}$$

□

For $N \in \mathbb{Z}$, denote

$$E_N = \inf \{E(u) : u \in X, Q(u) = N\}.$$

The above lemma gives a lower bound for E_N. The upper bound may be derived by choosing suitable test functions.

Lemma 11.2.. *For $n = 1, 2, 4$, we have*

$$E_N \le c(n) |N|^{\frac{4n-1}{4n}} \quad \text{for all integers } N.$$

For $n \ne 1, 2, 4$, we have

$$E_N \le c(n) |N|^{\frac{4n-1}{4n}} \quad \text{for all even integers } N.$$

We start with some basic facts.

- If $u \in X$, $\phi : \mathbb{R}^{4n-1} \to \mathbb{R}^{4n-1}$ is an orthogonal transformation, then $u \circ \phi \in X$ and $Q(u \circ \phi) = \operatorname{sgn}(\det \phi) \cdot Q(u)$.

Indeed, we have

$$(u \circ \phi)^* \omega_{S^{2n}} = \phi^* u^* \omega_{S^{2n}} = \phi^* \mathrm{d}\tau = \mathrm{d}\phi^* \tau.$$

Here $\tau = \mathrm{d}^* (\Gamma * u^* \omega_{S^{2n}}) \in L^2$. Hence

$$\begin{aligned} Q(u \circ \phi) &= \frac{1}{|S^{2n}|^2} \int_{\mathbb{R}^{4n-1}} \phi^* u^* \omega_{S^{2n}} \wedge \phi^* \tau \\ &= \frac{1}{|S^{2n}|^2} \int_{\mathbb{R}^{4n-1}} \phi^* (u^* \omega_{S^{2n}} \wedge \tau) \\ &= \frac{\operatorname{sgn}(\det \phi)}{|S^{2n}|^2} \int_{\mathbb{R}^{4n-1}} u^* \omega_{S^{2n}} \wedge \tau \\ &= \operatorname{sgn}(\det \phi) \cdot Q(u). \end{aligned}$$

- If $u \in X$, $\psi \in C^\infty(S^{2n}, S^{2n})$, then $\psi \circ u \in X$ and $Q(\psi \circ u) = (\deg \psi)^2 Q(u)$.

Indeed, denote $\alpha = \psi^* \omega_{S^{2n}}$. Then

$$(\psi \circ u)^* \omega_{S^{2n}} = u^* \alpha = \mathrm{d}\widetilde{\tau}$$

for some $\widetilde{\tau} \in L^2$. It follows from Proposition 10.6 that

$$\begin{aligned} Q(\psi \circ u) &= \frac{1}{|S^{2n}|^2} \int_{S^{2n}} u^* \psi^* \omega_{S^{2n}} \wedge \widetilde{\tau} \\ &= \left(\frac{1}{|S^{2n}|} \int_{S^{2n}} \psi^* \omega_{S^{2n}} \right)^2 Q(u) \\ &= (\deg \psi)^2 Q(u). \end{aligned}$$

- Assume $x_1, x_2 \in \mathbb{R}^{4n-1}$, $\xi \in S^{2n}$, $r_1, r_2 > 0$ such that $|x_1 - x_2| > r_1 + r_2$, $u_1, u_2 \in X$ such that $u_1(x) = \xi$ for $|x - x_1| \ge r_1$, $u_2(x) = \xi$ for $|x - x_2| \ge r_2$. Let

$$u(x) = \begin{cases} u_1(x), \ x \in B_{r_1}(x_1), \\ u_2(x), \ x \in B_{r_2}(x_2), \\ \xi, \quad \text{otherwise.} \end{cases}$$

  Then $u \in X$ and $Q(u) = Q(u_1) + Q(u_2)$.

Indeed, $u_1^*\omega_{S^{2n}} = \mathrm{d}\tau_1$, $u_2^*\omega_{S^{2n}} = \mathrm{d}\tau_2$ for some $\tau_1, \tau_2 \in L^2$. Hence

$$u^*\omega_{S^{2n}} = u_1^*\omega_{S^{2n}} + u_2^*\omega_{S^{2n}} = \mathrm{d}\left(\tau_1 + \tau_2\right).$$

Hence

$$\begin{aligned} Q\left(u\right) &= \frac{1}{\left|S^{2n}\right|^2} \int_{\mathbb{R}^{4n-1}} \left(u_1^*\omega_{S^{2n}} + u_2^*\omega_{S^{2n}}\right) \wedge \left(\tau_1 + \tau_2\right) \\ &= Q\left(u_1\right) + Q\left(u_2\right) + \frac{1}{\left|S^{2n}\right|^2} \int_{\mathbb{R}^{4n-1}} u_1^*\omega_{S^{2n}} \wedge \tau_2 \\ &+ \frac{1}{\left|S^{2n}\right|^2} \int_{\mathbb{R}^{4n-1}} u_2^*\omega_{S^{2n}} \wedge \tau_1. \end{aligned}$$

Fix a $\delta > 0$ such that $r_1 + r_2 + 2\delta < |x_1 - x_2|$. Then $\mathrm{d}\tau_2 = 0$ on $B_{r_1+\delta}\left(x_1\right)$. It follows that $\tau_2 = \mathrm{d}\gamma_2$ for some $\gamma_2 \in W^{1,2}\left(B_{r_1+\delta}\left(x_1\right)\right)$. Note that on $B_{r_1+\delta}\left(x_1\right)$,

$$u_1^*\omega_{S^{2n}} \wedge \tau_2 = u_1^*\omega_{S^{2n}} \wedge \mathrm{d}\gamma_2 = \mathrm{d}\left(u_1^*\omega_{S^{2n}} \wedge \gamma_2\right).$$

Hence

$$\begin{aligned} \int_{\mathbb{R}^{4n-1}} u_1^*\omega_{S^{2n}} \wedge \tau_2 &= \int_{B_{r_1+\delta}(x_1)} u_1^*\omega_{S^{2n}} \wedge \tau_2 = \int_{B_{r_1+\delta}(x_1)} \mathrm{d}\left(u_1^*\omega_{S^{2n}} \wedge \gamma_2\right) \\ &= \int_{\mathbb{R}^{4n-1}} \mathrm{d}\left(u_1^*\omega_{S^{2n}} \wedge \gamma_2\right) = 0 \end{aligned}$$

by Lemma 9.9.

Proof of Lemma 11.2. We simply deal with the case $n \neq 1, 2, 4$. The case when $n = 1, 2, 4$ may be treated by similar methods. It follows from the previous facts that $E_{-N} = E_N$. Hence we may assume $N > 0$. By [Hu, corollary 3.6 on p214] we may find a $v_0 \in C^\infty\left(S^{4n-1}, S^{2n}\right)$ such that $Q\left(v_0\right) = 2$ and $v_0|_{S_+^{4n-1}} = \mathbf{n}$, the north pole of S^{2n}. Let $u_0\left(x\right) = v_0\left(\pi_{\mathbf{n}}^{-1}\left(x\right)\right)$. Here $\pi_{\mathbf{n}}$ is the stereographic projection with respect to the north pole of S^{4n-1}. For any even N, we may find a unique $m \in \mathbb{N}$ such that

$$m^2 \leq \frac{N}{2} < \left(m+1\right)^2.$$

Let $k = \frac{N}{2} - m^2$. Then $0 \leq k \leq 2m$. By scaling and packing we can find a $\psi \in C^\infty\left(S^{2n}, S^{2n}\right)$ such that $\psi\left(\mathbf{n}\right) = \mathbf{n}$, $\deg\psi = m$ and $|\mathrm{d}\psi| \leq c\left(n\right) m^{\frac{1}{2n}}$. Let

$$u\left(x\right) = \begin{cases} \psi\left(u_0\left(m^{-\frac{1}{2n}}x\right)\right), \text{ for } |x| \leq m^{\frac{1}{2n}} + 1, \\ u_0\left(x - \left(m^{\frac{1}{2n}} + 1 + 4j\right)e_1\right), \text{ for } \left|x - \left(m^{\frac{1}{2n}} + 1 + 4j\right)e_1\right| \leq 1, 1 \leq j \leq k \\ \mathbf{n}, \quad \text{otherwise}, \end{cases}$$

where $e_1 = \left(1, 0, \cdots, 0\right) \in \mathbb{R}^{4n-1}$. Then $Q\left(v\right) = 2m^2 + 2k = N$. Moreover since $|\mathrm{d}u| \leq c\left(n\right)$, we see that

$$\begin{aligned} E\left(u\right) &\leq c\left(n\right) m^{\frac{4n-1}{2n}} + c\left(n\right) k \\ &\leq c\left(n\right) m^{\frac{4n-1}{2n}} + c\left(n\right) m \\ &\leq c\left(n\right) m^{\frac{4n-1}{2n}} \leq c\left(n\right) N^{\frac{4n-1}{4n}}. \end{aligned}$$

□

12. The Decomposition Lemma

In this section, we prove the crucial decomposition lemma. Roughly speaking, the lemma says that we may break the domain space into infinitely many blocks, on the boundary of each block the map is almost constant, and hence, we can assign a Hopf–Whitehead invariant for it. By collecting nonzero "degree" blocks suitably, we may achieve a good understanding of the minimizing sequence of maps for the Faddeev energy (Theorem 13.1). Note that such a decomposition lemma for maps from $\mathbb{R}^3$ to S^2 was proven in [LY1] using the lifting through the Hopf fibration $S^3 \to S^2$. In higher dimensions, we will use the Hodge decomposition of differential forms in place of the lifting.

Let us introduce some notation. For $x \in \mathbb{R}^m$ we write

$$|x|_\infty = \max_{1 \le i \le m} |x_i| .$$

For $R > 0$, $y \in \mathbb{R}^m$,

$$Q_R(y) = \{x \in \mathbb{R}^m : |x - y|_\infty \le R\} .$$

$Q_R = Q_R(0)$. Denote

$$\mathbb{Z}^m = \{x \in \mathbb{R}^m : x_i \in \mathbb{Z} \text{ for } 1 \le i \le m\}$$

as the lattice of all integer points. Then

$$\mathbb{R}^m = \bigcup_{\xi \in 2R\mathbb{Z}^m} Q_R(\xi) .$$

Here $2R\mathbb{Z}^m$ means the scaling of the lattice $\mathbb{Z}^m$ by factor $2R$. The union of boundaries of these cubes is given by

$$\Sigma_R = \{x \in \mathbb{R}^m : x_i = (2j+1) R \text{ for some } 1 \le i \le m \text{ and integer } j\} .$$

Lemma 12.1.. *Assume $u \in X$. That is, $u \in W^{1,1}_{loc}(\mathbb{R}^{4n-1}, S^{2n})$ with*

$$E(u) = \int_{\mathbb{R}^{4n-1}} \left(|du|^{4n-2} + |u^* \omega_{S^{2n}}|^2 \right) dx \le \Lambda < \infty.$$

Let

$$\tau = d^* (\Gamma * u^* \omega_{S^{2n}}) .$$

Here Γ is the fundamental solution of $-\Delta$ on $\mathbb{R}^{4n-1}$. Then for every $\varepsilon > 0$, there exists $R = R(n, \varepsilon, \Lambda) > 0$, $y \in Q_{R/4}$ and $\kappa_\xi \in \mathbb{Z}$ for every $\xi \in 2R\mathbb{Z}^{4n-1}$ such that

$$\sum_{\xi \in 2R\mathbb{Z}^{4n-1}} \left| \frac{1}{|S^{2n}|^2} \int_{Q_R(\xi)+y} u^* \omega_{S^{2n}} \wedge \tau - \kappa_\xi \right| \le \varepsilon.$$

In particular, except for finitely many ξ's, $\kappa_\xi = 0$ and, when $\varepsilon < 1$,

$$\sum_{\xi \in 2R\mathbb{Z}^{4n-1}} \kappa_\xi = Q(u) .$$

Proof. Since $\|u^* \omega_{S^{2n}}\|_{L^{\frac{2n-1}{n}}(\mathbb{R}^{4n-1})} \le c(n) \|\mathrm{d}u\|^{2n}_{L^{4n-2}(\mathbb{R}^{4n-1})}$, it follows from Hölder's inequality that

$$\|u^* \omega_{S^{2n}}\|_{L^{\frac{4n-1}{2n}}(\mathbb{R}^{4n-1})} \le c(n, \Lambda) .$$

Hence

$$\|\tau\|_{L^{\frac{4n-1}{2n-1}}(\mathbb{R}^{4n-1})} + \|D\tau\|_{L^{\frac{4n-1}{2n}}(\mathbb{R}^{4n-1})} \le c(n, \Lambda) .$$

It follows from the Fubini type estimate (Section 3 of [HL]) that we may find some $y \in Q_{R/4}$ such that

$$u|_{\Sigma_R+y} \in W^{1,4n-2}_{loc}(\Sigma_R+y), \quad \tau|_{\Sigma_R+y} \in W^{1,\frac{4n-1}{2n}}_{loc}(\Sigma_R+y),$$

and

$$\begin{aligned}&\int_{\Sigma_R+y}\left(|\mathrm{d}u|^{4n-2}+|\tau|^{\frac{4n-1}{2n-1}}+|D\tau|^{\frac{4n-1}{2n}}\right)\mathrm{d}S\\ &\leq \frac{c(n)}{R}\int_{\mathbb{R}^{4n-1}}\left(|\mathrm{d}u|^{4n-2}+|\tau|^{\frac{4n-1}{2n-1}}+|D\tau|^{\frac{4n-1}{2n}}\right)\mathrm{d}x\\ &\leq \frac{c(n,\Lambda)}{R}.\end{aligned}$$

By translation we may assume $y=0$. Pick up a cube $Q_R(\xi)$ with $\xi \in 2R\mathbb{Z}^{4n-1}$. Without loss of generality, we may assume $\xi = 0$. We have

$$\int_{\partial Q_R}\left(|\mathrm{d}u|^{4n-2}+|\tau|^{\frac{4n-1}{2n-1}}+|D\tau|^{\frac{4n-1}{2n}}\right)\mathrm{d}S \leq \frac{c(n,\Lambda)}{R}.$$

Claim 12.2.. *There exists $u_1 \in W^{1,4n-1}(Q_{2R}\backslash Q_R, S^{2n})$ such that $u_1|_{\partial Q_R} = u|_{\partial Q_R}$, $u_1|_{\partial Q_{2R}} = \mathrm{const}$ and*

$$\|du_1\|_{L^{4n-1}(Q_{2R}\backslash Q_R)} \leq c(n)\|du\|_{L^{4n-2}(\partial Q_R)}.$$

Here we set

$$u_1(x) = \begin{cases} u(x), & x \in Q_R,\\ u_1(x), & x \in Q_{2R}\backslash Q_R,\\ u_1|_{\partial Q_{2R}}, & x \in \mathbb{R}^{4n-1}\backslash Q_{2R}.\end{cases}$$

Indeed, consider the map $\phi : \mathbb{R}^{4n-1} \to \mathbb{R}^{4n-1}$ given by $\phi(x) = |x|\frac{x}{|x|_\infty}$. Then ϕ is bi-Lipschitz with $|\phi(x)|_\infty = |x|$. In particular, $\phi(\partial B_R) = \partial Q_R$. Let $v(x) = u(\phi(x))$ for $x \in B_R$. Then

$$\int_{\partial B_R}|\mathrm{d}v|^{4n-2}\,\mathrm{d}S \leq \frac{c(n,\Lambda)}{R}.$$

It follows from Lemma 9.7 and scaling that, when $R = R(n,\Lambda)$ is large enough, there exists a $v_1 \in W^{1,4n-1}(B_{2R}\backslash B_R, S^{2n})$ with $v_1|_{\partial B_R} = v$, $v_1|_{\partial B_{2R}} = \mathrm{const}$ such that

$$\|\mathrm{d}v_1\|_{L^{4n-1}(B_{2R}\backslash B_R)} \leq c(n)\|\mathrm{d}v\|_{L^{4n-2}(\partial B_R)} \leq c(n)\|\mathrm{d}u\|_{L^{4n-2}(\partial Q_R)}.$$

Let $u_1 = v_1 \circ \phi^{-1}$. Then u_1 satisfies all the requirements in Claim 12.2.

Claim 12.3.. *There exists some $\tau_1 \in L^{\frac{4n-1}{2n-1}}(\mathbb{R}^{4n-1})$ with $\tau_1|_{Q_R} = \tau$, $\tau_1|_{\mathbb{R}^{4n-1}\backslash Q_{2R}} = 0$, $d\tau_1 = u_1^*\omega_{S^{2n}}$ and*

$$\begin{aligned}&\|\tau_1\|_{L^{\frac{4n-1}{2n-1}}(Q_{2R}\backslash Q_R)}\\ &\leq c(n)\left(\|du\|^{2n}_{L^{4n-2}(\partial Q_R)} + R^{\frac{2n-1}{4n-1}}\|\tau\|_{L^{\frac{4n-1}{2n-1}}(\partial Q_R)} + R^{\frac{2n}{4n-1}}\|D\tau\|_{L^{\frac{4n-1}{2n}}(\partial Q_R)}\right).\end{aligned}$$

Indeed we may write

$$\tau = \sum_{\lambda\in\Lambda(4n-1,2n-1)} f_\lambda(x)\,\mathrm{d}x_\lambda.$$

Then we define

$$\widetilde{f}_\lambda(x) = \begin{cases} f_\lambda(x), & x \in Q_R, \\ \frac{2R-|x|_\infty}{R} f_\lambda\left(\frac{Rx}{|x|_\infty}\right), & x \in Q_{2R} \backslash Q_R, \\ 0, & x \in \mathbb{R}^{4n-1} \backslash Q_{2R}, \end{cases}$$

and

$$\tau_2 = \sum_{\lambda \in \Lambda(4n-1,2n-1)} \widetilde{f}_\lambda(x)\, \mathrm{d}x_\lambda.$$

It follows that $\tau_2 \in L^{\frac{4n-1}{2n-1}}(\mathbb{R}^{4n-1})$, $D\tau_2 \in L^{\frac{4n-1}{2n}}(\mathbb{R}^{4n-1})$, and

$$\|\tau_2\|_{L^{\frac{4n-1}{2n-1}}(Q_{2R}\backslash Q_R)} \le c(n) R^{\frac{2n-1}{4n-1}} \|\tau\|_{L^{\frac{4n-1}{2n-1}}(\partial Q_R)},$$

$$\|D\tau_2\|_{L^{\frac{4n-1}{2n}}(Q_{2R}\backslash Q_R)} \le c(n) \left(R^{\frac{2n-1}{4n-1}} \|\tau\|_{L^{\frac{4n-1}{2n-1}}(\partial Q_R)} + R^{\frac{2n}{4n-1}} \|D\tau\|_{L^{\frac{4n-1}{2n}}(\partial Q_R)} \right).$$

Note that $(\mathrm{d}\tau_2)|_{Q_R} = u^*\omega_{S^{2n}}$ and $(\mathrm{d}\tau_2)|_{\mathbb{R}^{4n-1}\backslash Q_{2R}} = 0$. Let $v(x) = u_1(\phi(x))$. Then $v \in W^{1,1}_{loc}(\mathbb{R}^{4n-1}, S^{2n})$,

$$\int_{B_R} \left(|\mathrm{d}v|^{4n-2} + |v^*\omega_{S^{2n}}|^2\right) \mathrm{d}x + \int_{B_{2R}\backslash B_R} |\mathrm{d}v|^{4n-1}\, \mathrm{d}x < \infty,$$

and $v|_{\mathbb{R}^{4n-1}\backslash B_{2R}} = \text{const}$. It follows from Theorem 10.8 that $\mathrm{d}v^*\omega_{S^{2n}} = 0$. Let $\eta_2 = \phi^*\tau_2$. It follows from Lemma 9.3 that $\eta_2 \in L^{\frac{4n-1}{2n-1}}(\mathbb{R}^{4n-1})$ and $\mathrm{d}\eta_2 = \phi^*\mathrm{d}\tau_2$. In particular, $(\mathrm{d}\eta_2)|_{B_R} = v^*\omega_{S^{2n}}$, $(\mathrm{d}\eta_2)|_{\mathbb{R}^{4n-1}\backslash B_{2R}} = 0$, and

$$\|\mathrm{d}\eta_2\|_{L^{\frac{4n-1}{2n}}(B_{2R}\backslash B_R)} \le c(n) \left(R^{\frac{2n-1}{4n-1}} \|\tau\|_{L^{\frac{4n-1}{2n-1}}(\partial Q_R)} + R^{\frac{2n}{4n-1}} \|D\tau\|_{L^{\frac{4n-1}{2n}}(\partial Q_R)} \right).$$

Note that $\mathrm{d}(v^*\omega_{S^{2n}} - \mathrm{d}\eta_2) = 0$. Hence for any forms $\varphi \in C_c^\infty(\mathbb{R}^n)$,

$$\begin{aligned} &\int_{B_{2R}\backslash B_R} \langle v^*\omega_{S^{2n}} - \mathrm{d}\eta_2, \mathrm{d}^*\varphi \rangle\, \mathrm{d}x \\ &= \int_{\mathbb{R}^{4n-1}} \langle v^*\omega_{S^{2n}} - \mathrm{d}\eta_2, \mathrm{d}^*\varphi \rangle\, \mathrm{d}x \\ &= 0. \end{aligned}$$

Using the fact

$$H^{2n}\left(\overline{B_{2R}}\backslash B_R, \partial B_R \cup \partial B_{2R}, \mathbb{R}\right) \cong H^{2n-1}\left(\overline{B_{2R}}\backslash B_R, \mathbb{R}\right) \cong H^{2n-1}\left(S^{4n-2}, \mathbb{R}\right) = 0,$$

it follows from the Hodge theory that we may find some $\eta \in W^{1,\frac{4n-1}{2n}}(B_{2R}\backslash B_R)$ such that

$$v^*\omega_{S^{2n}} - \mathrm{d}\eta_2 = \mathrm{d}\eta \text{ on } B_{2R}\backslash B_R, \quad i_R^*\eta = 0, \quad i_{2R}^*\eta = 0,$$

and

$$\begin{aligned} &\|\eta\|_{L^{\frac{4n-1}{2n-1}}(B_{2R}\backslash B_R)} \\ &\le c(n) \|v^*\omega_{S^{2n}} - \mathrm{d}\eta_2\|_{L^{\frac{4n-1}{2n}}(B_{2R}\backslash B_R)} \\ &\le c(n) \left(\|\mathrm{d}u\|^{2n}_{L^{4n-2}(\partial Q_R)} + R^{\frac{2n-1}{4n-1}} \|\tau\|_{L^{\frac{4n-1}{2n-1}}(\partial Q_R)} + R^{\frac{2n}{4n-1}} \|D\tau\|_{L^{\frac{4n-1}{2n}}(\partial Q_R)} \right). \end{aligned}$$

Here $i_R : \partial B_R \to \mathbb{R}^{4n-1}$ and $i_{2R} : \partial B_{2R} \to \mathbb{R}^{4n-1}$ are the identity maps. Let

$$\eta = \begin{cases} \eta, & \text{on } B_{2R} \backslash B_R, \\ 0, & \text{on } B_R \cup \left(\mathbb{R}^{4n-1} \backslash B_{2R}\right). \end{cases}$$

Then it follows that for any form $\varphi \in C_c^\infty (\mathbb{R}^n)$,

$$\begin{aligned} & \int_{\mathbb{R}^{4n-1}} \eta \wedge \mathrm{d}\varphi \\ = & \int_{B_{2R}\backslash B_R} \eta \wedge \mathrm{d}\varphi \\ = & \int_{B_{2R}\backslash B_R} -\mathrm{d}(\eta \wedge \varphi) + \mathrm{d}\eta \wedge \varphi \\ = & \int_{B_{2R}\backslash B_R} (v^*\omega_{S^{2n}} - \mathrm{d}\eta_2) \wedge \varphi - \int_{\partial B_{2R}} i_{2R}^* (\eta \wedge \varphi) + \int_{\partial B_R} i_R^* (\eta \wedge \varphi) \\ = & \int_{\mathbb{R}^{4n-1}} (v^*\omega_{S^{2n}} - \mathrm{d}\eta_2) \wedge \varphi. \end{aligned}$$

Hence

$$\mathrm{d}\eta = v^*\omega_{S^{2n}} - \mathrm{d}\eta_2 \quad \text{on } \mathbb{R}^{4n-1}.$$

Let $\eta_1 = \eta + \eta_2$. Then $\mathrm{d}\eta_1 = v^*\omega_{S^{2n}}$. Denote $\tau_1 = (\phi^{-1})^* \eta_1$. Then $\tau_1 \in L^{\frac{4n-1}{2n-1}} (\mathbb{R}^{4n-1})$. By Lemma 9.3, we have

$$\mathrm{d}\tau_1 = \left(\phi^{-1}\right)^* \mathrm{d}\eta_1 = \left(\phi^{-1}\right)^* v^*\omega_{S^{2n}} = u_1^*\omega_{S^{2n}},$$

$\tau_1|_{Q_R} = \tau,\ \tau_1|_{\mathbb{R}^{4n-1}\backslash Q_{2R}} = 0$, and

$$\begin{aligned} & \|\tau_1\|_{L^{\frac{4n-1}{2n-1}}(Q_{2R}\backslash Q_R)} \\ & \leq c(n) \left(\|\mathrm{d}u\|^{2n}_{L^{4n-2}(\partial Q_R)} + R^{\frac{2n-1}{4n-1}} \|\tau\|_{L^{\frac{4n-1}{2n-1}}(\partial Q_R)} + R^{\frac{2n}{4n-1}} \|D\tau\|_{L^{\frac{4n-1}{2n}}(\partial Q_R)} \right). \end{aligned}$$

It follows from Proposition 10.10 that we have

$$Q(u_1) = \frac{1}{|S^{2n}|^2} \int_{\mathbb{R}^{4n-1}} u_1^*\omega_{S^{2n}} \wedge \tau_1.$$

Hence

$$
\begin{aligned}
&\left|\frac{1}{\left|S^{2n}\right|^2}\int_{B_R} u^*\omega_{S^{2n}}\wedge\tau-Q\left(u_1\right)\right| \\
&\leq\left|\frac{1}{\left|S^{2n}\right|^2}\int_{B_{2R}\backslash B_R} u_1^*\omega_{S^{2n}}\wedge\tau_1\right| \\
&\leq c\left(n\right)\left\|u_1^*\omega_{S^{2n}}\right\|_{L^{\frac{4n-1}{2n}}\left(Q_{2R}\backslash Q_R\right)}\left\|\tau_1\right\|_{L^{\frac{4n-1}{2n-1}}\left(Q_{2R}\backslash Q_R\right)} \\
&\leq c\left(n\right)\left\|\mathrm{d}u\right\|_{L^{4n-2}\left(\partial Q_R\right)}^{2n}\left(\left\|\mathrm{d}u\right\|_{L^{4n-2}\left(\partial Q_R\right)}^{2n}+R^{\frac{2n}{4n-1}}\left\|D\tau\right\|_{L^{\frac{4n-1}{2n}}\left(\partial Q_R\right)}+R^{\frac{2n-1}{4n-1}}\left\|\tau\right\|_{L^{\frac{4n-1}{2n-1}}\left(\partial Q_R\right)}\right) \\
&\leq c\left(n\right)\left\|\mathrm{d}u\right\|_{L^{4n-2}\left(\partial Q_R\right)}^{2}\int_{\partial Q_R}\left|\mathrm{d}u\right|^{4n-2}\mathrm{d}S+c\left(n\right)\left\|\mathrm{d}u\right\|_{L^{4n-2}\left(\partial Q_R\right)}^{\frac{6n-2}{4n-1}}\cdot\left(\int_{\partial Q_R}\left|\mathrm{d}u\right|^{4n-2}\mathrm{d}S\right. \\
&\left.+R\int_{\partial Q_R}\left|D\tau\right|^{\frac{4n-1}{2n}}\mathrm{d}S\right)+c\left(n\right)\left\|\mathrm{d}u\right\|_{L^{4n-2}\left(\partial Q_R\right)}^{\frac{2n}{4n-1}}\left(\int_{\partial Q_R}\left|\mathrm{d}u\right|^{4n-2}\mathrm{d}S+R\int_{\partial Q_R}\left|\tau\right|^{\frac{4n-1}{2n-1}}\mathrm{d}S\right) \\
&\leq\frac{c\left(n,\Lambda\right)}{R^{\frac{2n}{4n-1}}}\cdot R\int_{\partial Q_R}\left(\left|\mathrm{d}u\right|^{4n-2}+\left|\tau\right|^{\frac{4n-1}{2n-1}}+\left|D\tau\right|^{\frac{4n-1}{2n}}\right)\mathrm{d}S,
\end{aligned}
$$

if $R\geq 1$. We may set $\kappa_0=Q\left(u_1\right)\in\mathbb{Z}$. Then we get

$$
\begin{aligned}
&\sum_{\xi\in 2R\mathbb{Z}^{4n-1}}\left|\frac{1}{\left|S^{2n}\right|^2}\int_{Q_R(\xi)} u^*\omega_{S^{2n}}\wedge\tau-\kappa_\xi\right| \\
&\leq\frac{c\left(n,\Lambda\right)}{R^{\frac{2n}{4n-1}}}\cdot R\int_{\Sigma_R}\left(\left|\mathrm{d}u\right|^{4n-2}+\left|\tau\right|^{\frac{4n-1}{2n-1}}+\left|D\tau\right|^{\frac{4n-1}{2n}}\right)\mathrm{d}S \\
&\leq\frac{c\left(n,\Lambda\right)}{R^{\frac{2n}{4n-1}}}\leq\varepsilon,
\end{aligned}
$$

when R is large enough.

As a consequence,

$$
\sum_{\xi\in 2R\mathbb{Z}^{4n-1}}\left|\kappa_\xi\right|\leq\frac{1}{\left|S^{2n}\right|^2}\int_{Q_R(\xi)}\left|u^*\omega_{S^{2n}}\wedge\tau\right|\mathrm{d}x+\varepsilon<\infty.
$$

This implies $\kappa_\xi=0$ except for finitely many ξ's. On the other hand,

$$
\left|Q\left(u\right)-\sum_{\xi\in 2R\mathbb{Z}^{4n-1}}\kappa_\xi\right|\leq\sum_{\xi\in 2R\mathbb{Z}^{4n-1}}\left|\frac{1}{\left|S^{2n}\right|^2}\int_{Q_R(\xi)} u^*\omega_{S^{2n}}\wedge\tau-\kappa_\xi\right|\leq\varepsilon.
$$

Using the fact that $Q\left(u\right)-\sum_{\xi\in 2R\mathbb{Z}^{4n-1}}\kappa_\xi$ is an integer, we see that, when $\varepsilon<1$,

$$
Q\left(u\right)=\sum_{\xi\in 2R\mathbb{Z}^{4n-1}}\kappa_\xi.
$$

□

13. Existences of Minimizers

After the fore-going preparation, we are ready to prove the main result of the second part of this article, Theorem 13.1 below. This theorem describes the behavior of a minimizing sequence of maps for the Faddeev model. Based on this result and the sublinear growth law, we will obtain several existence statements in Section 13.1. It is worth pointing out that even for the Faddeev model for maps from $\mathbb{R}^3$ to S^2, Theorem 13.1 improves the substantial inequality in [LY1] to an equality. Such a result is based on some special operations on maps with finite Faddeev energy given in Lemma 13.2 and establishes a subadditivity property for the Faddeev knot energy spectrum.

Recall that

$$X=\left\{u\in W_{loc}^{1,1}\left(\mathbb{R}^{4n-1},S^{2n}\right)\,\middle|\,E\left(u\right)=\int_{\mathbb{R}^{4n-1}}\left(\left|\mathrm{d}u\right|^{4n-2}+\left|u^{*}\omega_{S^{2n}}\right|^{2}\right)\mathrm{d}x<\infty\right\}.$$

For $N\in\mathbb{Z}$, we set

$$E_N=\inf\{E\left(u\right)\mid u\in X_N\}\quad\text{where } X_N=\left\{u\in X\mid Q\left(u\right)=N\right\}.$$

Theorem 13.1.. *Assume that N is an nonzero integer such that $X_N\neq\phi$, $\{u_i\}\subset X_N$ such that $E\left(u_i\right)\to E_N$ as $i\to\infty$. Then there exists an integer m with $1\leq m\leq c\left(n\right)E_N$, m nonzero integers $N_1,\cdots,N_m$ and $y_{i1},\cdots,y_{im}\in\mathbb{R}^{4n-1}$ such that*

- $N=N_1+\cdots+N_m$.
- $|y_{ij}-y_{ik}|\to\infty$ *as* $i\to\infty$ *for* $1\leq j,k\leq m$, $j\neq k$.
- *If we set $v_{ij}\left(x\right)=u_i\left(x-y_{ij}\right)$ for $1\leq j\leq m$, then there exists a $v_j\in X$ such that*

$$\begin{aligned}v_{ij}&\to v_j \text{ a.e.}\\ dv_{ij}&\rightharpoonup dv_j \text{ in } L^{4n-2}\left(\mathbb{R}^{4n-1}\right),\\ v_{ij}^{*}\omega_{S^{2n}}&\rightharpoonup v_j^{*}\omega_{S^{2n}} \text{ in } L^{2}\left(\mathbb{R}^{4n-1}\right)\end{aligned}$$

  as $i\to\infty$ and

$$Q\left(v_j\right)=N_j,\quad E_{N_j}=E\left(v_j\right)\geq c\left(n\right)>0$$

  for all j.
-
$$E_N=\sum_{j=1}^{m}E_{N_j}.$$

In particular, if $E_N<E_{N'}+E_{N''}$ for $N=N'+N''$, $N',N''\neq0$, then E_N is achieved.

Before carrying out the proof of this theorem, we make some general discussion. Assume $u_i\in X$ with $E\left(u_i\right)\leq\Lambda<\infty$. Then, after passing to a subsequence, we may find a $u_\infty\in X$ such that $u_i\to u_\infty$ a.e., $\mathrm{d}u_i\rightharpoonup\mathrm{d}u_\infty$ in $L^{4n-2}\left(\mathbb{R}^{4n-1}\right)$, and $u_i^{*}\omega_{S^{2n}}\rightharpoonup u_\infty^{*}\omega_{S^{2n}}$ in $L^{2}\left(\mathbb{R}^{4n-1}\right)$.

Indeed we may find a $u_\infty\in W_{loc}^{1,4n-2}\left(\mathbb{R}^{4n-1},S^{2n}\right)$ such that, after passing to a subsequence, we have $u_i\to u_\infty$ a.e. and $\mathrm{d}u_i\rightharpoonup\mathrm{d}u_\infty$ in $L^{4n-2}\left(\mathbb{R}^{4n-1}\right)$. Next we claim for every $1\leq k\leq 2n$, $\lambda\in\Lambda\left(2n+1,k\right)$,

$$\mathrm{d}u_{i,\lambda_1}\wedge\cdots\wedge\mathrm{d}u_{i,\lambda_k}\to\mathrm{d}u_{\infty,\lambda_1}\wedge\cdots\wedge\mathrm{d}u_{\infty,\lambda_k},$$

in sense of distribution as $i\to\infty$. Here $u_{i,j}$ is the jth component of the vector u_i. The claim is true for $k=1$. Assume it is true for $k-1$. Then for $\lambda\in\Lambda\left(2n+1,k\right)$, since $k-1\leq 2n-1<4n-2$, we see

$$\left\|\mathrm{d}u_{i,\lambda_2}\wedge\cdots\wedge\mathrm{d}u_{i,\lambda_k}\right\|_{L^{\frac{4n-2}{k-1}}\left(\mathbb{R}^{4n-1}\right)}\leq c\left(n,\Lambda\right).$$

Combining with the induction hypothesis, we get $\mathrm{d}u_{\infty,\lambda_2}\wedge\cdots\wedge\mathrm{d}u_{\infty,\lambda_k}\in L^{\frac{4n-2}{k-1}}\left(\mathbb{R}^{4n-1}\right)$ and

$$\mathrm{d}u_{i,\lambda_2}\wedge\cdots\wedge\mathrm{d}u_{i,\lambda_k}\rightharpoonup\mathrm{d}u_{\infty,\lambda_2}\wedge\cdots\wedge\mathrm{d}u_{\infty,\lambda_k}\quad\text{in } L^{\frac{4n-2}{k-1}}\left(\mathbb{R}^{4n-1}\right).$$

Hence

$$u_{i,\lambda_1}\mathrm{d}u_{i,\lambda_2}\wedge\cdots\wedge\mathrm{d}u_{i,\lambda_k}\rightharpoonup u_{\infty,\lambda_1}\mathrm{d}u_{\infty,\lambda_2}\wedge\cdots\wedge\mathrm{d}u_{\infty,\lambda_k}\quad\text{in } L^{\frac{4n-2}{k-1}}\left(\mathbb{R}^{4n-1}\right).$$

It follows from Lemma 9.2 that

$$\begin{aligned}\mathrm{d}u_{i,\lambda_1}\wedge\cdots\wedge\mathrm{d}u_{i,\lambda_k}&=\mathrm{d}\left(u_{i,\lambda_1}\mathrm{d}u_{i,\lambda_2}\wedge\cdots\wedge\mathrm{d}u_{i,\lambda_k}\right)\\&\to\mathrm{d}\left(u_{\infty,\lambda_1}\mathrm{d}u_{\infty,\lambda_2}\wedge\cdots\wedge\mathrm{d}u_{\infty,\lambda_k}\right)\\&=\mathrm{d}u_{\infty,\lambda_1}\wedge\cdots\wedge\mathrm{d}u_{\infty,\lambda_k}\end{aligned}$$

in sense of distribution. The claim follows.

Using the fact

$$\left\|\Lambda_{2n}\left(\mathrm{d}u\right)\right\|_{L^2(\mathbb{R}^{4n-1})}\leq\left\|u^*\omega_{S^{2n}}\right\|_{L^2(\mathbb{R}^{4n-1})}\leq\sqrt{\Lambda},$$

we see that, for $\lambda\in\Lambda\left(2n+1,2n\right)$, $\mathrm{d}u_{\infty,\lambda_1}\wedge\cdots\wedge\mathrm{d}u_{\infty,\lambda_{2n}}\in L^2\left(\mathbb{R}^{4n-1}\right)$ and

$$\mathrm{d}u_{i,\lambda_1}\wedge\cdots\wedge\mathrm{d}u_{i,\lambda_{2n}}\rightharpoonup\mathrm{d}u_{\infty,\lambda_1}\wedge\cdots\wedge\mathrm{d}u_{\infty,\lambda_{2n}}\quad\text{in } L^2\left(\mathbb{R}^{4n-1}\right).$$

This together with the fact $u_i\to u_\infty$ a.e. implies $u_\infty^*\omega_{S^{2n}}\in L^2\left(\mathbb{R}^{4n-1}\right)$ and $u_i^*\omega_{S^{2n}}\rightharpoonup u_\infty^*\omega_{S^{2n}}$ in $L^2\left(\mathbb{R}^{4n-1}\right)$ as $i\to\infty$.

If we let

$$\tau_i=\mathrm{d}^*\left(\Gamma*u_i^*\omega_{S^{2n}}\right),\quad\tau_\infty=\mathrm{d}^*\left(\Gamma*u_\infty^*\omega_{S^{2n}}\right),$$

then

$$\begin{aligned}\tau_i&\rightharpoonup\tau_\infty\quad\text{in } L^{\frac{2(4n-1)}{4n-3}}\left(\mathbb{R}^{4n-1}\right),\\D\tau_i&\rightharpoonup D\tau_\infty\quad\text{in } L^2\left(\mathbb{R}^{4n-1}\right),\\\tau_i&\rightharpoonup\tau_\infty\quad\text{in } W^{1,2}\left(B_r\right)\text{ for every } r>0.\end{aligned}$$

Hence

$$u_i^*\omega_{S^{2n}}\wedge\tau_i\rightharpoonup u_\infty^*\omega_{S^{2n}}\wedge\tau_\infty\quad\text{in } L^1\left(B_r\right)$$

for all $r>0$.

Proof of Theorem 13.1. Since $N\neq0$, it follows from Lemma 11.1 that

$$E_N\geq c\left(n\right)\left|N\right|^{\frac{4n-1}{4n}}>0.$$

We may assume that i is large enough such that $E\left(u_i\right)\leq2E_N$. Let $\varepsilon>0$ be a tiny number to be fixed later. It follows from Lemma 12.1 that we may find some $R=R\left(n,\varepsilon,E_N\right)>0$, $y_i\in Q_{R/4}$, and integers $\kappa_{i,\xi}$ for $\xi\in2R\mathbb{Z}^{4n-1}$, such that

$$\sum_{\xi\in2R\mathbb{Z}^{4n-1}}\left|\frac{1}{\left|S^{2n}\right|^2}\int_{Q_R(\xi)+y_i}u_i^*\omega_{S^{2n}}\wedge\tau_i-\kappa_{i,\xi}\right|\leq\varepsilon.$$

Here $\tau_i=\mathrm{d}^*\left(\Gamma*u_i^*\omega_{S^{2n}}\right)$. By translation we may assume $y_i=0$. It follows from the calculation in the proof of Lemma 11.1 that

$$\int_{\mathbb{R}^{4n-1}}\left|u_i^*\omega_{S^{2n}}\wedge\tau_i\right|\mathrm{d}x\leq c\left(n\right)E_N^{\frac{4n}{4n-1}}.$$

Hence

$$\begin{aligned}
&\sum_{\xi\in 2R\mathbb{Z}^{4n-1}} |\kappa_{i,\xi}| \\
&\leq \frac{1}{|S^{2n}|^2}\int_{\mathbb{R}^{4n-1}} |u_i^*\omega_{S^{2n}}\wedge\tau_i|\,\mathrm{d}x + \sum_{\xi\in 2R\mathbb{Z}^{4n-1}}\left|\frac{1}{|S^{2n}|^2}\int_{Q_R(\xi)} u_i^*\omega_{S^{2n}}\wedge\tau_i - \kappa_{i,\xi}\right| \\
&\leq c(n)\,E_N^{\frac{4n}{4n-1}}.
\end{aligned}$$

Hence

$$\#\left\{\xi\in 2R\mathbb{Z}^{4n-1}\,|\,\kappa_{i,\xi}\neq 0\right\}\leq c(n)\,E_N^{\frac{4n}{4n-1}}.$$

After passing to a subsequence we may assume

$$\#\left\{\xi\in 2R\mathbb{Z}^{4n-1}\,|\,\kappa_{i,\xi}\neq 0\right\}=l.$$

For each i, we may order $\{\xi\in 2R\mathbb{Z}^{4n-1}:\kappa_{i,\xi}\neq 0\}$ and $\xi_{i1},\cdots,\xi_{il}$. After passing to a subsequence we may assume for all $1\leq j,k\leq l$, $\lim_{i\to\infty}|\xi_{ij}-\xi_{ik}|=\infty$ or $\lim_{i\to\infty}(\xi_{ij}-\xi_{ik})=\zeta_{jk}\in 2R\mathbb{Z}^{4n-1}$ exists. Passing to another subsequence we may assume for all $1\leq j,k\leq l$, either $\lim_{i\to\infty}|\xi_{ij}-\xi_{ik}|=\infty$ or $\xi_{ij}-\xi_{ik}=\zeta_{jk}$ for all i. We may also assume that $\kappa_{i,\xi_j}=\kappa_j$ for $1\leq j\leq l$ and all i's. Let $I=\{1,\cdots,l\}$. We say that $j,k\in I$ are equivalent if $\xi_{ij}-\xi_{ik}=\zeta_{jk}$. This defines an equivalence relation on I. Let $I_1,\cdots,I_m$ be the equivalent classes. For each $1\leq a\leq m$, we fix a $k_a\in I_a$. Let

$$N_a=\sum_{j\in I_a}\kappa_j=\sum_{j\in I_a}\kappa_{i,\xi_j}$$

for all i. Then

$$N_1+\cdots+N_m=\sum_{j=1}^{l}\kappa_{i,\xi_j}=\sum_{\xi\in 2R\mathbb{Z}^{4n-1}}\kappa_{i,\xi}=Q(u_i)=N.$$

Let $y_{ia}=\xi_{ik_a}\in 2R\mathbb{Z}^{4n-1}$. Then for $1\leq a,b\leq m$, $a\neq b$,

$$|y_{ia}-y_{ib}|=|\xi_{ik_a}-\xi_{ik_b}|\to\infty,$$

as $i\to\infty$. Let $v_{ia}(x)=u_i(x-y_{ia})$, $\tau_{ia}=\mathrm{d}^*(\Gamma * v_{ia}^*\omega_{S^{2n}})$. Then

$$\sum_{\xi\in 2R\mathbb{Z}^{4n-1}}\left|\frac{1}{|S^{2n}|^2}\int_{Q_R(\xi)} v_{ia}^*\omega_{S^{2n}}\wedge\tau_{ia}-\kappa_{i,\xi+y_{ia}}\right|\leq\varepsilon.$$

After passing to a subsequence if necessary, by the discussion following the statement of the theorem, we may find $v_a\in X$ such that as $i\to\infty$,

$$v_{ia}\to v_a \text{ a.e., } \mathrm{d}v_{ia}\rightharpoonup\mathrm{d}v_a \text{ in } L^{4n-2}\left(\mathbb{R}^{4n-1}\right),\ v_{ia}^*\omega_{S^{2n}}\rightharpoonup v_a^*\omega_{S^{2n}} \text{ in } L^2\left(\mathbb{R}^{4n-1}\right),$$

and

$$\tau_{ia}\rightharpoonup\tau_a\quad\text{in } W^{1,2}(B_r) \text{ for every } r>0.$$

Here $\tau_a=\mathrm{d}^*(\Gamma * v_a^*\omega_{S^{2n}})$. In particular,

$$v_{ia}^*\omega_{S^{2n}}\wedge\tau_{ia}\rightharpoonup v_a^*\omega_{S^{2n}}\wedge\tau_a\quad\text{in } L^1(B_r)$$

for all $r>0$. Note that it is clear that $\lim_{i\to\infty}\kappa_{i,\xi+y_{ia}}=\kappa_{\xi,a}$ always exists. Moreover

$$\kappa_{\xi,a}=\begin{cases}\kappa_j \text{ if } \xi=\zeta_{jk_a} \text{ for } j\in I_a,\\ 0,\quad\text{otherwise.}\end{cases}$$

Since $\frac{1}{|S^{2n}|^2}\int_{Q_R(\xi)} v_{ia}^*\omega_{S^{2n}}\wedge\tau_{ia}\to\frac{1}{|S^{2n}|^2}\int_{Q_R(\xi)} v_a^*\omega_{S^{2n}}\wedge\tau_a$ as $i\to\infty$, we see that

$$\sum_{\xi\in 2R\mathbb{Z}^{4n-1}}\left|\frac{1}{|S^{2n}|^2}\int_{Q_R(\xi)} v_a^*\omega_{S^{2n}}\wedge\tau_a-\kappa_{\xi,a}\right|\le\varepsilon.$$

Hence

$$\begin{aligned}|Q(v_a)-N_a|&=\left|Q(v_a)-\sum_{j\in I_a}\kappa_j\right|\\&\le\sum_{\xi\in 2R\mathbb{Z}^{4n-1}}\left|\frac{1}{|S^{2n}|^2}\int_{Q_R(\xi)} v_a^*\omega_{S^{2n}}\wedge\tau_a-\kappa_{\xi,a}\right|\le\varepsilon.\end{aligned}$$

This implies $Q(v_a)=N_a$ if we choose $\varepsilon<1$. Moreover, if we choose $\varepsilon\le\frac{1}{2}$, then

$$\left|\frac{1}{|S^{2n}|^2}\int_{Q_R} v_a^*\omega_{S^{2n}}\wedge\tau_a-\kappa_{j_a}\right|\le\frac{1}{2}.$$

Using the fact that $\kappa_{j_a}\ne 0$, we see that $\int_{Q_R}|v_a^*\omega_{S^{2n}}\wedge\tau_a|\,\mathrm{d}x\ge c(n)>0$. Hence the calculation in Lemma 11.1 implies $E(v_a)\ge c(n)>0$. Finally, fix $r>0$. Then for i large enough, we have

$$\begin{aligned}E(u_i)&\ge\sum_{a=1}^m\int_{B_r(y_{i,a})}\left(|\mathrm{d}u_i|^{4n-2}+|u_i^*\omega_{S^{2n}}|^2\right)\mathrm{d}x\\&=\sum_{a=1}^m\int_{B_r}\left(|\mathrm{d}v_{i,a}|^{4n-2}+\left|v_{i,a}^*\omega_{S^{2n}}\right|^2\right)\mathrm{d}x.\end{aligned}$$

Letting $i\to\infty$, we see that

$$E_N\ge\sum_{a=1}^m\int_{B_r}\left(|\mathrm{d}v_a|^{4n-2}+|v_a^*\omega_{S^{2n}}|^2\right)\mathrm{d}x.$$

Letting $r\to\infty$, we see that

$$E_N\ge\sum_{a=1}^m E(v_a)\ge\sum_{a=1}^m E_{N_a}.$$

Using $E(v_a)\ge c(n)>0$, we see that $m\le c(n)E_N$. To finish the argument, we observe that it follows from Corollary 13.3 below that $\sum_{a=1}^m E_{N_a}\ge E_N$. Hence $E_N=\sum_{a=1}^m E_{N_a}$ and $E_{N_a}=E(v_a)$ for all a's. □

Lemma 13.2.. *For every $u\in X$, there exists a sequence $u_i\in X$ and a sequence of positive numbers b_i such that*

$$u_i\to u\ \textit{a.e.},\quad du_i\to du\ \textit{in}\ L^{4n-2}\left(\mathbb{R}^{4n-1}\right),\quad u_i^*\omega_{S^{2n}}\to u^*\omega_{S^{2n}}\ \textit{in}\ L^2\left(\mathbb{R}^{4n-1}\right)$$

and

$$u_i(x',x_{4n-1})\equiv\text{const}\quad\textit{for}\ x_{4n-1}<-b_i.$$

Here $x=(x',x_{4n-1})$ with x' representing the first $4n-2$ coordinates.

To prove the lemma, we first introduce some coordinates on $\mathbb{R}^{4n-1}$. Note that we may use the stereographic projection with respect to the north pole $\mathbf{n}$ on S^{4n-2} to get

$$S^{4n-2}\backslash\{\mathbf{n}\} \to \mathbb{R}^{4n-2} : x \mapsto \xi, \quad \xi = \frac{x'}{1 - x_{4n-1}}.$$

In this way, we get a coordinate system on $S^{4n-2}\backslash\{\mathbf{n}\}$. For $x \in \mathbb{R}^{4n-1}\backslash\{(0,a) : a \geq 0\}$, we may take $r = |x|$ and ξ as the stereographic projection of $\frac{x}{|x|}$ with respect to $\mathbf{n}$. In this way, we get a coordinate system (r, ξ). The Euclidean metric is written as

$$g_{\mathbb{R}^{4n-1}} = \mathrm{d}r \otimes \mathrm{d}r + \frac{4r^2}{\left(1 + |\xi|^2\right)^2} \sum_{i=1}^{4n-2} \mathrm{d}\xi_i \otimes \mathrm{d}\xi_i.$$

We will use freely the coordinates x and (r, ξ). For $a > 0$, we denote

$$V_a = \{(r, \xi) : 0 < r < \infty, |\xi| < a\} \subset \mathbb{R}^{4n-1}$$

as the corresponding cone with origin as the vertex. Note that

$$V_1 = \left\{x \in \mathbb{R}^{4n-1} : x_{4n-1} < 0\right\}.$$

To continue we define a function

$$\phi(\xi) = \begin{cases} 0, \quad \xi \in B_{\frac{1}{8}}, \\ 2\left(|\xi| - \frac{1}{8}\right)\frac{\xi}{|\xi|}, \quad \xi \in B_{\frac{1}{4}}\backslash B_{\frac{1}{8}}, \\ \xi, \quad \xi \in B_{\frac{1}{2}}\backslash B_{\frac{1}{4}}. \end{cases}$$

We also write

$$F(r, \xi, \zeta) = F_\zeta(r, \xi) = (r, \phi(\xi) + \zeta)$$

for $0 < r < \infty$, $\xi \in B_{\frac{1}{2}}$ and $\zeta \in B_{\frac{1}{16}}$. It follows from the discussion in [HL, Section 3] that for *a.e.* $\zeta \in B_{\frac{1}{16}}$, $u \circ F_\zeta \in W^{1,2}_{loc}\left(V_{\frac{1}{2}}\right)$. Moreover

$$\begin{aligned} &\int_{V_{\frac{1}{2}}} \left(|\mathrm{d}(u \circ F_\zeta)|^{4n-2} + |(u \circ F_\zeta)^* \omega_{S^{2n}}|^2\right) \mathrm{d}x \\ &\leq c(n) \int_{\{0<r<\infty, \xi \in B_{\frac{1}{2}}\}} \left(|\mathrm{d}u|^{4n-2} + |u^*\omega_{S^{2n}}|^2\right)(r, \phi(\xi) + \zeta) \cdot r^{4n-2} \mathrm{d}r\mathrm{d}\xi. \end{aligned}$$

Hence

$$\begin{aligned} &\int_{B_{\frac{1}{16}}} \mathrm{d}\zeta \int_{V_{\frac{1}{2}}} \left(|\mathrm{d}(u \circ F_\zeta)|^{4n-2} + |(u \circ F_\zeta)^* \omega_{S^{2n}}|^2\right) \mathrm{d}x \\ &\leq c(n) \int_{\{0<r<\infty, \zeta \in B_1\}} \left(|\mathrm{d}u|^{4n-2} + |u^*\omega_{S^{2n}}|^2\right)(r, \zeta) \cdot r^{4n-2} \mathrm{d}r\mathrm{d}\zeta \\ &\leq c(n) \int_{V_1} \left(|\mathrm{d}u|^{4n-2} + |u^*\omega_{S^{2n}}|^2\right) \mathrm{d}x. \end{aligned}$$

It follows that we may find a $\zeta \in B_{\frac{1}{16}}$ such that

$$\int_{V_{\frac{1}{2}}} \left(\left| \mathrm{d} \left(u \circ F_\zeta \right) \right|^{4n-2} + \left| \left(u \circ F_\zeta \right)^* \omega_{S^{2n}} \right|^2 \right) \mathrm{d}x$$
$$\leq c(n) \int_{V_1} \left(|\mathrm{d}u|^{4n-2} + |u^* \omega_{S^{2n}}|^2 \right) \mathrm{d}x.$$

Let

$$v_1(r,\xi) = \begin{cases} u(r, \phi(\xi - \zeta) + \zeta), & \xi \in B_{\frac{1}{2}}(\zeta), \\ u(r,\xi), & \xi \notin B_{\frac{1}{2}}(\zeta). \end{cases}$$

Then $v_1 \in X$,

$$\int_{V_1} \left(|\mathrm{d}v_1|^{4n-2} + |v_1^* \omega_{S^{2n}}|^2 \right) \mathrm{d}x \leq c(n) \int_{V_1} \left(|\mathrm{d}u|^{4n-2} + |u^* \omega_{S^{2n}}|^2 \right) \mathrm{d}x$$

and

$$v_1(r,\xi) = u(r,\zeta) \quad \text{for } \xi \in B_{\frac{1}{16}},$$
$$v_1|_{\mathbb{R}^{4n-1} \backslash V_1} = u.$$

Let

$$v_2(r,\xi) = \begin{cases} v_1\left(r, \frac{\xi}{256}\right), & \xi \in B_{16}, \\ v_1\left(r, \left(\frac{511}{256}\left(|\xi| - 16\right) + \frac{1}{16}\right) \frac{\xi}{|\xi|}\right), & \xi \in B_{32} \backslash B_{16}, \\ v_1(r,\xi), & \xi \notin B_{32}. \end{cases}$$

We have $v_2 \in X$,

$$\int_{V_{32}} \left(|\mathrm{d}v_2|^{4n-2} + |v_2^* \omega_{S^{2n}}|^2 \right) \mathrm{d}x \leq c(n) \int_{V_{32}} \left(|\mathrm{d}u|^{4n-2} + |u^* \omega_{S^{2n}}|^2 \right) \mathrm{d}x$$

and

$$v_2(r,\xi) = u(r,\zeta) \quad \text{for } \xi \in B_{16},$$
$$v_2|_{\mathbb{R}^{4n-1} \backslash V_{32}} = u.$$

Let

$$f(r) = u(r,\zeta) \quad \text{for } 0 < r < \infty.$$

Then

$$\int_0^\infty \left| f'(r) \right|^{4n-2} r^{4n-2} \mathrm{d}r \leq c(n) \int_{V_1} \left(|\mathrm{d}u|^{4n-2} + |u^* \omega_{S^{2n}}|^2 \right) \mathrm{d}x < \infty.$$

Hence $|f'(r)| = |f'(r)| r \cdot \frac{1}{r} \in L^1([1,\infty))$. It follows that $\lim_{r\to\infty} f(r)$ exists. Without loss of generality we may assume that

$$\lim_{r\to\infty} f(r) = -\mathbf{n}.$$

Here $\mathbf{n}$ is the north pole of S^{2n}. We may find $R > 1$ large enough such that for $r \geq R$, $f(r)$ lies in lower half sphere. Let $\pi_{\mathbf{n}} : S^{2n} \backslash \{\mathbf{n}\} \to \mathbb{R}^{2n}$ be the stereographic projection with respect to $\mathbf{n}$. Define

$$g(r) = \pi_{\mathbf{n}}(f(r)) \quad \text{for } r \geq R.$$

Then $g(r) \to 0$ as $r \to \infty$, $|g(r)| \le 1$, and

$$\int_R^\infty |g'(r)|^{4n-2} r^{4n-2} \mathrm{d}r \le c(n) \int_{V_1} \left(|\mathrm{d}u|^{4n-2} + |u^* \omega_{S^{2n}}|^2 \right) \mathrm{d}x.$$

It follows from Hardy's inequality that

$$\begin{aligned} \int_R^\infty |g(r)|^{4n-2} \mathrm{d}r &\le c(n) \int_R^\infty |g'(r)|^{4n-2} r^{4n-2} \mathrm{d}r \\ &\le c(n) \int_{V_1} \left(|\mathrm{d}u|^{4n-2} + |u^* \omega_{S^{2n}}|^2 \right) \mathrm{d}x. \end{aligned}$$

Let

$$\eta(x) = \begin{cases} 1, & \text{if } x_{4n-1} \ge |x'| - 1, \\ \frac{x_{4n-1}+2}{|x'|+1}, & \text{if } |x'| - 1 \ge x_{4n-1} \ge -2, \\ 0, & \text{if } x_{4n-1} \le -2. \end{cases}$$

Note that

$$|\mathrm{d}\eta(x)| \le \frac{c(n)}{|x|+1}.$$

Denote

$$w(x) = \eta\left(\frac{x}{2R}\right) g(|x|) \quad \text{for } |x| > R.$$

Then

$$\begin{aligned} &\int_{\mathbb{R}^{4n-1} \setminus B_R} \left(|\mathrm{d}w|^{4n-2} + |\Lambda_{2n}(\mathrm{d}w)|^2 \right) \mathrm{d}x \\ &\le c(n) \int_R^\infty |g(r)|^{4n-2} \mathrm{d}r + c(n) \int_R^\infty |g'(r)|^{4n-2} r^{4n-2} \mathrm{d}r \\ &\le c(n) \int_{V_1} \left(|\mathrm{d}u|^{4n-2} + |u^* \omega_{S^{2n}}|^2 \right) \mathrm{d}x. \end{aligned}$$

Finally, we set

$$v(x) = \begin{cases} v_2(x), & \text{if } x_{4n-1} \ge |x'| - 2R, \\ \pi_{\mathbf{n}}^{-1}(w(x)), & \text{if } x_{4n-1} \le |x'| - 2R. \end{cases}$$

Then it follows from the construction that $v \in X$,

$$\int_{V_{32}} \left(|\mathrm{d}v|^{4n-2} + |v^* \omega_{S^{2n}}|^2 \right) \mathrm{d}x \le c(n) \int_{V_{32}} \left(|\mathrm{d}u|^{4n-2} + |u^* \omega_{S^{2n}}|^2 \right) \mathrm{d}x,$$

and

$$v|_{\mathbb{R}^{4n-1} \setminus V_{32}} = u, \quad v(x) = -\mathbf{n} \text{ for } x_{4n-1} \le -4R.$$

For every $\varepsilon > 0$, by vertical translation we may assume

$$\int_{V_{32}} \left(|\mathrm{d}u|^{4n-2} + |u^* \omega_{S^{2n}}|^2 \right) \mathrm{d}x < \varepsilon.$$

Then for the above constructed v, we have

$$\begin{aligned} &\int_{\mathbb{R}^{4n-1}} \left(|\mathrm{d}v - \mathrm{d}u|^{4n-2} + |v^* \omega_{S^{2n}} - u^* \omega_{S^{2n}}|^2 \right) \mathrm{d}x \\ &\le c(n) \int_{V_{32}} \left(|\mathrm{d}u|^{4n-2} + |u^* \omega_{S^{2n}}|^2 \right) \mathrm{d}x \le c(n)\,\varepsilon. \end{aligned}$$

Lemma 13.2 follows.

Corollary 13.3.. *For $N_1, N_2 \in \mathbb{Z}$, if $X_{N_1}, X_{N_2} \neq \emptyset$, then $X_{N_1+N_2} \neq \emptyset$ and*

$$E_{N_1+N_2} \leq E_{N_1} + E_{N_2}.$$

Indeed, for any $\varepsilon > 0$ small, it follows from Lemma 13.2 that we can find $u_1 \in X_{N_1}$, $u_2 \in X_{N_2}$ such that $E(u_1) < E_{N_1} + \varepsilon$, $E(u_2) < E_{N_2} + \varepsilon$, $u_1(x', x_{4n-1}) = -\mathbf{n}$ for $x_{4n-1} < 0$ and $u_2(x', x_{4n-1}) = -\mathbf{n}$ for $x_{4n-1} > 0$. Here $\mathbf{n}$ is the north pole of S^{2n}. Define

$$u(x) = \begin{cases} u_1(x), & \text{when } x_{4n-1} > 0, \\ u_2(x), & \text{when } x_{4n-1} < 0. \end{cases}$$

Then clearly $u \in X$ and $E(u) = E(u_1) + E(u_2) < E_{N_1} + E_{N_2} + 2\varepsilon$. We will show that $Q(u) = N_1 + N_2$. It follows that $E_{N_1+N_2} \leq E_{N_1} + E_{N_2} + 2\varepsilon$. Letting $\varepsilon \to 0^+$, we get the corollary. Indeed, denote

$$i : \mathbb{R}^{4n-2} \to \mathbb{R}^{4n-1} : x' \mapsto (x', 0)$$

as the natural put in map. Since $u_1^* \omega_{S^{2n}} \in L^{\frac{2(4n-1)}{4n+1}}$ and $u_1^* \omega_{S^{2n}} = 0$ on $\mathbb{R}_-^{4n-1}$, it follows from the Hodge theory that we may find $\tau_1 \in L^2(\mathbb{R}_+^{4n-1})$ with $D\tau_1 \in L^{\frac{2(4n-1)}{4n+1}}(\mathbb{R}_+^{4n-1})$ and $i^*\tau_1 = 0$. Let $\tau_1 = 0$ on $\mathbb{R}_-^{4n-1}$. Then the same argument as in the proof of Claim 12.3 shows that $\mathrm{d}\tau_1 = u_1^* \omega_{S^{2n}}$ on $\mathbb{R}^{4n-1}$. Similarly we may find $\tau_2 \in L^2(\mathbb{R}^{4n-1})$ such that $\mathrm{d}\tau_2 = u_2^* \omega_{S^{2n}}$ and $\tau_2|_{\mathbb{R}_+^{4n-1}} = 0$. Note that

$$\mathrm{d}(\tau_1 + \tau_2) = u_1^* \omega_{S^{2n}} + u_2^* \omega_{S^{2n}} = u^* \omega_{S^{2n}}.$$

It follows from Proposition 10.6 that

$$\begin{aligned} Q(u) &= \frac{1}{|S^{2n}|^2} \int_{\mathbb{R}^{4n-1}} u^* \omega_{S^{2n}} \wedge (\tau_1 + \tau_2) \\ &= \frac{1}{|S^{2n}|^2} \int_{\mathbb{R}^{4n-1}} u_1^* \omega_{S^{2n}} \wedge \tau_1 + u_2^* \omega_{S^{2n}} \wedge \tau_2 \\ &= Q(u_1) + Q(u_2) \\ &= N_1 + N_2. \end{aligned}$$

13.1. **Some discussion.** Here we describe some consequences of Theorem 13.1. For $n = 1, 2, 4$, we know for all $N \in \mathbb{Z}$, $X_N \neq \emptyset$ and

$$c(n)^{-1} |N|^{\frac{4n-1}{4n}} \leq E_N \leq c(n) |N|^{\frac{4n-1}{4n}}.$$

In particular, one can find $N_0 > 0$ with

$$E_{N_0} = \inf \{ E_N \mid N \in \mathbb{N} \}$$

and E_{N_0} is attainable. Let

$$\mathbb{S} = \{ N \in \mathbb{Z} : E_N \text{ is attainable} \}.$$

Then for every $N \neq 0$, there exist nonzero $N_1, \cdots, N_m \in \mathbb{S}$ with $N = N_1 + \cdots + N_m$ and

$$E_N = E_{N_1} + \cdots + E_{N_m}.$$

It follows from this and the fact $E_N \leq c(n) |N|^{\frac{4n-1}{4n}}$ that $\mathbb{S}$ must be infinite (otherwise E_N would grow at least linearly).

The situation for $n \neq 1, 2, 4$ is more subtle. In this case, we do not know whether $X_N \neq \emptyset$ when N is an odd integer (see Conjecture 1). If Conjecture 1 is verified, then similar conclusions as above are true with all N's being even. On the other hand, if

$X_N \neq \emptyset$ for some odd integer N, then it follows from Lemma 13.2 and the proof of Lemma 11.2 that for all integers N, $X_N \neq \emptyset$ and

$$c(n)^{-1} |N|^{\frac{4n-1}{4n}} \leq E_N \leq c(n) |N|^{\frac{4n-1}{4n}}.$$

Again the set $\mathbb{S} = \{N \in \mathbb{Z} \mid E_N \text{ is attainable}\}$ must be infinite.

14. Skyrme Model Revisited

In this section, we will prove a similar subadditivity property for the Skyrme energy spectrum (Corollary 14.2). As a consequence, the substantial inequality derived in [E1, E2, LY1] is improved to an equality (Theorem 14.3).

Recall that for a map $u \in W^{1,1}_{loc}(\mathbb{R}^3, S^3)$, the Skyrme energy is given by

$$E(u) = \int_{\mathbb{R}^3} \left(|\mathrm{d}u|^2 + |\mathrm{d}u \wedge \mathrm{d}u|^2 \right) \mathrm{d}x.$$

Denote

$$X = \left\{ u \in W^{1,1}_{loc}\left(\mathbb{R}^3, S^3\right) \mid E(u) < \infty \right\}.$$

The main aim of this section is to prove the following.

Lemma 14.1.. *For every $u \in X$, there exists a sequence $u_i \in X$ and a sequence of positive numbers b_i such that*

$$u_i \to u \text{ a.e.}, \quad du_i \to du \text{ in } L^2\left(\mathbb{R}^3\right), \quad du_i \wedge du_i \to du \wedge du \text{ in } L^2\left(\mathbb{R}^3\right)$$

and

$$u_i(x_1, x_2, x_3) \equiv \text{const} \quad \text{for } x_3 < -b_i.$$

For $N \in \mathbb{Z}$, we let

$$X_N = \left\{ u \in X \,\middle|\, \deg(u) = \frac{1}{|S^3|} \int_{\mathbb{R}^3} u^* \omega_{S^3} = N \right\} \tag{14.1}$$

and

$$E_N = \inf \{E(u) \mid u \in X_N\}. \tag{14.2}$$

A simple corollary of the lemma is the following

Corollary 14.2.. *For $N_1, N_2 \in \mathbb{Z}$,*

$$E_{N_1+N_2} \leq E_{N_1} + E_{N_2}.$$

Theorem 14.3.. *Assume N is an nonzero integer and $u_i \in X_N$ such that $E(u_i) \to E_N$. Then there exists an integer m with $1 \leq m \leq c \cdot E_N$, m nonzero integers $N_1, \cdots, N_m$ and $y_{i1}, \cdots, y_{im} \in \mathbb{R}^3$ such that*

- *$N = N_1 + \cdots + N_m$.*
- *$|y_{ij} - y_{ik}| \to \infty$ as $i \to \infty$ for $1 \leq j, k \leq m$, $j \neq k$.*
- *If we set $v_{ij}(x) = u_i(x - y_{ij})$ for $1 \leq j \leq m$, then there exists a $v_j \in X$ such that*

$$v_{ij} \to v_j \text{ a.e.}$$
$$dv_{ij} \rightharpoonup dv_j \text{ in } L^2\left(\mathbb{R}^3\right),$$
$$v_{ij}^* \omega_{S^3} \rightharpoonup v_j^* \omega_{S^3} \text{ in } L^2\left(\mathbb{R}^3\right)$$

as $i \to \infty$ *and*

$$\frac{1}{|S^3|}\int_{\mathbb{R}^3} v_j^* \omega_{S^3} = N_j, \quad E_{N_j} = E(v_j) \geq c > 0$$

for all j.

- $$E_N = \sum_{j=1}^{m} E_{N_j}.$$

In particular, if $E_N < E_{N'} + E_{N''}$ *for* $N = N' + N''$, $N', N'' \neq 0$, *then* E_N *defined in (14.2) is attainable.*

This theorem follows from similar arguments for Theorem 13.1 (see [E1, E2, LY1]). Unlike the integral formula for the Hopf–Whitehead invariant, the formula for the topological degree given in (14.1) is purely local and it makes the discussion relatively simpler.

Now we turn to the proof of Lemma 14.1. First we introduce some coordinates on $\mathbb{R}^3$. Note that we may use the stereographic projection with respect to $(0,0,1)$ on S^2 to get

$$S^2 \setminus \{(0,0,1)\} \to \mathbb{R}^2 : x \mapsto \xi, \quad \xi = \left(\frac{x_1}{1-x_3}, \frac{x_2}{1-x_3}\right).$$

In this way, we get a coordinate system on $S^2 \setminus \{(0,0,1)\}$. For $x \in \mathbb{R}^3 \setminus \{(0,0,a) : a \geq 0\}$, we may use coordinate $r = |x|$ and ξ as the stereographic projection of $\frac{x}{|x|}$ with respect to $(0,0,-1)$. In this way, we get a coordinate (r, ξ_1, ξ_2). The Euclidean metric is written as

$$g_{\mathbb{R}^3} = \mathrm{d}r \otimes \mathrm{d}r + \frac{4r^2}{\left(1+|\xi|^2\right)^2}\left(\mathrm{d}\xi_1 \otimes \mathrm{d}\xi_1 + \mathrm{d}\xi_2 \otimes \mathrm{d}\xi_2\right).$$

We will use freely the coordinates x and (r, ξ). For $a > 0$, we denote

$$V_a = \{(r,\xi) : 0 < r < \infty, |\xi| < a\} \subset \mathbb{R}^3$$

as the corresponding cone with origin as the vertex. Note that $V_1 = \{x \in \mathbb{R}^3 : x_3 < 0\}$.

To continue, we define a function,

$$\phi(\xi) = \begin{cases} 0, & \xi \in B_{\frac{1}{8}}, \\ 2\left(|\xi| - \frac{1}{8}\right)\frac{\xi}{|\xi|}, & \xi \in B_{\frac{1}{4}} \setminus B_{\frac{1}{8}}, \\ \xi, & \xi \in B_{\frac{1}{2}} \setminus B_{\frac{1}{4}}. \end{cases}$$

We also write

$$F(r,\xi,\zeta) = F_\zeta(r,\xi) = (r, \phi(\xi) + \zeta)$$

for $0 < r < \infty$, $\xi \in B_{\frac{1}{2}}$ and $\zeta \in B_{\frac{1}{16}}$. It follows from the discussion in [HL, section 3] that for *a.e.* $\zeta \in B_{\frac{1}{16}}$, $u \circ F_\zeta \in W^{1,2}_{loc}\left(V_{\frac{1}{2}}\right)$. Moreover

$$\begin{aligned} &\int_{V_{\frac{1}{2}}} \left(|\mathrm{d}(u \circ F_\zeta)|^2 + |\mathrm{d}(u \circ F_\zeta) \wedge \mathrm{d}(u \circ F_\zeta)|^2\right)\mathrm{d}x \\ &\leq c\int_{\{0<r<\infty, \xi \in B_{\frac{1}{2}}\}} \left(|\mathrm{d}u|^2 + |\mathrm{d}u \wedge \mathrm{d}u|^2\right)(r, \phi(\xi)+\zeta) \cdot r^2 \mathrm{d}r\mathrm{d}\xi. \end{aligned}$$

Hence

$$\begin{aligned}
&\int_{B_{\frac{1}{16}}} \mathrm{d}\zeta \int_{V_{\frac{1}{2}}} \left(\left|\mathrm{d}\left(u\circ F_\zeta\right)\right|^2 + \left|\mathrm{d}\left(u\circ F_\zeta\right)\wedge \mathrm{d}\left(u\circ F_\zeta\right)\right|^2\right)\mathrm{d}x \\
&\le c\int_{\{0<r<\infty,\zeta\in B_1\}} \left(|\mathrm{d}u|^2 + |\mathrm{d}u\wedge \mathrm{d}u|^2\right)(r,\zeta)\cdot r^2 \mathrm{d}r\mathrm{d}\zeta \\
&\le c\int_{V_1} \left(|\mathrm{d}u|^2 + |\mathrm{d}u\wedge \mathrm{d}u|^2\right)\mathrm{d}x.
\end{aligned}$$

It follows that we may find some $\zeta\in B_{\frac{1}{16}}$ such that

$$\begin{aligned}
&\int_{V_{\frac{1}{2}}} \left(\left|\mathrm{d}\left(u\circ F_\zeta\right)\right|^2 + \left|\mathrm{d}\left(u\circ F_\zeta\right)\wedge \mathrm{d}\left(u\circ F_\zeta\right)\right|^2\right)\mathrm{d}x \\
&\le c\int_{V_1} \left(|\mathrm{d}u|^2 + |\mathrm{d}u\wedge \mathrm{d}u|^2\right)\mathrm{d}x.
\end{aligned}$$

Let

$$v_1(r,\xi) = \begin{cases} u\left(r,\phi(\xi-\zeta)+\zeta\right), & \xi\in B_{\frac{1}{2}}(\zeta), \\ u(r,\xi), & \xi\notin B_{\frac{1}{2}}(\zeta). \end{cases}$$

Then $v_1\in X$,

$$\int_{V_1} \left(|\mathrm{d}v_1|^2 + |\mathrm{d}v_1\wedge \mathrm{d}v_1|^2\right)\mathrm{d}x \le c\int_{V_1} \left(|\mathrm{d}u|^2 + |\mathrm{d}u\wedge \mathrm{d}u|^2\right)\mathrm{d}x,$$

and

$$\begin{aligned}
&v_1(r,\xi) = u(r,\zeta) \quad \text{for } \xi\in B_{\frac{1}{16}}, \\
&v_1|_{\mathbb{R}^3\backslash V_1} = u.
\end{aligned}$$

Let

$$v_2(r,\xi) = \begin{cases} v_1\left(r,\frac{\xi}{256}\right), & \xi\in B_{16}, \\ v_1\left(r,\left(\frac{511}{256}(|\xi|-16)+\frac{1}{16}\right)\frac{\xi}{|\xi|}\right), & \xi\in B_{32}\backslash B_{16}, \\ v_1(r,\xi), & \xi\notin B_{32}. \end{cases}$$

We have $v_2\in X$,

$$\int_{V_{32}} \left(|\mathrm{d}v_2|^2 + |\mathrm{d}v_2\wedge \mathrm{d}v_2|^2\right)\mathrm{d}x \le c\int_{V_{32}} \left(|\mathrm{d}u|^2 + |\mathrm{d}u\wedge \mathrm{d}u|^2\right)\mathrm{d}x,$$

and

$$\begin{aligned}
&v_2(r,\xi) = u(r,\zeta) \quad \text{for } \xi\in B_{16}, \\
&v_2|_{\mathbb{R}^3\backslash V_{32}} = u.
\end{aligned}$$

Let

$$f(r) = u(r,\zeta) \quad \text{for } 0<r<\infty.$$

Then

$$\int_0^\infty |f'(r)|^2 r^2 \mathrm{d}r \le c\int_{V_1} \left(|\mathrm{d}u|^2 + |\mathrm{d}u\wedge \mathrm{d}u|^2\right)\mathrm{d}x < \infty.$$

Hence $|f'(r)| = |f'(r)| r \cdot \frac{1}{r} \in L^1([1,\infty))$. It follows that $\lim_{r\to\infty} f(r)$ exists. Without loss of generality we may assume

$$\lim_{r\to\infty} f(r) = (0,0,0,-1).$$

We may find $R > 1$ large enough such that for $r \geq R$, $f(r)$ lies in lower half sphere. Let $\mathbf{n} = (0,0,0,1)$ and $\pi_{\mathbf{n}} : S^3 \backslash \{\mathbf{n}\} \to \mathbb{R}^3$ be the stereographic projection with respect to $\mathbf{n}$, define

$$g(r) = \pi_{\mathbf{n}}(f(r)) \quad \text{for } r \geq R.$$

Then $g(r) \to 0$ as $r \to \infty$, $|g(r)| \leq 1$ and

$$\int_R^\infty |g'(r)|^2 r^2 \mathrm{d}r \leq c \int_{V_1} \left(|\mathrm{d}u|^2 + |\mathrm{d}u \wedge \mathrm{d}u|^2\right) \mathrm{d}x.$$

It follows from Hardy's inequality that

$$\int_R^\infty |g(r)|^2 \mathrm{d}r \leq c \int_R^\infty |g'(r)|^2 r^2 \mathrm{d}r \leq c \int_{V_1} \left(|\mathrm{d}u|^2 + |\mathrm{d}u \wedge \mathrm{d}u|^2\right) \mathrm{d}x.$$

Let

$$\eta(x) = \begin{cases} 1, & \text{if } x_3 \geq \sqrt{x_1^2 + x_2^2} - 1, \\ \frac{x_3+2}{\sqrt{x_1^2+x_2^2+1}}, & \text{if } \sqrt{x_1^2+x_2^2} - 1 \geq x_3 \geq -2, \\ 0, & \text{if } x_3 \leq -2. \end{cases}$$

Note that

$$|\mathrm{d}\eta(x)| \leq \frac{c}{|x|+1}.$$

Denote

$$w(x) = \eta\left(\frac{x}{2R}\right) g(|x|) \quad \text{for } |x| > R.$$

Then

$$\begin{aligned} &\int_{\mathbb{R}^3 \backslash B_R} \left(|\mathrm{d}w|^2 + |\mathrm{d}w \wedge \mathrm{d}w|^2\right) \mathrm{d}x \\ &\leq c \int_R^\infty |g(r)|^2 \mathrm{d}r + c \int_R^\infty |g'(r)|^2 r^2 \mathrm{d}r \\ &\leq c \int_{V_1} \left(|\mathrm{d}u|^2 + |\mathrm{d}u \wedge \mathrm{d}u|^2\right) \mathrm{d}x. \end{aligned}$$

Finally, we let

$$v(x) = \begin{cases} v_2(x), & \text{if } x_3 \geq \sqrt{x_1^2+x_2^2} - 2R, \\ \pi_{\mathbf{n}}^{-1}(w(x)), & \text{if } x_3 \leq \sqrt{x_1^2+x_2^2} - 2R. \end{cases}$$

Then, it follows from the construction, that $v \in X$,

$$\int_{V_{32}} \left(|\mathrm{d}v|^2 + |\mathrm{d}v \wedge \mathrm{d}v|^2\right) \mathrm{d}x \leq c \int_{V_{32}} \left(|\mathrm{d}u|^2 + |\mathrm{d}u \wedge \mathrm{d}u|^2\right) \mathrm{d}x,$$

and

$$v|_{\mathbb{R}^3 \backslash V_{32}} = u, \quad v(x_1,x_2,x_3) = (0,0,0,-1) \text{ for } x_3 \leq -4R.$$

For every $\varepsilon > 0$, after a vertical translation, we may assume

$$\int_{V_{32}} \left(|\mathrm{d}u|^2 + |\mathrm{d}u \wedge \mathrm{d}u|^2\right) \mathrm{d}x < \varepsilon.$$

Then for the above constructed v, we have

$$\int_{\mathbb{R}^3} \left(|\mathrm{d}v - \mathrm{d}u|^2 + |\mathrm{d}v \wedge \mathrm{d}v - \mathrm{d}u \wedge \mathrm{d}u|^2 \right) \mathrm{d}x$$
$$\leq c \int_{V_{32}} \left(|\mathrm{d}u|^2 + |\mathrm{d}u \wedge \mathrm{d}u|^2 \right) \mathrm{d}x \leq c\varepsilon.$$

Lemma 14.1 follows.

15. Conclusions

In this paper, we have carried out a systematic study of the Faddeev type knot energies in the most general Hopf dimensions governing maps from $\mathbb{R}^{4n-1}$ into S^{2n}. These maps are topologically stratified by the Hopf–Whitehead invariant, Q, which may be represented by a Chern–Simons type integral invariant. Two different types of energies are considered. The first type, referred to as the Nicole–Faddeev–Skyrme (NFS) model, contains a potential energy term and a conformally invariant kinetic energy term and allows a direct resolution in the spirit of the concentration-compactness principle due to the validity of an energy-cutting lemma. The second type, referred to as the Faddeev model, does not contain a potential energy term or a conformally invariant kinetic term and challenges a direct approach in a similar fashion. Nevertheless, we are able to show that both models follow the same energetic and topological decomposition relations in a global minimization process which closely resemble the energy conservation and charge conservation relations observed in a nuclear fission process. Furthermore, both types of models obey the same fractionally-powered universal growth laws relating knot energy to knot topology. These results lead us to the conclusion that, for either the NFS model or the Faddeev model, there is an infinite set of integers, $\mathbb{S}$, such that for each $N \in \mathbb{S}$, there exists a global energy minimizer among the maps in the topological class given by $Q = N$. Besides, in the compact setting where the domain space is S^{4n-1}, both models allow the existence of a global energy minimizer among the topological class $Q = N$ at any realizable Hopf–Whitehead number N.

Acknowledgements. F. Hang was supported in part by NSF under grant DMS-0647010 and a Sloan Research Fellowship. F. Lin was supported in part by NSF under grant DMS-0700517. Y. Yang was supported in part by NSF under grant DMS-0406446 and an Othmer senior faculty fellowship at Polytechnic University.

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