Exponential Growth Solutions of Elliptic Equations

Fengbo Hang *and* Fanghua Lin

(*Courant Institute, 251 Mercer street, New York, NY 10012*)

(fengbo@cims.nyu.edu linf@cims.nyu.edu)

Abstract We show that for a class of second order divergence form elliptic equations on an infinite strip with the Dirichlet boundary condition, the space of a fixed order exponential growth solutions is of finite dimension. An optimal estimation of the dimension is given. Examples also show that the finiteness property may not be true if one drops some of the conditions we made in our result.

Keywords elliptic equations, exponential growth function, Poincare's inequality, mean value inequality

1991MR Subject Classification 35B05

1 Introduction

For a differential equation on an unbounded domain, it is natural to consider solutions which satisfy certain growth conditions. The first step to understand the structure of these solutions is to prove spaces of solutions of a given growth are finite dimensional. In [1],[2], T.Colding and W.Minicozzi considered the polynomial growth harmonic functions on a complete Riemannian manifold. They showed that if the Ricci curvature is nonnegative, then the space of a fixed order polynomial growth harmonic functions has finite dimension, which proves a conjecture of S.T.Yau. Later in [3] P.Li gave another approach via the mean value inequality. In the present paper, we consider the second order divergence form elliptic equations on an infinite strip with zero Dirichlet boundary value. Here infinite strip means the product of the real line and a bounded domain. For the equation $\triangle u = 0$ on $\mathbb{R} \times (0,1)$ with the zero Dirichlet boundary condition, we know any solution must be a linear combination of $u_k(x,y) = e^{k\pi x}\sin(k\pi y),\ k \in \mathbb{Z}$. This suggests, for a general elliptic equation, it is reasonable to consider those solutions which have exponential growth in the unbounded direction. More precisely, we want to show the space of a fixed order exponential growth solutions has finite dimension. Now let us describe the main result. Let $\Omega_0 \subset \mathbb{R}^n$ be open and bounded. Volume(Ω_0)$\leq V_0 < \infty$, diam(Ω_0)$\leq D_0 < \infty$. $\Omega = \mathbb{R} \times \Omega_0$ is the infinite strip. We always assume

$$a^{ij}, b^i, c \in L^\infty(\Omega, \mathbb{R}) \text{ and } \Lambda > 0, |a^{ij}(x)| \leq \Lambda, \text{ for } x \in \Omega, 1 \leq i, j \leq n+1. \tag{1.1}$$

$$\text{there exists } \lambda > 0 \text{ s.t. } \lambda |\xi|^2 \leq a^{ij}(x)\xi^i \xi^j \text{ for all } x \in \Omega \text{ and } \xi \in \mathbb{R}^{n+1}. \tag{1.2}$$

Consider the following problem:

$$\begin{cases} -\partial_j(a^{ij}\partial_i u) + b^i \partial_i u + cu = 0 \text{ in } \Omega. \\ u : \Omega \to \mathbb{R} \text{ s.t. } u|_{(-a,a)\times\Omega_0} \in H^1((-a,a)\times\Omega_0) \text{ for any } a > 0. \\ u|_{\mathbb{R}\times\partial\Omega_0} = 0. \end{cases} \tag{1.3}$$

Let $d \geq 1$, define $\mathcal{A}_d = \{u \backslash u$ is a solution of equation(1.3), there exists $c = c(u) > 0$ s.t. $|u(x)| \leq ce^{d|x_1|}$, for all $x \in \Omega\}$.

Main Theorem. There exists $\varepsilon_0 = \varepsilon_0(n, \lambda, V_0) > 0$ s.t. if $c(x) \geq -\varepsilon_0$, $|b^i(x)| \leq \sqrt{\varepsilon_0}$, for all $x \in \Omega$, then dim$\mathcal{A}_d \leq Cd^n$, where $C = C(n, \lambda, \Lambda, V_0)$.

From the example of the Laplace equation on $\mathbb{R}\times(0,1)$ with the zero Dirichelet boundary condition, we know that the power n in the above dimension estimate is optimal. It is interesting that the smallness condition on the b^i and the negative part of c is necessary. Indeed by the arguments in [4] and [5], we may have a divergence form elliptic equation on the unit ball, whose coefficients are α Holder continuous for any $\alpha \in (0,1)$, such that it has a smooth nonzero solution which vanishes identically outside the half ball. Then by a simply patching of equations on the infinite strip, we may find a divergence form elliptic equation with coefficients α Holder continuous for any $\alpha \in (0,1)$, such that the space of bounded solutions are infinite dimensional. In view of the above theorem, we know this example is possible because the smallness condition is not satisfied. It is an interesting question to ask whether the main theorem remains true if we assume a^{ij} to be Lipschitz continuous and drop the smallness condition on the remaining coefficients. We also remark that the similar results can not be true for domains of type $\mathbb{R}^k \times \Omega_0$, $k \geq 2$.

The paper is written as follows. In section 2 below, we describe the basic properties of solutions of equation(1.3). Section 3 gives some characterizations of exponential growth monotone functions. Section 4 shows how to construct good solutions from arbitrary given ones. Then we give some necessary estimates in section 5 and section 6, where we use certain kind of compactness coming from the Poincare's inequality. In section 7, we give another approach to the dimension estimate by the mean value inequality. Both approaches are interesting because they may be useful for other problems related to the one considered here.

We would like to thank Prof. Weiyue Ding for the suggestions related to P.Li's paper [3] and an improvement of the earlier version of the paper.

2 Basic properties of solutions

In this section we will show that if the first order term coefficients and the negative part of the zero order coefficient of the equation(1.3) are small, then on any finite strip the L^2 integral of any solution has a definite part concentrating at the two ends. For $a > 0$, denote $D_a = (-a, a) \times \Omega_0$.

Lemma 2.1 *There exists $\varepsilon_0 = \varepsilon_0(n, \lambda, V_0) > 0$ s.t. if $c(x) \geq -\varepsilon_0$, $|b^i(x)| \leq \sqrt{\varepsilon_0}$ for all $x \in \Omega$, then for any solution u of equation(1.3), any $b > a > 0$, we have*

$$\int_{D_a} |Du|^2 \leq \frac{C}{(b-a)^2} \int_{D_b \setminus D_a} u^2,$$

$$\int_{D_a} u^2 \leq \frac{C}{(b-a)^2} \int_{D_b \setminus D_a} u^2,$$

$$\int_{D_b \setminus D_a} u^2 \geq \left(1 + \frac{C}{(b-a)^2}\right)^{-1} \int_{D_b} u^2,$$

where $C = C(n, \lambda, \Lambda, V_0) > 0$.

Proof. Choose $\eta_0 \in C_c^\infty(\mathbb{R})$ s.t.

$$0 \leq \eta_0 \leq 1,\ \eta_0|_{(-a,a)} = 1,\ \eta_0|_{\mathbb{R}\setminus(-b,b)} = 0 \text{ and } |\eta_0'(t)| \leq \frac{2}{b-a} \text{ for all } t \in \mathbb{R},$$

and let $\eta(x) = \eta_0(x_1),\ x \in \Omega$. It is easy to see

$$\eta|_{D_a} = 1,\ \eta|_{\Omega\setminus D_b} = 0,\ |D\eta| \leq \frac{2}{b-a}.$$

Let $\phi = \eta^2 u$ be the testing function, then using equation(1.3) and the integration by parts, one has

$$\int_\Omega \eta^2 a^{ij}\partial_i u \partial_j u + 2\eta u a^{ij}\partial_i u \partial_j \eta + b^i \partial_i u \eta^2 u + c\eta^2 u^2 = 0.$$

Hence

$$\lambda\int_\Omega \eta^2|Du|^2 \le C(n,\Lambda)\int_\Omega \eta|u||Du||D\eta| - \int_\Omega \eta^2 u b^i\partial_i u - \int_\Omega c\eta^2u^2$$

$$\le C(n,\Lambda)\int_\Omega \eta|u||Du||D\eta| + C(n)\sqrt{\varepsilon_0}\int_\Omega \eta^2|u||Du| + \varepsilon_0\int_\Omega \eta^2u^2$$

$$\le C(n,\lambda,\Lambda)\int_\Omega |D\eta|^2u^2 + \frac{\lambda}{2}\int_\Omega \eta^2|Du|^2 + C(n,\lambda)\varepsilon_0\int_\Omega \eta^2u^2$$

Therefore, one concludes

$$\int_\Omega \eta^2|Du|^2 \le C(n,\lambda,\Lambda)\int_\Omega |D\eta|^2u^2 + C(n,\lambda)\varepsilon_0\int_\Omega \eta^2u^2$$

$$\le C(n,\lambda,\Lambda)\int_\Omega |D\eta|^2u^2 + C(n,\lambda,V_0)\varepsilon_0\int_\Omega \eta^2|D'u|^2.$$

Here we have applied the Poincare's inequality to u on every slice $\{x_1\}\times\Omega_0$ (see p164 of [6]), and D' denotes the last n directional derivatives. Choose ε_0 s.t. $C(n,\lambda,V_0)\varepsilon_0 = \frac{1}{2}$, then we get

$$\int_\Omega \eta^2|Du|^2 \le C(n,\lambda,\Lambda)\int_\Omega |D\eta|^2u^2 \le \frac{C(n,\lambda,\Lambda)}{(b-a)^2}\int_{D_b\setminus D_a} u^2.$$

$$\int_{D_a} |Du|^2 \le \frac{C(n,\lambda,\Lambda)}{(b-a)^2}\int_{D_b\setminus D_a} u^2.$$

Using Poincare's inequality again, one obtains

$$\int_{D_a} u^2 \le \frac{C(n,\lambda,\Lambda,V_0)}{(b-a)^2}\int_{D_b\setminus D_a} u^2.$$

The third inequality in lemma 2.1 follows from the inequality above. **Q.E.D.**

Corollary 2.1 *There exists $\varepsilon_0 = \varepsilon_0(n,\lambda,V_0) > 0$ s.t. if $c(x) \ge -\varepsilon_0$, $|b^i(x)| \le \sqrt{\varepsilon_0}$ for all $x\in\Omega$, then any nonzero solution u of equation(1.3) must satisfy*

$$\liminf_{b\to\infty}\, b^{-1}\log\int_{D_b} u^2 \ge C_0(n,\lambda,\Lambda,V_0) > 0.$$

In particular, any bounded solution must be zero.

Proof. Let ε_0 be as in the lemma 2.1. From the second inequality in the conclusions we have for any $a, h > 0$, $\int_{D_a} u^2 \leq Ch^{-2} \int_{D_{a+h}} u^2$, where $C = C(n, \lambda, \Lambda, V_0)$. Choose h s.t. $Ch^{-2} = e^{-1}$, then $\int_{D_a} u^2 \leq e^{-1} \int_{D_{a+h}} u^2$. By a simple iteration, we have $\int_{D_a} u^2 \leq e^{-k} \int_{D_{a+kh}} u^2$ for any integer $k \geq 0$. Given any $b \geq a$, find a k s.t. $a + (k+1)h > b \geq a + kh$, then

$$\int_{D_b} u^2 \geq \int_{D_{a+kh}} u^2 \geq e^k \int_{D_a} u^2 \geq e^{\frac{b-a}{h}-1} \int_{D_a} u^2.$$

Since $u \neq 0$, we may find an a s.t. $\int_{D_a} u^2 > 0$. So

$$b^{-1} \log \int_{D_b} u^2 \geq \frac{1}{h} + \frac{\log \int_{D_a} u^2 - \frac{a}{h} - 1}{b}.$$

The conclusion follows by setting $C_0 = \frac{1}{h}$. **Q.E.D.**

3 Exponential growth monotone functions

In this section, we want to study properties of exponential growth nondecreasing functions which are similar to those been studied in [1] and [2] for the polynomial growth case.

Lemma 3.1 *Suppose* $f : [0, \infty) \to [0, \infty)$ *is a nondecreasing function, which is not identically zero. Let* $d > 0,\ \delta > 0,\ \sigma > e^{\delta d}$. *If there exists* $c \geq 0$ *s.t.* $f(t) \leq ce^{dt}$, *then there exists a sequence* $t_j \to \infty$ *s.t.* $0 < f(t_j + \delta) \leq \sigma f(t_j)$, *for all* $j \in \mathbb{N}$.

Proof. We need to show that there exists $t_j \to \infty$ s.t. $f(t_j + \delta) \leq \sigma f(t_j)$, for all $j \in \mathbb{N}$. This can be done by a contradiction argument. In fact, if it is not the case, then there exists $t_0 > 0$ s.t. $f(t + \delta) > \sigma f(t)$ for any $t \geq t_0$. Without losing of generality, we may assume $f(t_0) > 0$. By an iteriaton we know $f(t_0)\sigma^k \leq f(t_0 + k\delta) \leq ce^{dt_0}e^{k\delta d}$, for all natural number k, which implies $f(t_0) \leq ce^{dt_0}\left(\frac{e^{\delta d}}{\sigma}\right)^k$. Let $k \to \infty$, we get $f(t_0) \leq 0$, a contradiction. **Q.E.D.**

Lemma 3.2 *Suppose* $f_1, f_2, \cdots, f_{2k} : [0, \infty) \to [0, \infty),\ k \in \mathbb{N}$, *each* f_j *is nondecreasing and not identically zero. Let* $d > 0,\ \delta > 0,\ \sigma > e^{2d\delta}$, *and suppose for all* j, *there exists* $c_j \geq 0$ *s.t.* $f_j(t) \leq c_j e^{dt}$. *Then there exists* $E \subset \{1, 2, \cdots, 2k\}$ *s.t.* $card(E) = k$ *and there exists a sequence* $t_l \to \infty$ *s.t.* $0 < f_i(t_l + \delta) \leq \sigma f_i(t_l)$, *for any* $i \in E,\ l \in \mathbb{N}$.

Proof. Let us consider $f(t) = f_1(t)f_2(t)\cdots f_{2k}(t)$. f is a nondecreasing function and not identically zero. $f(t) \leq ce^{2kdt}$, $c = c_1c_2\cdots c_{2k}$. Since $\sigma^k > e^{\delta\cdot(2kd)}$, from lemma 3.1 we know there exists $t_l \to \infty$ s.t. $0 < f(t_l + \delta) \leq \sigma^k f(t_l)$, for any l. Then there exists $E \subset \{1, 2, \cdots, 2k\}$ with $card(E) = k$ and $0 < f_i(t_l + \delta) \leq \sigma f_i(t_l)$, for any $i \in E$, for otherwise we would obtain $f(t_l + \delta) > \sigma^{k+1} f(t_l)$, which contradicts with the choices of t_l. **Q.E.D.**

4 Constructing good functions from given ones

In this section, we will show how to construct solutions with desired properties from arbitrary given ones. For any $a > 0$, any u, v locally square integrable functions on $\mathbb{R} \times \overline{\Omega_0}$, denote $I_a(u, v) = \int_{D_a} uv$.

Lemma 4.1 *Suppose $V \subset \mathcal{A}_d$ is a nonzero finite dimensional subspace, $m = dimV$, then there exists $a_0 \geq 3$ s.t. for any $a \geq a_0$, I_a is an inner product on V.*

Proof. Pick up a base $u_1, \cdots, u_m$ of V. For any $a > 0$, define $W_a = \{(\lambda_1, \cdots, \lambda_m) \backslash \lambda_j \in \mathbb{R}$ and $\sum_{j=1}^m \lambda_j u_j|_{D_a} = 0\}$. It is easy to see $a_1 < a_2 \Rightarrow W_{a_1} \supset W_{a_2}$. Thus there exists $\lim_{a \to \infty} \text{dim} W_a = l \leq m$. It follows there is an $a_0 \geq 3$ s.t. $a \geq a_0 \Rightarrow \text{dim} W_a = l$. The latter implies $W_a = W_{a_0}$. Now it is easy to check $W_{a_0} = \cap_{a>0} W_a = 0$, this tells us, for $a \geq a_0$, $u_1|_{D_a}, \cdots, u_m|_{D_a}$ is linearly independent. Then we may conclude I_a is an inner product on V. **Q.E.D.**

Remark. The above lemma follows directly from the unique continuation theorem if the coefficients a^{ij} are Lipschitz continuous, see [7] for more information.

Lemma 4.2 *Suppose $u_1, u_2, \cdots, u_{2k} \in \mathcal{A}_d$ are linearly independent, $d \geq 1$, $k \in \mathbb{N}$, $\delta > 0$, $\sigma > e^{(4d+2)\delta}$ all fixed. Let $k_0 = [\frac{k}{2\sigma}]$. Then for any $A > 0$, there exists an $a \in \mathbb{R}$ s.t. $a > A$ and we may find $v_1, v_2, \cdots, v_{k_0} \in \mathcal{A}_d$, which satisfy*

$$\int_{D_a} v_i v_j = \delta_{ij},\ 1 \leq i, j \leq k_0,$$

$$\int_{D_{a+\delta}} v_i v_j = 0,\ i \neq j,\ 1 \leq i, j \leq k_0,$$

$$\int_{D_{a+\delta}} v_i^2 \leq 2\sigma,\ 1 \leq i \leq k_0.$$

Proof. Let $V = \text{span}\,\{u_1, u_2, \cdots, u_{2k}\} \subset \mathcal{A}_d$. For any $a \geq a_0$ (the number in lemma 4.1), we use $P_{a,\pi}$ to denote the orthogonal projection of V with respect to I_a onto the subspace π. Define

$$w_{i,a} = P_{a,(\text{span}\{u_1,u_2,\cdots,u_{i-1}\})^\perp}(u_i) \text{ for } i = 1, 2, \cdots, 2k,$$

so $w_{i,a} = c_{a,i,1}u_1 + c_{a,i,2}u_2 + \cdots + c_{a,i,i-1}u_{i-1} + u_i \neq 0$. For $a' \geq a_0$, we have

$$w_{i,a'} = P_{a',(\text{span}\{u_1,\cdots,u_{i-1}\})^\perp}(u_i) = P_{a',(\text{span}\{u_1,\cdots,u_{i-1}\})^\perp}(w_{i,a})$$

$\Rightarrow 0 < \int_{D_{a'}} w_{i,a'}^2 \leq \int_{D_{a'}} w_{i,a}^2$, so $a_0 \leq a' < a \Rightarrow 0 < \int_{D_{a'}} w_{i,a'}^2 \leq \int_{D_a} w_{i,a}^2$. Let

$$f_i(a) = \begin{cases} \int_{D_a} w_{i,a}^2 & \text{when } a \geq a_0 \\ 0 & \text{when } 0 \leq a < a_0. \end{cases}$$

Then f_i is nondecreasing and not identically zero. For all $a \geq a_0$,

$$f_i(a) = \int_{D_a} w_{i,a}^2 \leq \int_{D_a} u_i^2 \leq c_i^2 \int_{D_a} e^{2d|x_1|} \mathrm{d}x_1 \mathrm{d}x'$$

$$\leq c_i^2 e^{2da} V_0 \cdot 2a \leq 2c_i^2 V_0 e^{(2d+1)a}.$$

From lemma 3.2, we may assume for $f_1, f_2, \cdots, f_k$, there exists a sequence $t_l \to \infty$ s.t. $0 < f_i(t_l + \delta) \leq \sigma f_i(t_l)$, for all $l \in \mathbb{N}$, $1 \leq i \leq k$. Choose $l_0 \in \mathbb{N}$ s.t. $t_{l_0} > A$. Let $a = t_{l_0}$, then we have $0 < f_i(a+\delta) \leq \sigma f_i(a)$ for $1 \leq i \leq k$. Denote $w_1 = w_{1,a+\delta}$, $\cdots$, $w_k = w_{k,a+\delta}$, then

$$0 < \int_{D_{a+\delta}} w_i^2 \leq \sigma \int_{D_a} w_{i,a}^2 \leq \sigma \int_{D_a} w_i^2, \ 1 \leq i \leq k,$$

$$\int_{D_{a+\delta}} w_i w_j = 0, \ i \neq j, \ 1 \leq i, j \leq k.$$

After a scaling we may assume

$$\int_{D_{a+\delta}} w_i w_j = \delta_{ij}, \ 1 \leq i, j \leq k,$$

$$\int_{D_a} w_i^2 \geq \frac{1}{\sigma}, \ 1 \leq i \leq k.$$

Let $V_1 = \text{span}\{w_1, \cdots, w_k\} \subset V$, from linear algebras we know there exists $v_1, v_2, \cdots, v_k$, which forms a base for V_1 s.t. $I_{a+\delta}(v_i, v_j) = \delta_{ij}$; $I_a(v_i, v_j) = 0$ for $i \neq j$. Now

$$\frac{k}{\sigma} \leq \sum_{i=1}^{k} \int_{D_a} w_i^2 = \sum_{i=1}^{k} \int_{D_a} v_i^2.$$

Let $k_0 = [\frac{k}{2\sigma}]$. Using the fact that $\int_{D_a} v_i^2 \leq \int_{D_{a+\delta}} v_i^2 = 1$, we get $\text{card}(\left\{ i \backslash \int_{D_a} v_i^2 \geq \frac{1}{2\sigma} \right\}) \geq k_0$. Without losing of generality, we may assume $v_1, \cdots, v_{k_0}$ satisfy $\int_{D_a} v_i^2 \geq \frac{1}{2\sigma}$, $i = 1, 2, \cdots, k_0$. After a scaling we obtain the desired $v_1, \cdots, v_{k_0}$. **Q.E.D.**

5 First estimate of $\dim \mathcal{A}_d$

Lemma 5.1 *Suppose $c(x) \geq -\varepsilon_0$, $|b^i(x)| \leq \sqrt{\varepsilon_0}$, where ε_0 is the constant in lemma 2.1. Fix $\theta > 1$, $a \geq 2$. If $v_1, \cdots, v_{\mathcal{N}}$ are solutions of equation(1.3), $d_0 \geq 1$, and*

$$\int_{D_a} v_i v_j = \delta_{ij}, \ \int_{D_{a+\frac{1}{d_0}}} v_i^2 \leq \theta, \ 1 \leq i, j \leq \mathcal{N},$$

then $\mathcal{N} \leq C d_0^{n+1}$, where $C = C(n, \lambda, \Lambda, \theta, D_0)$.

Proof. We may extend any solution by zero outside Ω when necessary. From lemma 2.1 we have $\int_{D_a} |Dv_i|^2 \leq C d_0^2$. Let $l \in \mathbb{N}$ to be determined, then divide $((-a, -a+1) \times \mathbb{R}^n) \cup ((a-1, a) \times \mathbb{R}^n)$ into cubes of side length l^{-1}. Define $\mathcal{C} = \{I \backslash I$ is a cube as above which has nonempty intersection with $\Omega\}$. We have $card(\mathcal{C}) \leq C l^{n+1}$, $C = C(n, D_0)$. Define

$$V = \text{span}\{v_1, v_2, \cdots, v_{\mathcal{N}}\}, \ \mathcal{M} : V \to \bigoplus_{I \in \mathcal{C}} \mathbb{R} : v \mapsto (\int_I v)_{I \in \mathcal{C}}.$$

We may choose a base $w_1, \cdots, w_k$ for $\ker \mathcal{M} \subset V$, which is orthonormal with respect to I_a, and complete it to an orthonormal base for V, $w_1, \cdots, w_k, \cdots, w_{\mathcal{N}}$. For $1 \leq i \leq k$ and any $I \in \mathcal{C}$,

$$\int_I w_i^2 \leq \frac{C(n)}{l^2} \int_I |Dw_i|^2 .$$

Thus by lemma 2.1, one has

$$\left(1 + C\frac{1}{(a-(a-1))^2}\right)^{-1} \int_{D_a} w_i^2 \leq \int_{D_a \backslash D_{a-1}} w_i^2 \leq \frac{C(n)}{l^2} \int_{D_a} |Dw_i|^2,$$

from which one gets

$$1 = \int_{D_a} w_i^2 \leq \frac{C}{l^2} \int_{D_a} |Dw_i|^2, \quad C = C(n, \lambda, \Lambda, D_0).$$

Summing the above inequality from 1 to k, we obtain

$$k \leq \frac{C}{l^2} \sum_{i=1}^{k} \int_{D_a} |Dw_i|^2 \leq \frac{C}{l^2} \sum_{i+1}^{\mathcal{N}} \int_{D_a} |Dw_i|^2$$

$$= \frac{C}{l^2} \sum_{i=1}^{\mathcal{N}} \int_{D_a} |Dv_i|^2 \leq \frac{C}{l^2} d_0^2 \mathcal{N} = \left(\frac{\sqrt{C} d_0}{l}\right)^2 \mathcal{N}.$$

Let $l=[\sqrt{2C}d_0]+1$, we get $k\le\frac{\mathcal{N}}{2}$. Now

$$\mathcal{N}=k+\mathcal{N}-k\le\frac{\mathcal{N}}{2}+\mathrm{card}(\mathcal{C})\Rightarrow\mathcal{N}\le 2\mathrm{card}(\mathcal{C})\le Cl^{n+1}\le Cd_0^{n+1}.$$

Q.E.D.

Corollary 5.1 *Under the conditions in the above lemma, we have* $\dim\mathcal{A}_d\le Cd^{n+1},\ C=C(n,\lambda,\Lambda,D_0)$.

Proof. It follows from lemma 4.2 and lemma 5.1. **Q.E.D.**

6 Accurate estimate of $\dim\mathcal{A}_d$

Lemma 6.1 *Suppose* $c(x)\ge-\varepsilon_0$, $|b^i(x)|\le\sqrt{\varepsilon_0}$, *where* ε_0 *is the constant in lemma 2.1. Let* $\theta>1$ *be fixed,* $4\le K\le16$, $d_m=K^m$ *for* $m\in\mathbb{Z}$, $m\ge0$. *If* $v_1,\cdots,v_{\mathcal{N}}$ *are solutions of equation(1.3) and* $a\ge5-2^{1-m}$ *s.t.*

$$\int_{D_a}v_iv_j=\delta_{ij},\ \int_{D_{a+\frac{1}{d_m}}}v_i^2\le\theta,\ 1\le i,j\le\mathcal{N},$$

then $\mathcal{N}\le\overline{C}d_m^n,\ \overline{C}=\overline{C}(n,\lambda,\Lambda,\theta,D_0)$.

Proof. By induction on m. If $m=0$, then $a\ge3,\ d_0=1$, the conclusion follows from lemma 5.1. Say we have it for $m-1$, where $m\ge1$. Let $a\ge5-2^{1-m}$,

$$\int_{D_a}v_iv_j=\delta_{ij}\ ,\ \int_{D_{a+\frac{1}{d_m}}}v_i^2\le\theta,\ 1\le i,j\le\mathcal{N}.$$

Then $\int_{D_a}|Dv_i|^2\le Cd_m^2,\ C=C(n,\lambda,\Lambda,\theta,D_0)$. Set $\varepsilon=\frac{1}{ld_{m-1}}$, where $l\in\mathbb{N}$ is to be determined. We divide $((-a,-a+\frac{1}{d_{m-1}})\times\mathbb{R}^n)\cup((a-\frac{1}{d_{m-1}},a)\times\mathbb{R}^n)$ into cubes of side length ε. Define $\mathcal{C}=\{$those cubes above which have nonempty intersection with $\Omega\}$. Then $\mathrm{card}(\mathcal{C})\le\frac{C}{d_{m-1}\varepsilon^{n+1}}=Cl^{n+1}d_{m-1}^n$. Define

$$V=\mathrm{span}\{v_1,v_2,\cdots,v_{\mathcal{N}}\},\ \mathcal{M}:V\to\bigoplus_{I\in\mathcal{C}}\mathbb{R}:v\mapsto(\int_I v)_{I\in\mathcal{C}}.$$

Denote $k = \dim(\ker\mathcal{M})$, $\overline{a} = a - \frac{1}{d_{m-1}} \geq 5 - 2^{1-m} - 2^{1-m} = 5 - 2^{2-m}$. Choose $w_1, w_2, \cdots, w_k \in \ker\mathcal{M}$ s.t. $I_a(w_i, w_j) = \delta_{ij}$; $I_{\overline{a}}(w_i, w_j) = 0$, for $i \neq j$. From the induction hypothesis we know that at most $\overline{C}d_{m-1}^n$ w_i's satisfy $\int_{D_{\overline{a}}} w_i^2 \geq \frac{1}{\theta}$, so

$$(k - \overline{C}d_{m-1}^n)(1 - \frac{1}{\theta}) \leq \sum_{i=1}^{k} \int_{D_a \setminus D_{\overline{a}}} w_i^2.$$

But for $1 \leq i \leq k$,

$$\int_{D_a \setminus D_{\overline{a}}} w_i^2 = \sum_{I \in \mathcal{C}} \int_I w_i^2 \leq \sum_{I \in \mathcal{C}} C\varepsilon^2 \int_I |Dw_i|^2 \leq C\varepsilon^2 \int_{D_a} |Dw_i|^2 .$$

Complete $w_1, w_2, \cdots, w_k$ to an orthonormal base for V with respect to I_a, say $w_1, w_2, \cdots, w_{\mathcal{N}}$. Then

$$(k - \overline{C}d_{m-1}^n)(1 - \frac{1}{\theta}) \leq C\varepsilon^2 \sum_{i=1}^{k} \int_{D_a} |Dw_i|^2 \leq C\varepsilon^2 \sum_{i=1}^{\mathcal{N}} \int_{D_a} |Dw_i|^2$$

$$= C\varepsilon^2 \sum_{i=1}^{\mathcal{N}} \int_{D_a} |Dv_i|^2 \leq C\varepsilon^2 d_m^2 \mathcal{N}$$

$$= \frac{CK^2}{l^2}\mathcal{N} \leq \frac{C}{l^2}\mathcal{N},\ C = C(n, \lambda, \Lambda, \theta, D_0).$$

Now we have

$$k - \overline{C}d_{m-1}^n \leq \frac{C}{l^2}\mathcal{N},\ C = C(n, \lambda, \Lambda, \theta, D_0).$$

Put $l = [\sqrt{2C}] + 1$; then $k - \overline{C}d_{m-1}^n \leq \frac{1}{2}\mathcal{N}$. But

$$\mathcal{N} \leq k + \mathrm{card}(\mathcal{C}) \leq k + Cl^{n+1}d_{m-1}^n \leq \frac{1}{2}\mathcal{N} + \overline{C}d_{m-1}^n + Cd_{m-1}^n$$

$$\Rightarrow \mathcal{N} \leq 2\overline{C}d_{m-1}^n + Cd_m^n,\ C = C(n, \lambda, \Lambda, \theta, D_0).$$

If we let $\frac{\overline{C}}{2} \geq C$ (from the beginning), then $\mathcal{N} \leq \overline{C}d_m^n$. **Q.E.D.**

Corollary 6.1 *Under the conditions in the above lemma, $\theta > 1$ is fixed, $r \geq 4$, $a \geq 5$, $v_1, v_2, \cdots, v_{\mathcal{N}}$ are solutions of equation(1.3) s.t.*

$$\int_{D_a} v_i v_j = \delta_{ij},\ \int_{D_{a+\frac{1}{r}}} v_i^2 \leq \theta,\ 1 \leq i, j \leq \mathcal{N}.$$

Then $\mathcal{N} \leq \overline{C}r^n$, where $\overline{C} = \overline{C}(n, \lambda, \Lambda, \theta, D_0)$.

Proof. There exists $l \in \mathbb{N}$ s.t. $4^l \le r \le 4^{l+1}$, then $4 \le r^{\frac{1}{l}} \le 4^{1+\frac{1}{l}} \le 16$. Putting $K = r^{\frac{1}{l}}$, we have $r = K^l$ and $4 \le K \le 16$. From lemma 6.1 we know $\mathcal{N} \le \overline{C}(K^l)^n = \overline{C}r^n$, where $\overline{C} = \overline{C}(n, \lambda, \Lambda, \theta, D_0)$. **Q.E.D.**

Proof of the main theorem. Suppose $u_1, u_2, \cdots, u_{2k} \in \mathcal{A}_d$ are linearly independent. Put $\delta = \frac{1}{4d+2}$, $\sigma = e+1$ in lemma 4.2. We get $v_1, \cdots, v_{k_0} \in \mathcal{A}_d$, $a \ge 5$ s.t.

$$\int_{D_a} v_i v_j = \delta_{ij}, \int_{D_{a+\delta}} v_i^2 \le 2(e+1), \ 1 \le i, j \le k_0,$$

where $k_0 = [\frac{k}{2\sigma}]$. Now from corollary 6.1 we know $k_0 \le \overline{C}(4d+2)^n \le Cd^n$. It follows that $k \le 2\sigma k_0 + 2\sigma \le Cd^n$. Hence $\dim\mathcal{A}_d \le Cd^n$. **Q.E.D.**

7 The other approach to estimate $\dim\mathcal{A}_d$

Now we follow the idea in [3], by using the mean value inequality to give another approach to the dimension estimate.

Lemma 7.1 *Suppose $V \subset \mathcal{A}_d$ is a nonzero finite dimensional subspace, $m = dimV$, $\delta > 0$, $\theta > 2d\delta$, then for any $a_1 \ge a_0$(the constant in lemma 4.1), there exists $a \ge a_1$ s.t. $tr_{a+\delta} I_a \ge me^{-\theta}$, where $tr_{a+\delta} I_a$ means the trace of I_a under $I_{a+\delta}$.*

Proof. Suppose the lemma false, then there would exist $a_1 \ge a_0$ s.t. for any $a \ge a_1$, we have $tr_{a+\delta} I_a < me^{-\theta}$. Hence $det_{a+\delta} I_a \le e^{-m\theta}$. By iteration we get $det_{a_1+k\delta} I_{a_1} \le e^{-km\theta}$ for any natural number k. Fix an orthonormal base of V under I_{a_1}, say $u_1, \cdots, u_m$. $|u_i(x)| \le c_i e^{d|x_1|}$ for some $c_i > 0$. Then

$$\int_{D_b} u_i^2 \le C(c_i, V_0) b e^{2db} \ \text{ for any } b > 0,$$

from which we get

$$det_{a_1} I_{a_1+k\delta} \le C(a_1 + k\delta)^m e^{2md(a_1+k\delta)}, \ C = C(c_1, \cdots, c_m, m, V_0).$$

Combine with the former inequality we have $e^{km\theta} \le C(a_1 + k\delta)^m e^{2md(a_1+k\delta)}$, hence $1 \le C(a_1 + k\delta)^m e^{km(2d\delta-\theta)+2mda_1}$. Let $k \to \infty$, we get a contradiction. **Q.E.D.**

Lemma 7.2 *Suppose $c(x) \ge -\varepsilon_0$, $|b^i(x)| \le \sqrt{\varepsilon_0}$, where ε_0 is the constant in lemma 2.1. Let V be the same as in lemma 7.1. $u_1, \cdots, u_m$ is a base of V. Then for $a \ge a_0$(the number in lemma 4.1), $\delta \in (0, 1]$, we have*

$$\sum_{i=1}^{m} \int_{D_a} u_i^2 \le C\delta^{-n}\Theta,$$

where

$$\Theta = sup\{\int_{D_{a+\delta}} u^2 \ \setminus u = \sum_{i=1}^{m} \xi_i u_i,\ \xi_i \in \mathbb{R},\ \sum_{i=1}^{m} \xi_i^2 = 1\},\ C = C(n, \lambda, \Lambda, V_0).$$

Proof. First from Degiorgi-Nash-Moser theory we know any solution u is Holder continuous. For any $x \in D_a \setminus D_{a-1}$, denote $V_x = \{u \setminus u \in V, u(x) = 0\}$, then $\dim(V/V_x) \leq 1$. If $V_x \neq V$, we have $\dim(V/V_x) = 1$. Choose a base $v_1, \cdots, v_m$ of V s.t. $v_1, \cdots, v_{m-1} \in V_x$ and the base transformation matrix is an orthogonal matrix, then

$$\sum_{i=1}^{m} u_i(x)^2 = \sum_{i=1}^{m} v_i(x)^2 = v_m(x)^2 \leq \frac{C(n, \lambda, \Lambda)}{(\frac{a+\delta-|x_1|}{2})^{n+1}} \int_{B_{a+\delta-|x_1|}(x)} v_m^2,$$

here v_m is extended as zero outside Ω, and we use the local estimate at the boundary, see p202 of [6]. Thus we have

$$\sum_{i=1}^{m} u_i(x)^2 \leq C(n, \lambda, \Lambda)(a + \delta - |x_1|)^{-(n+1)} \int_{D_{a+\delta}} v_m^2$$

$$\leq C(n, \lambda, \Lambda)(a + \delta - |x_1|)^{-(n+1)} \Theta.$$

If $V_x = V$, the estimate is still true because the sum on the left is 0, and hence

$$\sum_{i=1}^{m} \int_{D_a \setminus D_{a-1}} u_i^2 \leq C(n, \lambda, \Lambda, V_0)\Theta \int_{a-1}^{a} (a + \delta - t)^{-(n+1)} dt$$

$$\leq C(n, \lambda, \Lambda, V_0)\Theta \int_{0}^{1} (\delta + t)^{-(n+1)} dt \leq C(n, \lambda, \Lambda, V_0)\Theta\delta^{-n}.$$

From lemma 2.1 we have

$$\int_{D_a} u_i^2 \leq C(n, \lambda, \Lambda, V_0) \int_{D_a \setminus D_{a-1}} u_i^2 \ \text{ for } 1 \leq i \leq m.$$

Combine this last fact with the above inequalities we get the conclusion. **Q.E.D.**

Proof of the main theorem. Suppose V is any nonzero finite dimensional subspace of $\mathcal{A}_d$, and $c \geq -\varepsilon_0$, $|b^i| \leq \sqrt{\varepsilon_0}$. ($\varepsilon_0$ is the number in lemma 2.1). $m = \dim V$. Given θ and δ s.t. $\theta > 2d\delta$, choose $a \geq a_0$ s.t. $tr_{a+\delta} I_a \geq me^{-\theta}$ as in lemma 7.1. Fix any orthonormal base for V under $I_{a+\delta}$, namely $u_1, \cdots, u_m$. From lemma 7.2 we know $\sum_{i=1}^{m} \int_{D_a} u_i^2 \leq C\delta^{-n}\Theta = C\delta^{-n}$.

Hence $me^{-\theta} \leq C\delta^{-n}$, which implies $m \leq Ce^{\theta}\delta^{-n}$. Put $\delta = \frac{1}{d}$, $\theta = 3$, we get $m \leq Cd^n$, where $C = C(n, \lambda, \Lambda, V_0)$. **Q.E.D.**

References

1 Colding T H, Minicozzi II W P. Harmonic functions on manifolds. Ann of Math, 1997, 146(3): 725–747

2 Colding T H, Minicozzi II W P. Weyl type bounds for harmonic functions. Invent Math, 1998, 131(2): 257–298

3 Li P. Harmonic sections of polynomial growth. Math Res Lett, 1997, 4(1): 35–44

4 Pliś A. A smooth linear elliptic differential equation without any solution in a sphere. Comm Pure Appl Math, 1961, 14(3): 599–617

5 Pliś A. On non-uniqueness in Cauchy problem for an elliptic second order differential equation. Bull Acad Polon Sci Sér Sci Math Astronom Phys, 1963, 11(3): 95–100

6 Gilbarg D, Trudinger N S. Elliptic partial differential equations of second order. 2nd edition, 3rd printing. Berlin: Springer-Verlag, 1998

7 Garofalo N, Lin Fanghua. Unique continuation for elliptic operators: a geometric variational approach. Comm Pure Appl Math, 1987, 40(3): 347-366