A REMARK ON ZHONG-YANG'S EIGENVALUE ESTIMATE

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Abstract. We prove that equality in Zhong-Yang's eigenvalue estimate can only hold on a circle. This answers an open question raised by Sakai and leads to the more interesting question whether a manifold is close to a circle or a line segment in the Gromov-Hausdorff sense when it almost satisfies the equality. As a first step we make a conjecture in dimension two.

Geometric inequalities are beautiful and powerful, especially when they are optimal. A classic example is the isoperimetric inequality: let D be a bounded domain in $\mathbb{R}^2$, then its area A and perimeter L satisfy the inequality

$$4\pi A \leq L^2.$$

Moreover we know exactly when the equality holds: if and only if D is a disc. For optimal geometric inequalities it is interesting to have a complete understanding of the equality case. Sometimes this can be easily achieved by checking the proof of the inequality. Take as an example the following elegant theorem due to Lichenerowicz [L]: let (M,g) be a compact Riemannian manifold of dimension n with $\mathrm{Ric} \geq n-1$, then $\lambda_1 \geq n$, here λ_1 is the first eigenvalue of the Laplacian operator on functions. It was proved by Obata [O] several years later that equality holds iff (M,g) is isometric to the standard sphere S^n. This is not difficult to prove by tracing back each inequality in Lichnerowicz's argument. In other cases characterizing the equality case may not be so easy. Take the Myers theorem proved in 1941: let (M,g) be a compact Riemannian manifold of dimension n with $\mathrm{Ric} \geq n-1$, then its diameter $d \leq \pi$ (see e.g. [P1]). To understand the equality case it is far from enough to simply analyze the proof of the inequality as doing so only gives some information along a geodesic. Some new idea is required. It was only proved in 1975 by Cheng [C] that (M,g) is isometric to the standard sphere S^n if $d = \pi$. His proof uses his eigenvalue comparison theorem. There is also an elementary proof in [Sh] using the Bishop-Gromov volume comparison theorem.

For a compact Riemannian manifold (M,g) with nonnegative Ricci curvature, Li-Yau [LY, Li] derived the beautiful inequality $\lambda_1 \geq \frac{\pi^2}{2d^2}$, here d is the diameter of M. By sharpening Li-Yau's method, Zhong-Yang [ZY] improved the inequality to $\lambda_1 \geq \frac{\pi^2}{d^2}$, which is optimal as equality holds on S^1. It is a natural question whether S^1 is the only case for equality. To answer this question it is not enough to go through Zhong-Yang's proof of the inequality. This was raised as an open problem by Sakai in [S].

The main purpose of this short note is to derive the following

Theorem 1. *Let (M,g) be a smooth compact Riemannian manifold with $\mathrm{Ric} \geq 0$ and $\lambda_1 = \frac{\pi^2}{d^2}$, here d is the diameter of M, then it is isometric to the circle of radius $\frac{d}{\pi}$.*

The solution depends on a different approach to Li-Yau's gradient estimate of the first eigenfunction. Li-Yau derived their gradient estimate by applying the maximum principle to a judiciously chosen auxiliary function. The proof focuses on a point where this auxiliary function attains its maximum. We derive a differential inequality on the dense open set which consists of all regular points of the eigenfunction and then apply the strong maximum principle.

Lemma 1. *Let u be a nonzero smooth function on a Riemannian manifold such that $-\Delta u = \lambda u$. Set $\phi = |\nabla u|^2 + \lambda u^2$. Then on $\Omega = \{\nabla u \neq 0\}$,*

$$\Delta\phi - \frac{\nabla\left(\phi - 2\lambda u^2\right)\cdot\nabla\phi}{2\left|\nabla u\right|^2} \geq 2\mathrm{Ric}(\nabla u, \nabla u).$$

Proof. Under a local orthonormal frame, we have

$$\frac{1}{2}\phi_i = \sum_j u_j u_{ji} + \lambda u u_i.$$

Hence

$$\left|\frac{1}{2}\nabla\phi - \lambda u \nabla u\right|^2 = \sum_i \left(\sum_j u_j u_{ji}\right)^2 \leq \left|D^2 u\right|^2 \left|\nabla u\right|^2.$$

This implies

$$\frac{1}{4}\left|\nabla\phi\right|^2 - \lambda u \nabla u\cdot\nabla\phi \leq \left|\nabla u\right|^2 \left(\left|D^2 u\right|^2 - \lambda^2 u^2\right).$$

Therefore on Ω,

$$\left|D^2 u\right|^2 - \lambda^2 u^2 \geq \frac{\left|\nabla\phi\right|^2 - 4\lambda u\nabla u\cdot\nabla\phi}{4\left|\nabla u\right|^2} = \frac{\nabla\left(\phi - 2\lambda u^2\right)\cdot\nabla\phi}{4\left|\nabla u\right|^2}.$$

By the Bochner formula we have

$$\begin{aligned}
\frac{1}{2}\Delta\phi &= \left|D^2 u\right|^2 + \nabla u\cdot\nabla\Delta u + \mathrm{Ric}(\nabla u, \nabla u) + \lambda\left|\nabla u\right|^2 + \lambda u\Delta u \\
&= \left|D^2 u\right|^2 - \lambda^2 u^2 + \mathrm{Ric}\left(\nabla u, \nabla u\right) \\
&\geq \frac{\nabla\left(\phi - 2\lambda u^2\right)\cdot\nabla\phi}{4\left|\nabla u\right|^2} + \mathrm{Ric}\left(\nabla u, \nabla u\right).
\end{aligned}$$

∎

Now we recall the work of Li-Yau and Zhong-Yang following the presentation in the book [SY, section 4 of chapter 3]. Let (M, g) be an n-dimensional compact Riemannian manifold with $\mathrm{Ric} \geq 0$. Suppose that u is a first eigenfunction and the first eigenvalue is λ_1. We can assume

$$1 = \max_M u > \min_M u = -k, \quad 0 < k \leq 1.$$

Let

$$\widetilde{u} = \frac{u - \frac{1-k}{2}}{\frac{1+k}{2}},$$

then

$$\begin{cases} -\Delta\widetilde{u} = \lambda_1(\widetilde{u} + a), a = \frac{1-k}{1+k} \in [0, 1), \\ \max_M \widetilde{u} = 1, \min_M \widetilde{u} = -1. \end{cases}$$

For small $\varepsilon > 0$, let $v_\varepsilon = \frac{\widetilde{u}}{1+\varepsilon}$. Then

$$\begin{cases} -\Delta v_\varepsilon = \lambda_1(v_\varepsilon + a_\varepsilon), a_\varepsilon = \frac{a}{1+\varepsilon}, \\ \max_M v_\varepsilon = \frac{1}{1+\varepsilon}, \min_M v_\varepsilon = -\frac{1}{1+\varepsilon}. \end{cases}$$

Li and Yau proved the following gradient estimate

$$\frac{|\nabla v_\varepsilon|^2}{1 - v_\varepsilon^2} \leq \lambda_1(1 + a_\varepsilon). \tag{0.1}$$

Zhong and Yang established a more precise estimate than (0.1). Set $v_\varepsilon = \sin\theta_\varepsilon$. The function θ_ε has its range in $[-\frac{\pi}{2} + \delta, \frac{\pi}{2} - \delta]$, where δ is specified by

$$\sin(\frac{\pi}{2} - \delta) = \frac{1}{1+\varepsilon}.$$

Define $\psi : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to \mathbb{R}$ by

$$\begin{cases} \psi(\theta) = \frac{\frac{4}{\pi}(\theta + \cos\theta\sin\theta) - 2\sin\theta}{\cos^2\theta}, \theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \\ \psi(\frac{\pi}{2}) = 1, \psi(-\frac{\pi}{2}) = -1. \end{cases}$$

It is clear that ψ is continuous on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and $\psi(-\theta) = -\psi(\theta)$. Moreover $\psi'(\theta) \geq 0$ and $|\psi(\theta)| \leq 1$. The main result of Zhong and Yang is the following estimate which improves (0.1),

$$|\nabla\theta_\varepsilon|^2 \leq \lambda_1 \left(1 + a_\varepsilon\psi(\theta_\varepsilon)\right). \tag{0.2}$$

From this result one can deduce $\lambda_1 \geq \frac{\pi^2}{d^2}$ as follows. Let p_0 and p_1 be two points such that $\theta_\varepsilon(p_0) = -\frac{\pi}{2} + \delta$ and $\theta_\varepsilon(p_1) = \frac{\pi}{2} - \delta$. Let γ be the shortest geodesic from p_0 to p_1, then

$$\begin{aligned} \lambda_1^{1/2} d &\geq \lambda_1^{1/2} L(\gamma) \\ &\geq \int_{-\frac{\pi}{2}+\delta}^{\frac{\pi}{2}-\delta} \frac{d\theta}{\sqrt{1 + a_\varepsilon\psi(\theta)}} \\ &= \int_0^{\frac{\pi}{2}-\delta} \left(\frac{1}{\sqrt{1 + a_\varepsilon\psi(\theta)}} + \frac{1}{\sqrt{1 - a_\varepsilon\psi(\theta)}} \right) d\theta \\ &= 2\int_0^{\frac{\pi}{2}-\delta} \left(1 + \sum_{j=1}^{\infty} \frac{(4j)!}{2^{4j}(2j)!^2} a_\varepsilon^{2j} \psi(\theta)^{2j} \right) d\theta \\ &\geq \pi - 2\delta + \frac{3}{4} a_\varepsilon^2 \int_0^{\frac{\pi}{2}-\delta} \psi(\theta)^2 \, d\theta. \end{aligned}$$

Letting $\varepsilon \to 0^+$, hence $\delta \to 0$ too, we get

$$\lambda_1^{1/2} d \geq \pi + \frac{3(1-k)^2}{4(1+k)^2} \int_0^{\frac{\pi}{2}} \psi(\theta)^2 d\theta.$$

Therefore $\lambda_1 \geq \frac{\pi^2}{d^2}$ and the inequality is strict unless $k = 1$ (i.e. $\min_M u = -1$).

We now prove Theorem 1. Suppose $\lambda_1 = \frac{\pi^2}{d^2}$. First we have $\min_M u = -1$. By scaling the metric we assume that $d = \pi$, then $\lambda_1 = 1$. Let $\phi = |\nabla u|^2 + u^2$. By

Lemma 1 we have on $\Omega = \{\nabla u \neq 0\}$,

$$\Delta\phi - \frac{\nabla\left(\phi - 2u^2\right)\cdot\nabla\phi}{2\left|\nabla u\right|^2} \geq 0. \tag{0.3}$$

By the maximum principle

$$\phi = \left|\nabla u\right|^2 + u^2 \leq \max_{\{\nabla u = 0\}}\left(\left|\nabla u\right|^2 + u^2\right) = 1.$$

Take two points p_0 and p_1 such that $u(p_0) = -1$ and $u(p_1) = 1$. Let $\gamma : [0, l] \to M$ be a (unit speed) minimizing geodesic from p_0 to p_1. Denote $f(t) = u(\gamma(t))$, then

$$\left|f'(t)\right| = \left|\nabla u(\gamma(t)) \cdot \gamma'(t)\right| \leq \left|\nabla u(\gamma(t))\right| \leq \sqrt{1 - f(t)^2}.$$

Hence for any $\varepsilon > 0$,

$$\pi \geq l \geq \int_{\{0 \leq t \leq l, f'(t) > 0\}} dt \geq \int_0^l \frac{f'(t)}{\sqrt{1 - f(t)^2} + \varepsilon} dt = \int_{-1}^1 \frac{1}{\sqrt{1 - x^2} + \varepsilon} dx. \tag{0.4}$$

Let $\varepsilon \to 0^+$ we see

$$\pi \geq l \geq \int_{\{0 \leq t \leq l, f'(t) > 0\}} dt \geq \int_{-1}^1 \frac{1}{\sqrt{1 - x^2}} dx = \pi.$$

It follows that $l = \pi$ and $f'(t) > 0$ for a.e. $t \in (0, \pi)$. Hence f is strictly increasing on $[0, \pi]$. From (0.4) we see

$$\int_0^\pi \frac{f'(t)}{\sqrt{1 - f(t)^2}} dt = \pi.$$

It follows that $f'(t) = \sqrt{1 - f(t)^2}$ for $t \in (0, \pi)$. By continuity we have

$$f'(t) = \sqrt{1 - f(t)^2} \quad \text{for all } t \in [0, \pi].$$

Hence $f^2 + f'^2 = 1$. Differentiating with respect to t, we get $ff' + f'f'' = 0$. Since $f'(t) > 0$ for $0 < t < \pi$, we see $f'' + f = 0$. As $f(0) = -1$ and $f'(0) = 0$, we see

$$f(t) = u(\gamma(t)) = -\cos t \quad \text{for } t \in [0, \pi].$$

It follows that $\left(D^2 u\right)(\gamma'(0), \gamma'(0)) = 1$. Since $\Delta u(p_0) = 1$ and $\left(D^2 u\right)_{p_0} \geq 0$, we must have

$$\left(D^2 u\right)_{p_0} = \lambda_{\gamma'(0)} \otimes \lambda_{\gamma'(0)},$$

here for any tangent vector X, λ_X is the dual cotangent vector given by $\lambda_X(Y) = \langle X, Y\rangle$ for any tangent vector Y. To continue, we make the following observation:

Lemma 2. *The set $\{u = \pm 1\}$ has at most four points.*

Proof. Let p be any point with $u(p) = 1$. For a minimizing geodesic $\gamma_p : [0, l_p] \to M$ from p_0 to p, the same argument as before shows $l_p = \pi$ and

$$\left(D^2 u\right)_{p_0} = \lambda_{\gamma_p'(0)} \otimes \lambda_{\gamma_p'(0)}.$$

It follows that $\gamma_p'(0) = \pm\gamma'(0)$. Hence $p = \exp_{p_0}\left(\pi\gamma_p'(0)\right)$ has at most two choices. The same argument works for $\{u = -1\}$. ∎

To finish the proof of Theorem 1, we only need to show the dimension of M must be 1. Indeed, if this is not the case, then $\dim M \geq 2$. Let $M^* = M \backslash \{u = \pm 1\}$, then M^* is still connected. We want to show $|\nabla u|^2 + u^2 = 1$ on M^*. Indeed, let

$$E = \left\{ p \in M^* : |\nabla u(p)|^2 + u(p)^2 = 1 \right\}.$$

Clearly E is closed. On the other hand, if $p \in E \subset \Omega$, by (0.3) and the strong maximum principle, $|\nabla u|^2 + u^2 \equiv 1$ near p. Hence E must be either an empty set or M^*. Since for any $t \in (0, \pi)$,

$$|\nabla u(\gamma(t))|^2 + (u(\gamma(t)))^2 \geq \sin^2 t + \cos^2 t = 1,$$

we see E is nonempty and therefore $E = M^*$.

Define $X = \frac{\nabla u}{|\nabla u|}$ on M^*. Since $|\nabla u|^2 + u^2 \equiv 1$, by differentiation we have

$$D^2 u(X, X) = -u. \tag{0.5}$$

The proof of Lemma 1 shows that

$$\left|D^2 u\right|^2 = u^2. \tag{0.6}$$

By (0.5) and (0.6),

$$D^2 u = -u \lambda_X \otimes \lambda_X.$$

A simple computation then shows that $D_X X = 0$. In particular all integral curves of X are geodesics. Denote $\Sigma = \{u = 0\}$. Since $|\nabla u| = 1$ on Σ we see Σ is a hypersurface (which may have more than one components). For any $p \in \Sigma$, let α_p be the maximal integral curve of $-X$ with $\alpha_p(0) = p$. Then α_p is a unit speed geodesic. Let $f_p(t) = u(\alpha_p(t))$. We have $f_p(0) = 0$ and $f_p'(t) = -\sqrt{1 - f_p(t)^2}$. It follows that $f_p(t) = -\sin t$ for $t \in \left[0, \frac{\pi}{2}\right)$. On the other hand, as a geodesic on M, α_p is defined on $[0, \infty)$. We have $u(\alpha_p(t)) = -\sin t$ for $t \in \left[0, \frac{\pi}{2}\right]$. In particular $u\left(\alpha_p\left(\frac{\pi}{2}\right)\right) = -1$. The same argument as before shows

$$\left(D^2 u\right)_{\alpha_p\left(\frac{\pi}{2}\right)} = \lambda_{\alpha_p'\left(\frac{\pi}{2}\right)} \otimes \lambda_{\alpha_p'\left(\frac{\pi}{2}\right)}.$$

Note that $p = \exp_{\alpha_p\left(\frac{\pi}{2}\right)}\left(-\frac{\pi}{2}\alpha_p'\left(\frac{\pi}{2}\right)\right)$. Since there are at most two points in the set $\{u = -1\}$, we may find q with $u(q) = -1$ and infinitely many $p \in \Sigma$ such that $\alpha_p\left(\frac{\pi}{2}\right) = q$. This clearly leads to a contradiction since $\alpha_p'\left(\frac{\pi}{2}\right)$ has at most two choices. Therefore M must be of one dimension and the main theorem follows.

We would like to point out that one has a similar result for the first nonzero eigenvalue of the Laplacian with respect to the Neumann boundary condition of a manifold with nonnegative Ricci curvature and convex boundary. More precisely:

> Let (M, g) be a smooth compact Riemannian manifold with nonnegative Ricci curvature and nonempty convex boundary, μ_1 be the first nonzero eigenvalue of Laplacian with respect to the Neumann boundary condition, d be the diameter of M, then $\mu_1 \geq \frac{\pi^2}{d^2}$. Moreover equality holds if and only if M is isometric to a line segment.

In this case the sharp inequality $\mu_1 \geq \frac{\pi^2}{d^2}$ was first established by Payne and Weinberger [PW] for a bounded convex open subset Ω with smooth boundary in the Euclidean space (but it does not seem to follow from their argument that Ω is a line segment if equality holds). In general the inequality $\mu_1 \geq \frac{\pi^2}{d^2}$ was derived in [LY, ZY]. The equality case follows from a minor modification of the above argument.

Sakai [S] also asked whether a closed manifold with nonnegative Ricci curvature is close to a circle in the Gromov-Hausdorff sense if λ_1 is close to π^2/d^2. This is a natural question in the context of "almost rigidity" on which there is now a large literature. To our knowledge the first almost rigidity result under a lower Ricci bound is the following theorem due to Colding [Co1, Co2]: a compact Riemannian manifold (M^n, g) with $\text{Ric} \geq n-1$ is close to S^n in the Gromov-Hausdorff sense iff $\text{Vol}(M)$ is close to $\text{Vol}(S^n)$. See Cheeger-Colding [ChC] for further results in this direction. In regarding to the Lichenerowicz-Obata theorem for a compact Riemannian manifold (M^n, g) with $\text{Ric} \geq n-1$, Petersen [P2] proved that if λ_{n+1} is closed to n then M is close to S^n in the Gromov-Hausdorff sense (it was known that λ_1 is not enough).

For $\varepsilon > 0$ small let Σ_ε be the boundary surface of the ε-neighborhood of a line segment with length π in $\mathbb{R}^3$, then $\lambda_1(\Sigma_\varepsilon) \to 1$ and Σ_ε converges to $[0, \pi]$ is the Growmov-Hausdorff sense as $\varepsilon \to 0^+$. This example suggests Sakai's question should be modified to allow the possibility of a line segment as the collapsing limit even for a sequence of closed manifolds. The modified conjecture seems very plausible in dimension two and more precisely we make a conjecture of the following Bonnesen-style inequality ([Os]) (one may make similar conjectures in higher dimensions or for manifolds).

Conjecture 1. *Assume Ω is a bounded convex open subset in $\mathbb{R}^2$ with smooth boundary, d is the diameter of Ω, w is the width of Ω, that is*

$$w = \inf_{e \in S^1} \left(\sup_{x \in \Omega} x \cdot e - \inf_{x \in \Omega} x \cdot e \right),$$

then the first eigenvalue with respect to Neumann boundary condition

$$\mu_1 \geq \frac{\pi^2}{d^2} + c\frac{w^2}{d^4}.$$

Here c is an absolute constant.

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