PROBLEM ONE

1. (Section 3.3, from 24-31) Determine if the following relations on the set \( \{1, 2, 3, \ldots \} \) are reflexive, transitive, symmetric, and/or antisymmetric.

(a) \((x, y) \in R \) if \( x - y = 2 \)

Solution:

<table>
<thead>
<tr>
<th>reflexive</th>
<th>transitive</th>
<th>symmetric</th>
<th>antisymmetric</th>
</tr>
</thead>
<tbody>
<tr>
<td>No</td>
<td>No</td>
<td>No</td>
<td>Yes</td>
</tr>
</tbody>
</table>

(b) \((x, y) \in R \) if \(|x - y| = 2 \)

<table>
<thead>
<tr>
<th>reflexive</th>
<th>transitive</th>
<th>symmetric</th>
<th>antisymmetric</th>
</tr>
</thead>
<tbody>
<tr>
<td>No</td>
<td>No</td>
<td>Yes</td>
<td>No</td>
</tr>
</tbody>
</table>

2. (Section 3.3, 32) Let \( X \) be a nonempty set. Define a relation on \( P(X) \) (the power set of \( X \)) by \((A, B) \in R \) if \( A \subset B \) (here I do NOT mean proper subset). Is this relation reflexive, transitive, symmetric, and/or antisymmetric?

Solution:

<table>
<thead>
<tr>
<th>reflexive</th>
<th>transitive</th>
<th>symmetric</th>
<th>antisymmetric</th>
</tr>
</thead>
<tbody>
<tr>
<td>Yes</td>
<td>Yes</td>
<td>No</td>
<td>Yes</td>
</tr>
</tbody>
</table>

3. (Section 3.4, from 1-8) Determine if the following relations are equivalence relations on the set \( X = \{1, 2, 3, 4, 5\} \). If the relation is an equivalence relation, list all equivalence classes.

(a) \( \{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (1, 3), (3, 1)\} \)

Solution: This is an equivalence relation with equivalence classes

\[ \{1, 3\}, \{2\}, \{4\}, \{5\} \]

(b) \( \{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (1, 3), (3, 1), (3, 4), (4, 3)\} \)

Solution: This is not an equivalence relation because it is not transitive. If it were transitive, then, since \((1, 3)\) and \((3, 4)\) are in \(R\), we would need \((1, 4) \in R\). This is not true.

(c) \( \{(x, y) \in X \times X : 4 \text{ divides } x - y\} \)

Solution: This is an equivalence relation with equivalence classes

\[ \{1, 5\}, \{2\}, \{3\}, \{4\} \]
4. (not from the text) Recall that \( \mathbb{N} = \{0, 1, 2, \ldots\} \). Is the relation 

\[ \{(x, y) \in \mathbb{N} \times \mathbb{N} : |x - y| \leq 5\} \]

an equivalence relation?

Solution: No it is not. The relation is reflexive and symmetric, but not transitive. Take, for instance, \( x = 0, y = 1, z = 6 \). Then we have 

\[ (x, y) \in R \text{ and } (y, z) \in R \text{ but } (x, z) \notin R \]

PROBLEM TWO

1. Draw the digraphs for the relations in problem one (previous page), question 3.

(a) 

(b)
2. What must be true of a digraph of a relation in order for the relation to be

(a) **reflexive**?
Solution: The digraph must have the property that each of its points has an arrow (in both directions) from itself to itself.

(b) **symmetric**?
Solution: Each arrow in the digraph must be bi-directional (going in both directions).

(c) **antisymmetric**?
Solution: Each arrow, other than loops from one point to itself, must be uni-directional (going in only one direction).

(d) **transitive**?
Solution: If we imagine traveling along the arrows from point to point, it must be true that whenever we can reach a point \( z \) from a point \( x \) in *two* steps, we can also reach the point \( z \) from the point \( x \) in *one* step.

3. Find the following quantities:

(a) \( 2^{43} \mod 17 \)
Solution:

\[
2^{43} \mod 17 = (2^3)(2^4)^{10} \mod 17 \\
= (8)(16)^{10} \mod 17 \\
= (8)(-1)^{10} \mod 17 \\
= (8)(1) \mod 17
\]

3
(b) $11^{201} \mod 13$

Solution:

$11^{201} \mod 13 = (11)(11^2)^{100} \mod 13$

$= (11)(121)^{100} \mod 13$

$= (11)4^{100} \mod 13$

$= (11)(4^2)^{50} \mod 13$

$= (11)(16)^{50} \mod 13$

$= (11)3^{50} \mod 13$

$= (11)(9)^{3^{48}} \mod 13$

$= (99)(3^3)^{16} \mod 13$

$= (8)(27)^{16} \mod 13$

$= (8)1^{16} \mod 13$

$= 8 \mod 13 = 8$

4. Using a hash table of length 8, find a data set consisting of 8 odd numbers such that, when we insert the data in order (as we did in class) into the hash table, exactly 6 collisions occur.

Solution: For example, using the initial data

$$\{1, 3, 9, 17, 25, 33, 41, 49\}$$

works (prove this to yourself!)
PROBLEM THREE
Suppose that we have two sets $A$ and $B$, represented as lists. At the end of each list is a special character, EOL. For example, $A$ could be the list 1,5,4,6,7,EOL. Consider the following algorithm which intersects the sets $A$ and $B$ and places the result in a set $C$. Assume that $C$ begins as a list whose only element is EOL.

LINE
1  $i = 1$
2  $j = 1$
3  while ($A(i) \neq \text{EOL}$) {
4     if ($A(i) \in B$ and $A(i) \notin C$) {
5         put $A(i)$ into the $j^{th}$ spot of $C$
6         put EOL into the $j + 1^{st}$ spot of $C$
7         $j = j + 1$
8     }
9     $i = i + 1$
10   }

1. **Trace through the algorithm using initial data** $A = \{9, 2, 3, 4\}$, $B = \{3, 1, 5, 2\}$.

   **Solution:**

   

<table>
<thead>
<tr>
<th>$i$</th>
<th>$j$</th>
<th>$C$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>?</td>
<td>EOL</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>EOL</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>2, 3</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>2, 3</td>
</tr>
<tr>
<td>4</td>
<td>3</td>
<td>2, 3</td>
</tr>
<tr>
<td>5</td>
<td>3</td>
<td>2, 3</td>
</tr>
</tbody>
</table>

2. **With the data set above, how many lines in the algorithm are executed?**

   **Solution:** 20

3. **Do you think the algorithm works? Why or why not?**

   **Solution:** I think that it does work, although (a) our initial data included lists which did not have an EOL character (in my analysis in questions 1 and 2, I assumed that the fifth character in each list was EOL, so that the algorithm would stop) and (b) the algorithm does not place an EOL character after the completed list $C$. Notice that the algorithm must only
step through the list $A$ because it is looking for elements in $A \cap B$ and these elements must, in particular, be in $A$. 