1. Suppose the stock price $X_t$ and the process $Y_t$ underlying the stochastic volatility are described by

$$dX_t = \mu X_t dt + \sigma_t X_t dW_t$$

$$\sigma_t = f(Y_t)$$

$$dY_t = \frac{1}{\varepsilon} \mu_Y(t, Y_t) dt + \frac{1}{\sqrt{\varepsilon}} \sigma_Y(t, Y_t) d\tilde{Z}_t$$

where $f$ is a positive function and $\tilde{Z}_t = \rho W_t + \sqrt{1 - \rho^2} Z_t$, with $W_t$ and $Z_t$ being two independent Brownian motions. Derive the PDE that governs the price of a European option with the payoff function $h(x)$ and maturity $T$.

2. For stochastic volatility models,

$$dX_t = \mu X_t dt + \sigma_t X_t dW_t$$

and $\sigma_t = f(Y_t)$ and $Y_t$ is described by a diffusion process. If we assume $\sigma_t$ and $W_t$ are independent, under certain conditions, one can show that the implied volatility curve $I(p, K)$ (for fixed stock price $x$, time $t$ and maturity $T$) is a locally convex function around $K_m = xe^{r(T-t)}$. We will demonstrate a special case of this using the following procedure. Under the assumption that

$$\bar{\sigma}^2 = \frac{1}{T-t} \int_t^T f^2(Y_s) ds$$

is a Bernoulli random variable, i.e.,

$$\bar{\sigma}^2 = \begin{cases} \sigma_1^2 & \text{with probability } p \\
\sigma_2^2 & \text{with probability } 1-p \end{cases}$$

(a) Show that, from the Hull-White pricing formula, we can determine the implied volatility from

$$C_{BS}(K; I(p, K)) = p C_{BS}(K; \sigma_1) + (1 - p) C_{BS}(K; \sigma_2)$$  \hspace{1cm} (1)$$

where $C_{BS}(K; \sigma)$ is the standard Black-Scholes pricing formula for a European call with strike $K$ and volatility $\sigma$ and $I(p, K)$ is the implied volatility.

(b) Define $g(p)$ by

$$g(p) \equiv p \frac{\partial C_{BS}}{\partial K}(\sigma_1) + (1-p) \frac{\partial C_{BS}}{\partial K}(\sigma_2) - \frac{\partial C_{BS}}{\partial K}(I(p, K)),$$

show that

$$\text{sign} \left( \frac{\partial I}{\partial K} \right) = \text{sign}(g(p))$$

and $g(0) = g(1) = 0$ (Note that $\partial C_{BS}/\partial \sigma > 0$)

(c) From Eq. (1), show that

$$C_{BS}(\sigma_1) - C_{BS}(\sigma_2) = \frac{\partial C_{BS}}{\partial \sigma}(I(p, K)) \frac{\partial I}{\partial p}$$

and further show that

$$\frac{d^2 g}{dp^2} = 2 \left( C_{BS}(\sigma_1) - C_{BS}(\sigma_2) \right)^2 \log \left( \frac{xe^{r(T-t)}/K}{(T-t)I^3} \right)$$
(d) By noticing $I > 0$, show that

$$\text{sign} \left( \frac{d^2 g}{dp^2} \right) = \text{sign} \left( \log \left( \frac{xe^{r(T-t)}}{K} \right) \right),$$

and further using (b) above show that the implied volatility $I (K)$ is locally convex around $K_m = xe^{r(T-t)}$, which is the forward price of the stock.

3. Let us generalize the two-state Markov chain. Suppose that, instead of merely jumping between two states, the process $Y_t$ jumps after exponentially holding times to random variables, uniformly distributed between $-1$ and $+1$. We assume that (1) the jump sizes and holding times are independent, so $Y_t$ is a pure jump Markov process in $[-1, +1]$, (2) the mean holding time is $1/\alpha$ (which means that the number of jumps $N_t$ before time $t$ is a Poisson process with intensity $\alpha$, i.e.,

$$P \{N_t = k\} = \frac{(\alpha t)^k}{k!} e^{-\alpha t} \quad \text{for integers } k > 0.$$

(a) For any bounded function $g$ on $(-1, 1)$, show that

$$\mathbb{E} [g (Y_t)] = g (y) e^{-\alpha t} + \left( \int g (z) p (z) dz \right) \alpha te^{-\alpha t} + O (t^2)$$

where $\mathbb{E} [g (Y_t)] = \mathbb{E} [g (Y_t) | N_t = 0] P \{N_t = 0\} + \mathbb{E} [g (Y_t) | N_t \geq 1] P \{N_t \geq 1\}$, and $p (y)$ is the density function for the uniformly distributed jumps, i.e., $p (y) = \frac{1}{2} 1_{(-1,1)} (y)$.

(b) By taking the limit,

$$\lim_{t \to 0^+} \frac{\mathbb{E} [g (Y_t)] - g (y)}{t}$$

show that the infinitesimal generator for this process is

$$\mathcal{L} g (y) = \alpha \int [g (z) - g (y)] p (z) dz$$

(c) Find the invariant distribution $p^*$ for the process $Y_t$.

(d) Defining

$$\langle g \rangle \equiv \int g (z) p^* (z) dz$$

show that

$$\mathbb{E} [g (Y_0) h (Y_t)] = \langle g \rangle \langle h \rangle + e^{-\alpha t} [\langle gh \rangle - \langle g \rangle \langle h \rangle]$$

for any continuous bounded functions $g$ and $h$. Therefore, as $t \to \infty$, $Y_t$ decorrelates from the initial $Y_0$ at the exponential rate $\alpha$.

(e) Find the solution $u$ that satisfies

$$\mathcal{L} u (y) = 0.$$