1. In the lecture, we showed that the solution of

\[
\begin{align*}
  w_t &= w_{xx} 
  \quad \text{for } t > 0 \text{ and } x > 0 \\
  w(x, t = 0) &= 0 \\
  w(x = 0, t) &= \phi(t)
\end{align*}
\]

can be expressed as

\[
w(x, t) = \int_0^t \frac{\partial}{\partial y} G(x, 0, t - s) \phi(s) \, ds
\]  

(1)

where \( G(x, y, s) \) is the probability that a random walker (i.e., \( dy = \sqrt{2}dW \)), starting at \( x \) at time 0, reaches \( y \) at time \( s \) without first hitting the boundary at 0. The following line of reasoning provides a different way of looking at this solution:

(a) Express, in terms of \( G \), the probability that the random walker, starting at \( x \) at time 0, hits the boundary before time \( t \). Differentiate in \( t \) to obtain the probability that it hits the boundary at time \( t \) (This is known as the first passage time density).

(b) Use the forward Kolmogorov equation and integration by parts to show that the first passage time density is \( \frac{\partial}{\partial y} G(x, 0, t) \).

(c) Deduce the formula (1).

2. For the process \( dy = \mu dt + dW \) with an absorbing boundary at \( y = 0 \),

(a) suppose the process starts at \( x > 0 \) at time 0, let \( G(x, y, t) \) be the probability that the random walker is at position \( y \) at time \( t \) without first hitting the boundary. Show that

\[
G(x, y, t) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{|y - x - \mu t|^2}{2t}} - \frac{1}{\sqrt{2\pi t}} e^{-2\mu x} e^{-\frac{|y + x - \mu t|^2}{2t}}
\]

i.e., to verify that this \( G \) solves the relevant forward Kolmogorov equation with appropriate boundary and initial conditions.

(b) Show that the first passage time density is

\[
\frac{1}{2} \frac{\partial}{\partial y} G(x, 0, t) = \frac{x}{t \sqrt{2\pi t}} e^{-\frac{|x + \mu t|^2}{2t}}
\]

3. Consider the heat equation \( u_t - u_{xx} = 0 \) in one space dimension, with discontinuous initial data

\[
u(x, 0) = \begin{cases} 
0 & \text{if } x < 0 \\
1 & \text{if } x > 0
\end{cases}
\]

(a) Show that

\[
u(x, t) = N\left(\frac{x}{\sqrt{4t}}\right)
\]

where

\[
N(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{y^2}{2}} \, dy
\]

i.e., the cumulative normal distribution.

(b) What is \( \max_x u_x(x, t) \) as a function of time \( t \)? Where is it achieved? What is \( \min_x u_x(x, t) \)? Sketch the graph of \( u_x \) as a function of \( x \) at a given time \( t > 0 \).
(c) Show that
\[ v(x, t) = \int_{-\infty}^{x} u(z, t) \, dz \]
solves
\[ \begin{cases} v_t - v_{xx} = 0 \\ v(x, 0) = \max \{x, 0\}. \end{cases} \]
Discuss the qualitative behavior of \( v(x, t) \) as a function of \( x \) for a given \( t \): how rapidly does \( v \) tend to 0 as \( x \to -\infty \)? What is the behavior of \( v \) as \( x \to \infty \)? What is the value of \( v(0, t) \)? Sketch the graph of \( v(x, t) \) as a function of \( x \) for given \( t > 0 \).

4. Give “solution formulas” for the following initial-boundary-value problems for the heat equation
\[ w_t - w_{xx} = 0 \text{ for } t > 0, \text{ and } x > 0 \]
with the following initial and boundary conditions:

(a) \( w_1(x = 0, t) = 0 \) and \( w_1(x, t = 0) = 1 \). Express the solution in terms of the cumulative normal distribution \( N(\cdot) \).

(b) \( w_2(x = 0, t) = 0 \) and \( w_2(x, t = 0) = (x - K)_+ \) with \( K > 0 \). Express your solution in terms of the function \( v(x, t) \) defined in Problem 3(c)

(c) \( w_3(x = 0, t) = 0 \) and \( w_3(x, t = 0) = (x - K)_+ \) with \( K < 0 \)

(d) \( w_4(x = 0, t) = 1 \) and \( w_4(x, t = 0) = 0 \).

Interpret each as the expected payoff of a suitable barrier-type option, whose underlying is described by \( dy = \sqrt{2} \, dW \) with initial condition \( y(0) = x \) and an absorbing barrier at 0.