Lecture 5

Infinitesimal Generators

First, consider time-homogeneous diffusion process

\[ dX_t = \mu(X_t) dt + \sigma(X_t) dW_t \]

\( g(x) \) — twice continuously differentiable with bounded derivatives.

\[ \sum g(x) = \frac{1}{2} \sigma(x)^2 \frac{d^2}{dx^2} g(x) + \mu(x) \frac{d}{dx} g(x) \]

\[ L^g(X_t) = \frac{1}{2} \sigma(x)^2 \frac{d^2}{dx^2} g(x) + \mu(x) \frac{d}{dx} g(x) \]

\[ \Rightarrow \mathbf{a} \text{ associated} \]

\[ \text{martingale: } M_t = g(X_t) - \int_0^t L^g(X_s) ds \]

\[ \Rightarrow \text{ a martingale. (The drift term is removed)} \]

\[ \mathbb{E} [X_0 = x] \]

\[ \mathbb{E} \left[ E \left[ g(X_t) - \int_0^t L^g(X_s) ds \right] \right] = M_0 = g(x) \]

\[ \Rightarrow \mathbb{E} \left[ g(X_t) \right] = g(x) + \mathbb{E} \left[ \int_0^t L^g(X_s) ds \right] \]
\[ \frac{d}{dt} \mathbb{E}[g(X_t)] \bigg|_{t=0} = \lim_{t \to 0} \frac{\mathbb{E}[g(X_t) - g(x)]}{t} \]

\[ = \lim_{t \to 0} \mathbb{E} \left[ \frac{1}{t} \int_0^t g(X_s) \, ds \right] \]

\[ = \log(e) \quad \text{[\textit{L}^\infty \text{-} \text{Lebesgue dominated convergence theorem}]} \]

L-infinite derivative

generator of
the Markov process

\[ dX_t = \mu(t, X_t) \, dt + \sigma(t, X_t) \, dW_t. \]

For nonhomogeneous diffusion, i.e.

\[ dX_t = \mu(t, X_t) \, dt + \sigma(t, X_t) \, dW_t \]

For \( g \) smooth and bounded, the corresponding generalization is

\[ M_t = g(t, X_t) - \int_0^t \left( \frac{\partial}{\partial X} g + L g \right)(s, X_s) \, ds \]

is a martingale. (Use Itô lemma)

\[ \Delta_t = \frac{1}{2} \sigma^2(t, X_t) \frac{\partial^2}{\partial x^2} + \mu(t, x) \frac{\partial}{\partial x} \]
\[ M_t = e^{-rt} g(t, x_t) - \int_0^t e^{-rs} \left( \frac{\partial g}{\partial x} + \log f \cdot \frac{\partial g}{\partial f} - rg \right) (s, x_s) ds \]

**Stochastic Volatility Models**

In the Black-Scholes theory,

- **Stock** — continuous and jumps (diffusion)
- **Option** — can be hedged continuously w/o transaction costs.
- **Constant volatility.**

Q: What if the volatility is not constant?

\[ \text{Stochastic volatility} \Rightarrow \text{Incomplete market.} \]

Implied Volatility and the smile curve

Q: What is the implied volatility?
Recall Black-Scholes:

$$ L_{BS}(0) = \frac{\partial}{\partial x} + \frac{1}{2} \sigma^2 x^2 \frac{\partial^2}{\partial x^2} + r \frac{\partial}{\partial x} - r $$

**European Call option:**

$$ P(c, t; x) = (x - K)^+ $$

Black-Scholes formula (Prices at time $t$ and $X_t = x$)

$$ C_{BS}(t, x) = x N(d_1) - K e^{-(r + \frac{1}{2} \sigma^2)(T-t)} N(d_2) $$

$$ d_1 = \frac{\log \frac{x}{K} + (r + \frac{1}{2} \sigma^2)(T-t)}{\sigma \sqrt{T-t}} $$

$$ d_2 = d_1 - \sigma \sqrt{T-t} $$

$$ T-t = \text{time to maturity} $$

$$ N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2} dy $$

$$ r = \text{interest rate} $$

Given an observed European call option price $C_{obs}$ for a contract with strike $K$ and maturity $T$,

prove:

$$ C_{BS}(t, x; K, T; \sigma) = C_{obs} $$

**I = \text{(Implied)}**

**implied volatility**

Q: Does such implied volatility exist?

$$ \frac{\partial C_{BS}}{\partial \sigma} = 0 \sigma e^{-d_1/2} \frac{1}{\sqrt{2\pi T-t}} > 0 $$

$$ C_{BS}(\sigma) \text{ is a monotonic fn of } \sigma $$
as long as \( C_{\text{obs}} > C_{BS}(t, x; K, T, \sigma = 0) \)

\[ \implies \text{a unique implied vol } I > 0. \]

NB: Q: Would the implied \( \sigma \) be different from put price of the same strike and maturity?

No! \{ \text{the put-call parity:} \}

\[ C_{BS}(t, x; K, T, \sigma) - P_{BS}(t, x; K, T, \sigma) = x - Ke^{-r(T-t)} \]

From \( C_{\text{obs}} \implies \text{implied vol } I = I(t, x; K, T) \)

NB: if \( C_{\text{obs}} = C_{BS}(t, x; K, T, \sigma) \) for some \( \sigma \),

then \( I = \sigma \)

Smile Effect:

- European option for all \( t \),
- current stock price \( x = 100 \)

\[ \begin{array}{c}
0.26 \quad 0.25 \\
0.24 \quad 0.23 \\
0.22 \quad 0.21 \\
0.2 \quad \vdots \\
0.15 \quad 0.12 \\
0.11 \quad 0.08 \\
0.05 \quad 0.02 \\
0.01 \quad 0.00 \end{array} \]

Smile curve \( I = I(K) \) \( \leftarrow \) From the B-S theory, it should be a constant.

\[ \text{Min of } I \sim \text{near the money} \]

i.e. \( 95\% < \frac{K}{x} \leq 105\% \)
Volatility skew

Skewness

The current index value

$X = 1411.71$

$T - t = 2$ months

In general, implied vol. surface

The option price $P(K, T)$

Q: How can we extract the local volatility surface from S
deep call option prices?

Assume $dx = r x dt + \sigma(x, t) \, dw$

$\sigma$ is a function of $x$:

$\text{the implied deterministic volatility}$
\[ P(K, T) = e^{-r(T-t)} \int_0^{\infty} (x-K)^+ \rho(x, T \mid x_0, t_0) \, dx \]

Transition probability density fn.

\[ P(K, T) = e^{-r(T-t)} \int_K^{\infty} (x-K) \rho(x, T \mid x_0, t_0) \, dx \]

\[ \frac{\partial \rho}{\partial K} = -e^{-r(T-t)} \int_K^{\infty} \rho(x, T \mid x_0, t_0) \, dx \]

\[ \frac{\partial^2 \rho}{\partial K^2} = e^{-r(T-t)} \rho(x, T \mid x_0, t_0) \]

\[ \therefore \rho(x, T \mid x_0, t_0) = e^{r(T-t)} \frac{\partial^2 \rho}{\partial K^2} \]

i.e. From prices \Rightarrow extraction of

the transition pdf. (a risk neutral pdf)

Recall the forward Kolmogorov eqn for \( \rho \) is

\[ \frac{\partial \rho}{\partial t} = \frac{1}{2} \frac{\partial^2}{\partial x^2} \left( \sigma^2 x^2 \rho \right) - \frac{\partial}{\partial x} (r x \rho) \]

\[ \text{Not determined yet} \quad \sigma = \sigma(x, T) \]

\[ \hat{\sigma}(x) = \sigma(x, T) \bigg|_{t=0}^T \]
\[ P(K,t) = e^{-r(T-t)} \int_K^{\infty} (x-K) p \, dx \]

\[ \frac{\partial P}{\partial T} = -rP + e^{-r(T-t)} \int_K^{\infty} (x-K) \left[ \frac{1}{2} \frac{\partial^2}{\partial x^2} (x^2 p) - \frac{1}{2} (x^2 p) \right] \, dx \]

Integrate by parts twice, \( \text{NB.} \frac{\partial}{\partial x} \rightarrow 0 \text{ as } x \rightarrow \infty \)

\[ = -rP + e^{-r(T-t)} \left( \frac{1}{2} x^2 p \right) \bigg|_K^{\infty} + r e^{-r(T-t)} \int_K^{\infty} x p \, dx \]

The last term:

\[ \int_K^{\infty} x p \, dx = \int_K^{\infty} (x-K+K) p \, dx = \int_K^{\infty} (x-K)p \, dx + K \int_K^{\infty} p \, dx \]

\[ = e^{-r(T-t)} P \]

\[ \frac{\partial P}{\partial K} = -e^{-r(T-t)} \int_K^{\infty} p \, dx \]

\[ \therefore \frac{\partial^2 P}{\partial T \partial K} = \frac{1}{2} \left[ \frac{\partial P}{\partial T} + rK \frac{\partial P}{\partial K} \right] \]

\[ = \frac{1}{2} \frac{K^2 \frac{\partial P}{\partial K}}{K} \]

\[ \therefore \sigma^2(K,t) = \left[ \frac{1}{2} \frac{K^2 \frac{\partial P}{\partial K}}{K^2} \right]^{\frac{1}{2}} \]

\[ \text{d.e. Given a set of prices } \Rightarrow \text{ Vol. surface } \delta(x,t) \]
NB: Interpolation (numerical) issues:

--- ill-posedness / regularization

NB: \( C_{t, T} \) is not a prediction about future volatility,

For example, a few days later, \( C_{t, T} \) may need re-fit.

But: Often today’s options are priced using yesterday’s option data.

NB: Use \( C_{t, T} \) to price non-traded (e.g. exotic) contracts
to be consistent with all instruments

i.e. price exotic consistent with vanillas

with the same volatility structure and

simultaneously hedged with these vanillas.

\[ \Rightarrow \text{Reduction of exposure to model errors} \]

Q: How to interpret the smile curve?

First, some bounds on the permissible slope of \( ICK \)

NB: \( C_{t, K} \) \( \uparrow \) as \( K \uparrow \) (otherwise, there is

\( C_{t, K_1} \) < \( C_{t, K_2} \) \( K_1 < K_2 \) potentially arbitrage opportunity)

buy \( C_{t, K_1} \), sell \( C_{t, K_2} \)
\[ \frac{\partial C^{\text{obs}}}{\partial K} = \frac{\partial C^{\text{bs}}}{\partial K} + \frac{\partial C^{\text{bs}}}{\partial \sigma} \frac{\theta_1}{\partial K} \leq 0 \]

\[ \frac{\partial_1}{\partial K} \leq -\frac{\left( \frac{\partial C^{\text{bs}}}{\partial K} \right)}{\left( \frac{\partial C^{\text{bs}}}{\partial \sigma} \right)} \quad (\text{Ns: } \frac{\partial C^{\text{bs}}}{\partial \sigma} > 0) \]

From put-call parity,

\[ \frac{\partial_1}{\partial K} \geq -\frac{\left( \frac{\partial \text{P}^{\text{bs}}}{\partial K} \right)}{\left( \frac{\partial \text{P}^{\text{bs}}}{\partial \sigma} \right)} \]

Using the BS formula, \( \Rightarrow \) Bounds:

\[ -\frac{1}{2 \sqrt{2\pi}} \left( 1 - N(d_2) \right) e^{-r(T-t) + \frac{\sigma^2}{2}} \leq \frac{\partial_1}{\partial K} \leq \frac{1}{2 \sqrt{2\pi}} N(d_2) e^{-r(T-t) + \frac{\sigma^2}{2}} \]

where

\[ d_1 = \frac{1}{\sigma \sqrt{T-t}} \left[ \log \frac{S}{K} + (r + \frac{1}{2} \sigma^2)(T-t) \right] \]

\[ d_2 = (d_1 - \sigma \sqrt{T-t}) \]

i.e. the slope cannot be too positive or too negative.

\textbf{Ns: Empirical observation:} Prices tend to go down, when the volatility goes up \( \Rightarrow \) negative, and vice versa.

\textbf{Ns: Smiles \Rightarrow } The Black-Scholes theory needs modification
Implied Deterministic Volatility

$\sigma = \sigma(t, X_t)$

and the stock price $dX_t = \mu X_t dt + \sigma(t, X_t) X_t dW_t$

then, the generalized Black-Scholes PDE:

$$\frac{\partial P}{\partial t} + \frac{1}{2} \sigma^2(t, X_t) X_t^2 \frac{\partial^2 P}{\partial X_t^2} + r X_t \frac{\partial P}{\partial X_t} - r P = 0$$

NB: the derivation is identical to that of Constant $\sigma$.

2. Hedging ratio $\Delta = \frac{\partial P}{\partial X_t}$ (the argument is the same as under Constant $\sigma$)

3. The market is still complete

$: the randomness of the volatility is a factor of the randomness of the lognormal model

$: risk-neutral measure $P^*$ under which the underlying is a geometric Brownian motion with drift $r$, i.e.

$$dX_t = r X_t dt + \sigma(t, X_t) X_t dW_t^*$$

$W_t^*$ is a $P^*$-Brownian motion

NB: this way we construct the volatility surface $\sigma(t, X_t)$.
Recall. Special case. \( \sigma_t(x) = \sigma_t \) — time-dependent vol.

\[
\bar{\sigma}^2 = \frac{1}{T-t} \int_t^T \sigma^2(s) ds \quad \text{Note time average of } \sigma^2 \\
\text{Not } \sigma
\]

No. 1° the BS formula holds with volatility parameter \( \sqrt{\bar{\sigma}^2} \)

i.e. the root-mean-square volatility

2° For fixed \( t, T \), Options priced using the BS formula with time-averaged volatility don’t exhibit

smile across strike prices

3° But there is change of implied volatility with time to maturity

\[
\bar{\sigma}^2 = \frac{1}{T-t} \int_t^T \sigma^2(s) ds \quad \text{changes as } T \text{ changes}
\]

\( \Rightarrow \) The volatility surface changes along \( T \)-axis
Stochastic Volatility Models

In general, the asset $X_t$:
$$dX_t = \mu X_t dt + \sigma_t X_t dW_t$$

$\sigma_t$ — volatility process
- diffusion process
- jump process

A Markov chain

Note 1: $\sigma_t > 0$ to be volatility.
Note 2: The implied deterministic volatility is perfectly correlated with $W_t$: $\sigma = \sigma(t, X_t)$

$\sigma_t$ can have its own random component

Mean-reverting stochastic volatility models

$\sigma_t = f(\gamma_t)$ if $f$ is some positive fn

the rate of mean reversion

$$dY_t = \alpha (\gamma - Y_t) dt + \cdots d\hat{\xi}_t$$

$m$: the long-run mean level of $Y_t$.

Brownian motion: $\hat{\xi}_t$ and $W_t$ can be correlated.
Example. Ornstein-Uhlenbeck process:

\[ dY_t = \alpha (m - Y_t) \, dt + \beta \, dZ_t \]

\[ Y_0 = \frac{m + (\gamma - m) e^{-\alpha t} + \beta \int_0^t e^{-\alpha (t-s)} \, ds}{1 - e^{-\alpha t}} \]

\[ Y_t \sim N \left[ m + (\gamma - m) e^{-\alpha t}, \frac{\beta^2}{2\alpha} (1 - e^{-2\alpha t}) \right] \]

\[ \forall \alpha, \; \text{invariant distribution} \]

\[ \text{No: no } \gamma \text{-dependence} \]

\[ Z_t \text{ and } W_t: \]

\[ d\mathbb{E}(W_t Z_t) = \rho \, dt, \quad -1 \leq \rho \leq 1 \]

\[ d(Z_t W_t) = Z_t \, dW_t + W_t \, dZ_t + d\mathbb{E}(W_t Z_t) \]

\[ Z_t = \rho W_t + \sqrt{1 - \rho^2} \, Z_t \quad \text{Wt, Zt are independent Brownian motions} \]

\[ \text{No: } \rho < 0 \text{ from financial data} \]

more general \( \rho = \rho(t) \) depending on time
Feller process (CIR)

\[ dy_t = (\kappa - \mu) y_t dt + \sqrt{\chi} dW_t \]

Models of Volatility:

- Hull-White: \( p = 0, f(y) = \frac{1}{y} \), Lognormal (not-mean-reverting)
- Scott: \( p = 0, f(y) = e^y \), Mean-reverting OU
- Stein-Stein: \( p = 0, f(y) = |y| \)
- Ball-Roma: \( p = 0, f(y) = \sqrt{y} \), CIR
- Heston: \( p > 0, f(y) = \sqrt{y} \), CIR

Qualitative Effects on the Stock Price Distribution

Under stochastic volatility:
- Exponential OU: \( f(y) = e^y \), \( Y_t \) - a mean-reverting OU.

\( p < 0 \Rightarrow \text{asymmetric tails} \)

Left - fatter.

Figure 2.3. Density functions for the stock price (under the subjective measure) in six months when the present value is 100. The solid line is estimated from simulation of an \( \expOu \) stochastic volatility model with \( \kappa = 1, \chi = \sqrt{2} \), long-run average volatility \( \delta = 0.1 \), and negative correlation \( \rho = -0.2 \). The dotted line is the corresponding Black–Scholes lognormal density function with volatility \( \delta \). The mean growth rate of the stock is \( \mu = 0.15 \).
Derivative Pricing with Stochastic Volatility

\[ \text{d}Y_t = \mu X_t \, \text{d}t + \sigma_t X_t \, \text{d}W_t \]

\[ \sigma_t = f(Y_t) \]

\[ \text{d}Y_t = \alpha (\mu - Y_t) \, \text{d}t + \beta \text{d}Z_t \quad \text{a mean reverting Ornstein-Uhlenbeck process} \]

- Recall the BS theory, the uncertainty introduced by \( \text{d}W \) can be hedged away using the underlying asset.

No arbitrage principle determines the price of the option.

- Note: there is an additional uncertainty introduced by \( \text{d}Z_t \).

**Q:** How can we price?

Use the underlying and an option with different maturity.

\[ p^{(t)}(t, x, y) \quad \text{the price of a European derivative with} \]

- maturity \( T_1 \) and payoff function \( h(t, X_t) \).

\[ p^{(t)}(t, x, y) \quad \text{the price of another European option with} \]

- the same payoff \( h \) but different maturity, \( T_2 > T_1 > t. \)
In addition, we have risky asset $X_t$, and riskless asset $\beta_t = e^{rt}$.

Replication:

$$P^{(T)}(T_1, X_{T_1}, Y_{T_1}) = a_{T_1} X_{T_1} + b_{T_1} \beta_{T_1} + c_{T_1} P^{(2)}(T_1, X_{T_1}, Y_{T_1})$$

Self-financing:

$$dP^{(1)}(t, X_t, Y_t) = a_t dX_t + b_t e^{rt} dt + c_t dP^{(2)}(t, X_t, Y_t)$$

No arbitrage $\Rightarrow$

$$P^{(1)}(t, X_t, Y_t) = a_t X_t + b_t e^{rt} + c_t P^{(2)}(t, X_t, Y_t) \quad \forall t < T_1$$

Does there exist such processes $a_t, b_t, c_t$?

Recall Ito's Lemma

$$dP(t, X_t, Y_t) = \frac{\partial P}{\partial t} dt + \frac{\partial P}{\partial x} dX_t + \frac{\partial P}{\partial y} dY_t$$

$$+ \frac{1}{2} \left( \frac{\partial^2 P}{\partial x^2} d\langle X \rangle_t + 2 \frac{\partial^2 P}{\partial x \partial y} d\langle XY \rangle_t + \frac{\partial^2 P}{\partial y^2} d\langle Y \rangle_t \right)$$

$$\langle X_t \rangle = \int_0^t \sigma_x^2 ds \quad \langle Y_t \rangle = \int_0^t \sigma_y^2 ds$$

$$d\langle XY \rangle_t = Cov (dX_t, dY_t)$$
For our case, we assume $dW+dt = \delta dt$ i.e. conditional uncertainty of stock and the uncertainty of vol.

LHS of Eq. (*)

\[
d\Phi(t, x, y) = \left[ \frac{\partial \rho^{(0)}}{\partial t} + \frac{1}{2} \left( \frac{\partial \rho^{(0)}}{\partial x} \right)^2 + \frac{1}{2} \left( \frac{\partial \rho^{(0)}}{\partial y} \right)^2 + \rho^{(0)} \left( \frac{\partial f}{\partial x} \frac{\partial \rho^{(0)}}{\partial x} \right) \right] dt
\]

\[+ \frac{\partial \rho^{(0)}}{\partial x} dx + \frac{\partial \rho^{(0)}}{\partial y} dy
\]

RHS of Eq. (*) (Apply the Ito Lemma to)

\[= \left[ ct \left( \frac{\partial}{\partial t} + M_t \right) \rho^{(0)} + b_t \rho^{(0)} e^{rt} \right] dt
\]

\[+ \left( ct + ct \frac{\partial \rho^{(0)}}{\partial x} \right) dx + c_t \frac{\partial \rho^{(0)}}{\partial y} dy
\]

NB: Equating $d\Phi_t$ terms (i.e. equating $d\tilde{\Phi}_t$ terms) $\Rightarrow$

\[\frac{\partial \rho^{(0)}}{\partial y} = ct \frac{\partial \rho^{(0)}}{\partial y}
\]

i.e. $ct = \frac{\partial \rho^{(0)}}{\partial y}$

Similarly, from $d\Phi_x$ terms (i.e. $d\tilde{\Phi}_x$ terms) $\Rightarrow$

\[\frac{\partial \rho^{(0)}}{\partial x} = ct - c_t \frac{\partial \rho^{(0)}}{\partial x}
\]

NB. Eq. (*) $\Rightarrow$ $b_t = \left( \rho^{(0)} - ax - c_t \rho^{(0)} \right) e^{-t}$
Substituting all these \( (b_t = \cdots, b_t = \cdots, b_t = \cdots) \) into the terms \( \Rightarrow \)

\[
\frac{\partial p_{c0}}{\partial t} + M_1 p_{c0} = \left( \frac{\partial p_{c1}}{\partial t} + \frac{\partial p_{c2}}{\partial t} \right) p_{c0} + \left( p_{c1} - \left( \frac{\partial p_{c1}}{\partial x} - \frac{\partial p_{c2}}{\partial x} \right) \right) x - \frac{\partial p_{c3}}{\partial y} p_{c3} \right) e^{-r} \cdot r \cdot e^{t}
\]

(A little rearrangement, \( \Rightarrow \))

\[
\left( \frac{\partial p_{c1}}{\partial y} \right)^{-1} M_2 p_{c1}(t, x, y) = \left( \frac{\partial p_{c3}}{\partial y} \right)^{-1} M_2 p_{c3}(t, x, y)
\]

\( M_2 = \frac{\partial}{\partial t} + M_1 + \left( t - \frac{\partial^2}{\partial y^2} - r \right) \)

depending on \( T_2 \) only

\( M_2 p_{c1} \) and \( p_{c3} \) can be arbitrary functions (i.e., different payoff functions).

\therefore it must be equal to a function that is independent of maturity, i.e.,

(in general, contract type)

\[
\frac{1}{\lambda^2} \left[ \lambda \left( \lambda^2 - \frac{1}{2} \lambda \mu + \frac{1}{2} \lambda \mu^2 \right) + \lambda \left( \beta \chi \right) + \frac{1}{2} \beta^2 \lambda \mu + \frac{1}{2} \beta^2 \lambda \mu^2 + r x \lambda \mu - r \lambda \right]
\]

\[
\left[ \lambda (\lambda^2 - \beta \wedge t, x, y) \right]
\]

the form is explained later

\[
\Lambda = \Psi \left( \frac{\mu - r}{\psi \psi} \right) + \Psi (x, y) \sqrt{1 - \rho^2}
\]

Why \( \sqrt{1 - \rho^2} \)?

later!

Final condition \( p_{c3}(t, x, y) = h(y) \)

\( \Rightarrow y \in (-\infty, +\infty) \)
Rewrite the above equation:

\[ \left( \frac{\partial \Phi}{\partial t} + \frac{1}{2} \text{Tr} \left( \frac{\partial^2 \Phi}{\partial x^2} + \frac{1}{2} \text{Tr} \left( \frac{\partial^2 \Phi}{\partial x \partial y} - I \right) \right) \right) \frac{\partial^2 \Phi}{\partial x \partial y} + \frac{\partial \Phi}{\partial x} \frac{\partial \Phi}{\partial y} = \Phi \]

\[ \text{correlation} \]

\[ \text{Low} \]

\[ \text{market price of volatility risk} \]

\[ \text{premium} \]

\[ \text{generator of diffusion process} \]

\[ \{ d \Phi, dW \} = \text{the risk premium factor from } d \Phi. \]

\[ \{ d \Phi, dW \} = \text{a perfect correlation as } dW. \]

\[ \text{there is no such contribution } \Phi \text{ if } 1 - P^2 = 0 \]

\[ \text{why our particular form of } \{ d \Phi, dW \} = \Phi \text{?} \]

\[ \Phi \text{norm: } d \Phi_t = \Phi \cdot dW + \sqrt{\Phi} \cdot dZ \]

\[ \frac{\partial \Phi}{\partial t} + \frac{1}{2} \left( \frac{\partial^2 \Phi}{\partial x^2} + \frac{1}{2} \text{Tr} \left( \frac{\partial^2 \Phi}{\partial x \partial y} - I \right) \right) \]

\[ + \left( \frac{\partial \Phi}{\partial x} \cdot \frac{\partial \Phi}{\partial x} + \frac{1}{2} \text{Tr} \left( \frac{\partial^2 \Phi}{\partial x \partial y} \right) \right) dW + \beta \left( \frac{\partial \Phi}{\partial x} \cdot \frac{\partial \Phi}{\partial y} + \frac{\partial ^2 \Phi}{\partial x \partial y} \right) dW + \mu \frac{\partial \Phi}{\partial x} dW \]

\[ \text{the difference in } \mu \frac{\partial \Phi}{\partial x} + \frac{\partial \Phi}{\partial x} \frac{\partial \Phi}{\partial y} \]

\[ \text{NB: } \Phi \text{ satisfies } \left( \frac{\partial^2 \Phi}{\partial x^2} + \text{Tr} \left( \frac{\partial^2 \Phi}{\partial x \partial y} \right) \right) \Phi = \beta \frac{\partial \Phi}{\partial y} \]

\[ \text{NB: } \lambda = \rho \frac{\mu - \gamma}{\gamma \rho} + \text{vol x vol } \]
\[
C(t, x, y) = \left[ \frac{\mu - r}{\sigma^2} \right] \left( x \phi \frac{\partial}{\partial x} + \beta \phi \frac{\partial}{\partial y} \right) + rP + \gamma \beta \sqrt{1-p^2} \sigma \phi \frac{\partial}{\partial y} dt + \left( x \phi \frac{\partial}{\partial x} + \beta \phi \frac{\partial}{\partial y} \right) dw_t + \beta \sqrt{1-p^2} \sigma \phi \frac{\partial}{\partial y} dw_t
\]

**NB: Market Price of Volatility Risk**

- \( \sigma = \frac{\partial C(t, x, y)}{\partial y} \) is not observed in the process of \( X_t, Y_t \)
  - it can be only seen from the derivative prices \( P \).

- Economic methods (maximum likelihood, moment methods) to find, \( \alpha, \beta, \gamma, \sigma, \varphi \) (after choosing an \( f(t, y) \)).

  Then use derivative data to estimate \( \sigma \). (say, annual)

- \( \sigma \) in the model

- Cross-sectional fitting

\[
\min_{\alpha, \beta, \gamma, \sigma} \sum \left( \text{Call of } C(t, x, y) \right)^2
\]

-庠: model predicted call option price

\[
\Rightarrow \text{This method: computationally very expensive.}
\]
Special Case: \((p = 0)\)

No. In equity markets, \(p < 0\)

Foreign exchange data \(p > 0\).

**Hull-White Model**

\[
\begin{align*}
\frac{dx}{dt} &= \mu x dt + \sigma x dW \\
\frac{d\sigma}{dt} &= \mu \sigma dt + \nu \sigma dZ \\
\mu\sigma &= \gamma(x, \sigma, t), \quad \nu\sigma = \sigma(t, x, \sigma, t) \\
\end{align*}
\]

- e.g. mean-reverting
- volatility \(\sigma \)

\[
\frac{dW}{dZ} = \rho dt
\]

A: \(f(z) = \eta\)

\[
\frac{\partial^2 p}{\partial z^2} + \frac{\partial p}{\partial z} + \rho \sigma^2 \frac{\partial^2 p}{\partial z \partial \sigma} + \frac{1}{2} \sigma^2 \frac{\partial^2 p}{\partial \sigma^2} + \rho \sigma \frac{\partial p}{\partial \sigma} + \left(\mu - \frac{1}{2}\sigma^2\right) \frac{\partial p}{\partial x} + \rho \sigma = 0
\]

**Assumptions**

1. no correlation \((x, \sigma)\) stock and volatility
2. drift-neutral dynamics, i.e. \(\lambda = 0\)
3. \(\mu, \sigma\) independent of stock \(x\).

**Def.** Hazard variable

\[
\nu = \bar{\sigma}^2 = \frac{1}{T-t} \int_t^T \sigma_t^2 dt \quad \text{averaged volatility}
\]

then, the option value \(\text{BS formula for vol } = \nu\)

\[
P(x_t, \sigma_t^2) = \int \mathcal{C}(\sigma_t) \mathcal{P}(\sigma_t | \sigma_t^2) d\sigma_t
\]

\(\text{Conditioned distribution of } \sigma \text{ given } \sigma_t \text{ at time } t.\)
Intuitive understanding:

\[ P(X_t, \sigma_t^2, t) = e^{-(t-t)} \int \phi(x_t) \Phi(X_t | X_t, \sigma_t^2) \, dx_t \]

Conditional distribution of \( X_t \) given \( X_t, \sigma_t^2 \) at \( t \).

\( p(x_t | x_t, \sigma_t^2) \) depends on the process of \( x, \sigma_t \).

Recall \( p(x | y) = \int g(x | z) h(z | y) \, dz \) for random variables \( p, g, h \) — conditional pdf.

\[ p(x_t | \sigma_t^2) = \int g(x_t | v) h(v | \sigma_t^2) \, dv \]

\[ \Phi(X_t, \sigma_t^2, t) = e^{-(t-t)} \int \phi(x_t) \Phi(X_t | X_t, \sigma_t^2) \, dx_t \, dv \]

\[ = \int e^{-(t-t)} \int \phi(x_t) \Phi(x_t | v) \, dx_t \, h(v | \sigma_t^2) \, dv \]

i.e. \( \sigma \) in Black-Scholes price of volatility.

This is true when \( \rho = 0, \mu, \nu, \sigma \) — independent of \( x_t \).

Lemma: Suppose that, in a risk-neutral world,

\[ dx_t = \mu_x \sigma_t^2 dt + \sigma_t^2 x_t \, dW_t \]

\[ d\sigma_t = \mu_{\sigma_t} \sigma_t^2 dt + \sigma_t \sigma_t \, dZ_t \]
assume $r$ constant, $\mu_t, \sigma_t$ independent of $X_t$

$dW, dZ$ independent Wiener processes.

Then the distribution of $\log \frac{X_t}{x}$ (when $X_0 = x$) conditional on $V$

is $N(\gamma, \frac{\sigma_t^2}{2})$ ($\gamma = \frac{1}{t} \int_0^T \mu_s \sigma_s^2 \, ds$)

NB: if $X_t$ and $\sigma_t$ are correlated, the lemma does not hold.

Financial implication:

NB: $P = E[c(w)]$

1. if $CD_0$ is convex, i.e.

$E[c(c)] > c(E[1])$

2. if $CD_0$ is concave

$E[c(c)] < c(E[1])$

NB: BSpline for a cell is convex for small $v$

(as a function) concave for high values of $v$
To determine when BS price is too high or too low, examine the curvature of $C''(X)$

\[ C''(X) = \frac{\sqrt{T-t}}{4V^{1/2}} N'(d_1) (d_{1d2} - 1) \]

\[ d_{1d2} = \frac{1}{\sigma \sqrt{T-t}} \left( \log \frac{X}{K} + \left( r + \frac{\sigma^2}{2} \right) (T-t) \right) \]

The sign of $C''(X)$ is determined by the sign of $(d_{1d2} - 1)$

The inflection pt: \( d_{1d2} = 1 \)

i.e. $V = V_{\text{critical}} = \frac{2}{T-t} \left[ N(1) + \left( \log \frac{X}{K} + r(T-t) \right)^2 - 1 \right]$.

When $V < V_{\text{critical}}$ \( C'' > 0 \) \( C \) — Convex in $V$

When $V > V_{\text{critical}}$ \( C'' < 0 \) \( C \) — Concave in $V$

At $V = V_{\text{critical}}$:

\[ X = Ke^{-r(T-t)} \]

\[ V > 0 \text{ (By default)} \]

\[ \Rightarrow C \text{ — Concave always} \]

\[ \Rightarrow \text{the actual option price} < \text{the BS price.} \]

\[ \text{At large } |\log \frac{X}{K}| \text{ i.e. } X \text{ — very small or very large} \]

\[ \text{deep out of money} \quad \text{deep in the money} \]

As $|\log \frac{X}{K}| \to \infty$, $V_{\text{critical}}$ becomes arbitrarily large.
\[ C \rightarrow \text{Convex} \]

\[ \Rightarrow \text{the actual price (i.e. } E[c^1]) \rightarrow \text{the BS price} \]

This picture is consistent with

\[ \text{N.B.: high price } \rightarrow \text{high implied vol.} \]

Summary

Features of the stochastic volatility approach.

1°. more realistic return distributions,
   - tails are fatter than lognormal
   - asymmetry of the distribution with noise sources correlated.

2°. Smile effects in option prices in stochastic volatility models. (The correlation controls skewness)

\[ dX_t = \pi_t dt + X_t \sigma_t \sqrt{1-r^2} dW_t + \rho dZ_t \]
\[ dY_t = \left[ \lambda (m_t - Y_t) - \beta \left( \frac{\mu - r}{\sigma(Y_t)} + Y_t \frac{1-r^2}{\sigma(Y_t)} \right) \right] dt + \beta dZ_t \]
\[ dW_t, dZ_t \text{ independent. } \sigma_t = f(Y_t) \]

Numerical computation from Chib.
But: 1°. Volatility is not directly observable.

- Difficult in estimating parameters in any model.

2°. No clear cut in choosing a right stochastic vol. model.

3°. Incomplete market.
   - i.e. derivatives cannot be perfectly hedged with the underlying asset.
   - A vol. risk premium has to be estimated from option prices.
Lecture 7

Volatility is mean-reverting. Atochastic Volatility

Volatility is not directly observed. Empirical studies are implied vol. often.

Volatility comes in bursts. i.e. the tendency of high volatility comes in bursts.

\[ \Rightarrow \text{a feature of mean reversion.} \]

Simple Examples

Markov chain

\[ Y_t: \text{a 2-state Markov chain} \]

\[ Y_t \in \{-1, +1\} \]

\[ \text{Random holding time} \]

\[ \begin{aligned} \text{In } dt & \quad \text{switching prob. } \alpha dt \\ \text{not switching prob. } & \quad 1 - \alpha dt \end{aligned} \]

\[ \alpha \text{ - Const.} \]

\[ \text{Exponentially waiting time.} \]

Transition prob. matrix

\[ P(t) = \begin{pmatrix} P(Y_{t+1} = -1 | Y_t = -1) & P(Y_{t+1} = +1 | Y_t = -1) \\ P(Y_{t+1} = -1 | Y_t = +1) & P(Y_{t+1} = +1 | Y_t = +1) \end{pmatrix} \]
then for small $st$ invariant, we have

$$Pst(\cdot) = \left( \begin{array}{cc} 1 - ast & ast \\ ast & 1 - ast \end{array} \right) + 0 \ast st$$

\[ \implies P(t+st) - P(t) = P(t) - P(t) \]

\[ = (P(st) - I) P(t) \]

\[ = \left( \begin{array}{cc} -ast & ast \\ ast & -ast \end{array} \right) P(t) + 0 \ast st \]

\[ \ast t \to 0, \quad \frac{dP}{dt} (t) = \left( \begin{array}{cc} -\lambda & \lambda \\ \lambda & -\lambda \end{array} \right) P(t) = \lambda P(t) \quad \lambda = \lambda \]

\[ \frac{dP}{dt} (t) = \left( \begin{array}{cc} -\lambda & \lambda \\ \lambda & -\lambda \end{array} \right) P(t) = \lambda P(t) \quad \lambda = \lambda \]

16: The holding time $\frac{1}{\lambda}$ switches is

$$f(t) = \left\{ \begin{array}{cl} \lambda e^{-\lambda t} & t > 0 \\ 0 & t \leq 0 \end{array} \right.$$ 

and the mean

$$E[H] = \frac{1}{\lambda} \quad \lambda - rate\ of\ mean\ zeroes$$

\[ \implies \text{large } \lambda, \text{ rapid switching} \]

\underline{Invariant\ distribution\ of\ } Y:\)

i.e. find an initial distribution for $Y_0$ sit.

for $t > 0$, $Y_t$ has the same distribution.

Q. How to find the invariant distribution?

\[ g - \text{arbitrary for} \quad \frac{d}{dt} E[g(Y_t)] = \frac{d}{dt} \left[ E[g(Y_t) | X_1, S_t] \right] = 0 \]

\[ (*) \]
Suppose the initial distribution \( p_0 = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} \).

Then  \( q(0) \Rightarrow \frac{d}{dt} \begin{pmatrix} \bar{p}^T \bar{p} \end{pmatrix} q = 0 \) \( \begin{pmatrix} \bar{p}^T \bar{p} \end{pmatrix} q = \begin{pmatrix} (\bar{p}^+1) q(+) + (\bar{p}^-1) q(-) \end{pmatrix} p_0^- \\
+ (\bar{p}^+1) q(+) + (\bar{p}^-1) q(-) p_0^+ \end{pmatrix} \)

\[
\frac{d}{dt} \begin{pmatrix} \bar{p} \end{pmatrix} = L \begin{pmatrix} \bar{p} \end{pmatrix}
\]

\[
\begin{pmatrix} \bar{p} \end{pmatrix} L \begin{pmatrix} \bar{p} \end{pmatrix} q = 0 \quad \forall q
\]

(i) Any vector in the row space is orthogonal to any vector in the null space.

(ii) \( \forall x \in \text{range of } A^T, \forall v \in \text{null space} \quad v^T A x = 0 \)

(iii) The null space is orthogonal to the null space.

(iv) The null space of \( A^T \) is orthogonal to the range of \( A \).

(v) \( Ax = 0 \) is consistent iff \( b^T q = 0 \) \( \forall q \) s.t. \( A^T q = 0 \)

(vi) \( b \) is orthogonal to every vector that is orthogonal to the column vector

\( \bar{p}_0 \) is a solution of the adjoint eigenvalue problem:

\[
(L^T \bar{p}_0) q = \begin{pmatrix} -1 \end{pmatrix}, \quad \bar{p}_0^T \bar{p}_0 = \frac{1}{2}
\]

NB: The invariant distribution solves the adjoint eigenvalue problem with eigenvector \( \bar{p}_0 \) and eigenvalue \( -1 \).
\[ L^p = 0 \Rightarrow b = (c_i \cdot c_i) \quad c_i = \text{c}_2 \quad \text{i.e.} \quad b = c(1) \]

\[ \text{const.} \quad \text{ady.} \]

i.e., the null vectors of the generator of the Markov chain are constant.

\[ L : \text{generator} \quad \text{i.e.} \]

\[ L g(p) = \lim_{t \to 0} \frac{E[g(Y_{t+})] - g(Y_t)}{t}, \quad \forall g \]

**Arnstein-Uhlenbeck Process**

\[ dY_t = \lambda (m - Y_t) \, dt + \beta \, dZ_t \]

\[ \lambda > 0 \]

\[ \text{a standard Brownian motion} \]

\[ Y_t \]

\[ \text{is a Gaussian process.} \]

The corresponding infinitesimal generator of the Markov process \( Y_t \):

\[ L = \lambda (m - Y) \frac{\partial}{\partial y} + \frac{1}{2} \beta^2 \frac{\partial^2}{\partial y^2} \]

Q. What is the invariant distribution?

\[ L - \text{infinitesimal generator} \]

\[ \frac{d}{dt} E[g(Y_t)] = E[Lg(Y_t)] \]

\[ g_0 \sim \text{invariant distribution}, \quad \text{then} \]

\[ \frac{d}{dt} E[g(Y_t)] = 0 \]

\[ \Rightarrow E[Lg(Y_t)] = 0 \quad \forall g \]
Let $\Phi(y)$ be the pdf of invariant distribution

$$
\int_{-\infty}^{+\infty} \Phi(y) L g(y) \, dy = 0
$$

i.e.

$$
\int_{-\infty}^{+\infty} \Phi(y) \left[ -2(m-y) \Phi + \frac{1}{2} \beta^2 \frac{d}{dy} \right] g(y) \, dy = 0
$$

Integrating by parts \Rightarrow

$$
\int_{-\infty}^{+\infty} \Phi(y) \left[ -2 \frac{d}{dy} (m-y) \Phi + \frac{1}{2} \beta^2 \frac{d}{dy} \right] g(y) \, dy = 0
$$

i.e.

$$
\int_{-\infty}^{+\infty} \Phi(y) L^+ \Phi(y) \, dy = 0
$$

the adjoint of $L$.

$$
L^+ \Phi = -2 \frac{d}{dy} (m-y) \Phi + \frac{1}{2} \beta^2 \frac{d}{dy}
$$

i.e. $\Phi(y)$ is valid for any $g(y)$

$$
L^+ \Phi = \frac{1}{2} \beta^2 \frac{d}{dy} \Phi - 2 \frac{d}{dy} (m-y) \Phi = 0
$$

$$
\frac{1}{2} \beta^2 \frac{d}{dy} \Phi - 2 (m-y) \Phi = C
$$

$$
\Phi(y) = C e^{-\frac{2(m-y)^2}{\beta^2}}
$$

$$
\int_{-\infty}^{+\infty} \Phi(y) \, dy = 1
$$

$$
\Phi(y) = \frac{1}{\sqrt{2\pi \sigma^2}} e^{-\frac{(y-x)^2}{2\sigma^2}} \sim N(m, \sigma^2)
$$

$$
\sigma^2 = \frac{\beta^2}{2\gamma^2}
$$

$$
\sigma = \frac{\beta}{\sqrt{2\gamma}}
$$

$$
E[(Y_t-m)(Y_s-m)] = \gamma^2 e^{-\frac{1}{2} \gamma |t-s|}
$$

$$
\frac{1}{\sigma} \text{ correlation time. Burt emotion}
$$

$\sigma_t = \exp(Y_t)$
Q1: What is the null space of the generator $L$? 

\[ L\phi = \frac{1}{2} \beta^2 \phi'' + \alpha (m-\gamma) \phi' = 0 \]

The solution is

\[ \phi(y) = C_1 \int_{-\infty}^{y} e^{-\frac{(m-\gamma)^2}{2V^2}} \, dz + C_2 \]

The integrand grows fast as $y \to \infty$. 

\[ \lim_{y \to \infty} \phi(y) = C_2 \]

\[ \text{i.e. the null space of } L \text{ is a constant} \]

---

Q2: Ergodic process, i.e.

\[ \lim_{t \to \infty} \frac{1}{t} \int_0^t g(Y_s) \, ds = \langle g \rangle = \int_{-\infty}^{\infty} g(y) \phi(y) \, dy \]

the long-time average of a bounded function $g$

---

\[ \text{statistical average} \]

\[ \text{with its invariant distribution} \]
The Returns Process

\[ \frac{dX_t}{X_t} - \mu dt = \sigma dw \] — volatility model.

the demeaned return process

For \( q_t \) — all processes, \( b_t = f(q) = e^{q_t} \)

\[ \alpha = 1 \] \n\[ \alpha = 200 \]

Volatility

\( 0 \) \n\( 0.1 \) \n\( 0.2 \) \n\( 0.3 \) \n\( 0.4 \) \n\( 0.5 \)

\( 0 \) \n\( 0.1 \) \n\( 0.2 \) \n\( 0.3 \) \n\( 0.4 \) \n\( 0.5 \) \n\( 0.6 \) \n\( 0.7 \) \n\( 0.8 \) \n\( 0.9 \) \n\( 1 \)

Returns

Time (years)

Size of fluctuations \( \sim \) constant

The S&P 500 return process \( \sim \) fast mean-reverting stochastic vol. model.

1996 S&P 500 returns computed from half-hourly data.
Asymptotics for Pricing European Derivatives

**Scalings:**

1. The rate of mean reversion — $\delta$

$$\varepsilon = \frac{1}{\delta}$$ — correlation time

— a small parameter

in fast mean reversion

2. The variance of the invariant distribution $Y_t$

$$\nu^2$$ — the long-run size of the volatility fluctuations

**Assumption**

$$\nu^2 \approx \text{Constant as } \varepsilon \to 0$$

**Rescale:**

$$dY_t = (\mu - \gamma Y_t)dt + \beta d\hat{\epsilon}_t$$

$$\nu^2 = \frac{\beta^2}{2\delta} \qquad \therefore \quad \beta = 2\nu^2 \delta$$

$$\therefore \quad \beta = \frac{\nu^2 \delta}{\varepsilon}$$

\( \int \) is bounded away from 0 to avoid degenerate diffusion

**i.e.** 0 volatility

$$dX_t = r X_t dt + \int f(Y_t) X_t dW_t$$

$$dY_t = \frac{1}{\varepsilon} (\mu - Y_t^\frac{n}{2}) dt + \frac{\nu^2 \delta}{\varepsilon} d\hat{\epsilon}_t$$

$$\hat{\epsilon}_t = \rho W_t + \sqrt{1-\rho^2} Z_t, \quad W_t, Z_t \text{ — independent Brownian Motion}$$
The Rescale Pricing Equation

A European derivative with payoff $h(x)$ and maturity $T$

The price $P^E(t, x, y)$ is given by

$$P^E(t, x, y) = \frac{\partial}{\partial t} \ln \left( \frac{h(x)}{1 + \lambda(t, x, y)} \right)$$

where $\lambda(t, x, y)$ is the market price of vol. risk.

Assumed to be a bounded fn. of $y$ alone.

The Notation. Nb. Three scales. $1 > \frac{1}{3} > \frac{1}{\sqrt{3}}$ > 1.

\[ L_0 = \frac{\partial}{\partial t} + \frac{1}{2} \sigma^2 \frac{\partial^2}{\partial x^2} + \sigma^2 \frac{\partial^2}{\partial y^2} + \frac{(\mu - r)}{\sqrt{3}} \frac{\partial}{\partial y} \]

\[ L_1 = \sqrt{2} \nu \sigma \lambda(x) \frac{\partial}{\partial x} - \frac{\sqrt{2} \nu \lambda(x)}{\sqrt{3}} \frac{\partial}{\partial y} \]

\[ L_2 = \frac{\partial}{\partial t} + \frac{1}{2} \sigma^2 \frac{\partial^2}{\partial x^2} + r x \frac{\partial}{\partial x} - r = \mathcal{L}_2 \]
\[\text{No. 1. } \delta \sigma \quad \text{— infinitesimal generator of the \text{AM} process}\]

\[\text{No. 2. } \lambda_1 \quad \text{— due to correlation} \rho \text{ and market price of risk.}\]

\[\text{No. 3. } \lambda_2 \quad \text{— the Black-Scholes operator of \text{vol. for}.}\]

\[\therefore \text{ the pricing PDE } \Rightarrow \]

\[
\left( \frac{1}{2} \frac{\partial^2}{\partial \xi^2} + \frac{1}{2} \lambda_1 + \lambda_2 \right) P_\xi^2 = 0
\]

\[
P^{\xi}(T; x, y) = h(x)
\]

\[\text{No. It is a singular perturbation problem.}\]

\[\text{Reason. } \varepsilon \frac{d}{dt} \quad \varepsilon \to 0.\]

\[\text{The Formal Expansion}\]

\[
P^\xi = P_0 + \varepsilon P_1 + \varepsilon^2 P_2 + \varepsilon^3 P_3 + \ldots
\]

\[P^{\xi}(T; x, y) = h(x) \quad P_0, P_1 \text{ -- fins of } f(t, x, y)\]

\[\text{No. } P_1(T; x, y) = 0\]

\[\text{Plug into Eq (1) } \Rightarrow \]

\[
\frac{1}{2} \delta \lambda_0 P_0 + \frac{1}{\lambda_1} \left( \delta_0 P_1 + \delta_1 P_0 \right)
\]

\[
+ (\delta_0 P_2 + \delta_1 P_1 + \delta_2 P_0)
\]

\[
+ \frac{1}{\lambda_1} (\delta_0 P_0 + \delta_1 P_2 + \delta_2 P_1) + \ldots = 0
\]
\( \text{generator of } \text{OLN: } L_0 = V^2 \frac{\partial^2}{\partial y^2} + (u - y) \frac{\partial}{\partial y} \)

\( \text{acting on } \gamma \text{- variable only} \)

\( \therefore P_0 \text{ is a constant with respect to } \gamma \text{- variable.} \)

(Recall the null space of the Markov process in general is a constant)

\( \therefore P_0 = P_0(u, x) \)

\( \therefore P_0 = P_0(u, x) \)

\( \text{O}(\frac{1}{r^2}): \quad L_0 P_1 + L_1 P_0 = 0 \)

\( L_1 \text{ has the derivative } \frac{\partial}{\partial y}. \)

\( (L_1 = \exp(-u, y) \frac{\partial}{\partial y} - \frac{2uv}{(u - y)} \frac{\partial}{\partial y}) \)

\( \therefore L_0 P_1 = 0 \)

Again \( P_1 = P_1(u, x) \), i.e. function of \( u, x \) only

\( \therefore \) The first 2 terms in the expansion

\( P_0 + \Delta \Delta P_1 \) will not depend on the present \( y \) or \( \gamma \).

\( (\therefore \text{They are } \gamma \text{- independent}) \)

\( \text{O}(1): \quad L_0 P_2 + L_1 P_1 + L_2 P_0 = 0 \)

\( = 0 \) (\( L_1 \text{ has } \frac{\partial}{\partial y}, P_1 = P_1(u, x) \) only)

\( \therefore L_0 P_2 + L_2 P_0 = 0 \) (1)
For fixed $\alpha$, $L_0 \rho_0$ is a function of $y$ ($C \equiv f(y)$)

Consider only $y$ dependence. $E_g (i) \Rightarrow$

$$L_0 Y + \Phi = 0 \quad \text{— Poisson eqn for} \ Y.$$

$$Y = Y(y)$$

Q: When does this eqn have a soln?

For soln to exist,

the centering condition, i.e.,

$$\left< \Phi \right> = \int_{-\infty}^{\infty} f(y) \Phi (y) \, dy = 0 \quad \text{— necessary condition for} \ L_0 Y + \Phi = 0 \quad \text{to have soln}.$$ 

**NB:** $\text{For } L_0, \quad \Phi(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{(y-m)^2}{2\sigma^2}}$

Why this condition?

By:

$$\Phi = - L_0 \Phi \quad \Rightarrow \quad \left< \Phi \right> = -\left< L_0 \Phi \right> \quad \Rightarrow \quad -\int L_0 \Phi \, \Phi \, dy$$

integrating by parts

$$\Rightarrow \int \Phi \, L_0^T \Phi \, dy = \int X(y) L_0^T \Phi \, dy \quad \Rightarrow \quad -\int L_0 \Phi \, \Phi \, dy$$

$$= 0 \quad \text{(} \therefore \text{)} L_0^T \Phi = 0 \quad \text{— Poisson eqn for} \ Y.$$
Formal path of \( L \dot{y} + g = 0 \):

\[
\chi(y) = \int_0^\infty \mathbb{E}(g(y_t) | Y_0 = y) \, dt
\]

Check:

\[
L \int_0^\infty \mathbb{E}(g(y_t) | Y_0 = y) \, dt = \int_0^\infty \mathbb{E}(g(y_t) | Y_0 = y) \, dt
\]

\[
= \int_0^\infty \mathbb{E}(g(y_t) | Y_0 = y) \, dt \quad \text{(Definition of infinitesimal generator)}
\]

\[
= \mathbb{E}(g(Y_0) | Y_0 = y)_{t=0} - \mathbb{E}(g(Y_0) | Y_0 = y)_{t=0}
\]

\[
\lim_{t \to 0} \mathbb{E}(g(Y_t) | Y_0 = y) = \langle g \rangle
\]

\[
\text{i.e. long-run average}
\]

\[
\text{i.e. \( \langle g \rangle \) i.e. w.t.\}
\]

\[
\text{invariant distribution}
\]

\[
= - \mathbb{E}(g(Y_0) | Y_0 = y)_{t=0}
\]

\[
= - g(y)
\]

\[
\text{i.e. } \Delta \chi = -g(y). \quad \text{QED.}
\]

Note: All paths of \( L \dot{y} + g(y) = 0 \) are

\[
\chi(y) = \int_0^\infty \mathbb{E}(g(y_t) | Y_0 = y) \, dt + \text{Const.}
\]

\[
\text{i.e. different paths differ by a Const.}
\]
The 0th-order Term

\[ L_0 P_2 + L_2 P_0 = 0 \]

The centering condition now is

\[ \langle L_0 P_0 \rangle = 0 \]

\[ \therefore \ P_0 = P_0(\xi, x) \ \text{i.e.} \ \text{indep. of } y \]

\[ \therefore \ \langle L_2 \rangle P_0 = 0 \]

\[ L_2 = \frac{\partial}{\partial t} + \left( f(x) \right) x \frac{\partial}{\partial x} + v x \frac{\partial}{\partial x} - r = \mathcal{L}_{\text{BS}}(\hat{f}(x)) \]

\[ \therefore \ \langle L_2 \rangle = \frac{\partial}{\partial t} + \frac{1}{2} \langle f(x) \rangle x \frac{\partial}{\partial x} + v x \frac{\partial}{\partial x} - r = \mathcal{L}_{\text{BS}}(\bar{\sigma}) \]

\[ \therefore \ \text{the zero-order } P_0(\xi, x) \text{ is} \]

\[ \text{the order of the BS equation with effective } \bar{\sigma}. \]

\[ \int \mathcal{L}_{\text{BS}}(\bar{\sigma}) P_0 = 0 \]

\[ P_0(\xi, x) = \text{free} \ \text{final data} \]

\[ \therefore \ \langle L_2 P_0 \rangle = 0 \ (\text{i.e. centering condition}) \]

\[ L_2 P_0 = L_2 P_0 - \langle L_2 P_0 \rangle = (L_2 - \langle L_2 \rangle) P_0 = \]

\[ \frac{1}{2} \left( f(x) - \bar{\sigma}^2 \right) x^2 \frac{\partial^2}{\partial x^2} P_0 \]
From $L_0 P_2 + L_2 P_0 = 0 \implies$

$$P_2(t,x,y) = -L_0^{-1} L_2 P_0$$

$$= -\frac{1}{2} L_0^{-1} (f(y) - \bar{\sigma}^2) x^2 \frac{\partial^2}{\partial x^2} P_0$$

if $L_0 \phi = f(y) - \bar{\sigma}^2$

i.e. $\phi$ is a poly

then $\phi = L_0^{-1} (f(y) - \bar{\sigma}^2)$

$$= L_0 (\phi) + \text{Const.}$$

i.e. the constant $y$ which can be $(t,x)$-dependent.

Q1. What is $\phi$? \[ \phi = f(y) - \bar{\sigma}^2 \]

\[ \implies \phi \text{ satisfies } \nu^2 \frac{d^2}{dy^2} \phi + (m - y) \frac{d}{dy} \phi = f(y) - \bar{\sigma}^2 \] (Eq. 60)

\[ \text{NB: } \int \frac{d}{dy} \left( \frac{\partial \phi}{\partial y} \right) = \int -\frac{d}{dy} \left( \frac{\partial}{\partial y} \left( \frac{\partial \phi}{\partial y} \right) \right) + \nu^2 \frac{d^2}{dy^2} \phi \]

\[ \phi = \frac{1}{\sqrt{2\pi}} e^{-\frac{(y-m)^2}{2\nu^2}} \]

\[ \implies \phi = -\frac{y-m}{\nu^2} \] (Eq. 60)

\[ \implies \phi(y) \frac{d}{dy} \phi = \nu^2 \int_{-\infty}^{y} (f(z) - \bar{\sigma}^2) \phi(z) dz \]

\[ \text{NB: } y \to \infty, \text{ all sides} \]

\[ \implies \frac{d}{dy} \phi = \nu^2 \int_{-\infty}^{y} (f(z) - \bar{\sigma}^2) \phi(z) dz \]
The 1st-order correction

O(\varepsilon) - term in the expansion:

\[ \lambda_0 P_3 + \lambda_1 P_2 + \lambda_2 P_1 = 0 \]

- Poisson eq. \Rightarrow\text{ Cauchy condition:}
  \[ \langle \lambda_1 P_2 + \lambda_2 P_1 \rangle = 0 \]

Recall \( P_2(t,x,y) = -\frac{1}{2} (\partial_x^2 \rho + \partial_x \partial_y \rho) x^2 \partial_x^2 P_0 \)

and \( P_1 = P_1(t,x) \text{ independent of } y \).

\[ \langle \lambda_2 \rangle = \Lambda_2 \sigma \]

\[ \Rightarrow -\frac{1}{2} \left( \langle \lambda_1 (x \partial y) \rangle x^2 \partial_x^2 P_0 + \langle \lambda_2 (x \partial y) \rangle P_1 \right) = 0 \]

(has \( \frac{\partial}{\partial y} \) only; \( \partial_x \rho = 0 \))

\[ \Lambda_2 \sigma P_1 = \frac{1}{2} \langle \lambda_2 (x \partial y) \rangle x^2 \partial_x^2 P_0 \]

\[ \sigma = \frac{\partial^2 \rho}{\partial x \partial y} = \frac{1}{2} \left( \nabla \rho \cdot \nabla \rho \right) \frac{\partial^2 \rho}{\partial x \partial y} = \frac{1}{2} \left( \nabla \rho \cdot \nabla \rho \right) \frac{\partial^2 \rho}{\partial x \partial y} \]

\[ \Lambda_2 \sigma P_1 = \frac{\partial^2 \rho}{\partial x \partial y} \frac{\partial^2 \rho}{\partial x \partial y} \frac{\partial^2 \rho}{\partial x \partial y} \]

\[ \int \Lambda_2 \sigma P_1 = \frac{\partial^2 \rho}{\partial x \partial y} \frac{\partial^2 \rho}{\partial x \partial y} \frac{\partial^2 \rho}{\partial x \partial y} \]

\[ \begin{cases} P_2(t,x) = 0 \end{cases} \quad \text{the equation for } P_1 \]
the 1st small correction

\[ \tilde{\rho}(t, x) = \mathcal{L}\rho(t, x) \] is the pole of

\[ \begin{cases} \mathcal{L}\rho(t, \bar{x}) \tilde{\rho}_1 = \mathcal{H}(t, x) \\ \tilde{\rho}_1 = 0 \end{cases} \]

therefore:
[2] \( \mathcal{H}(t, x) = V_2 \frac{\partial^2 \rho}{\partial x^2} + V_3 \frac{\partial^4 \rho}{\partial x^4} \)

\[ V_2 = \frac{\mu}{\alpha^2} \left( 2 \left\langle f \bar{h} \right\rangle - \left\langle \phi \bar{f} \right\rangle \right) \]

\[ V_3 = \frac{\mu^2}{\alpha^2} \left\langle f \bar{h} \right\rangle \]

\textbf{Q1. How to solve for } \tilde{\rho}_1 ?

\[ \text{NB. } \mathcal{L}\rho(t, \bar{x}) \left( - (t-t_0) \mathcal{H} \right) = \mathcal{H} - (t-t_0) \mathcal{L}\rho(t, \bar{x}) \mathcal{H} \]

\( \mathcal{L}\rho(t, \bar{x}) \mathcal{H} \) contains only terms of the type:

\[ \mathcal{L}\rho(t, \bar{x}) \left( \frac{\partial^n \rho}{\partial x^n} \right) = \frac{\partial^n \mathcal{L}\rho(t, \bar{x})}{\partial x^n} \mathcal{H} \rho = 0 \]

(check yourself)

\[ \therefore \mathcal{L}\rho(t, \bar{x}) \left( - (t-t_0) \mathcal{H} \right) = \mathcal{H} \]

\[ \therefore \tilde{\rho}(t, x) = - (t-t_0) \left( V_2 \frac{\partial^2 \rho}{\partial x^2} + V_3 \frac{\partial^4 \rho}{\partial x^4} \right) \]

\textbf{Q2. The corrected price is}

\[ P_0(t, x) - (t-t_0) \left( V_2 \frac{\partial^2 \rho}{\partial x^2} \rho_0 + V_3 \frac{\partial^4 \rho}{\partial x^4} \rho_0 \right) \]

\text{with } \mathcal{L}\rho(t, \bar{x}) \rho_0 = 0, \quad P_0(c, x) = h(x),

\text{i.e. } \rho_0 \text{ is the BS priced at } \bar{x}.
Universality of the correction:

Observation: Any stochastic volatility model will give rise to a 1st correction of this type. (If the stochastic volatility model is driven by an ergodic diffusion process)

NS: \( V_2 \) and \( V_3 \) determine the 1st correction.

\[ H(x, \xi) = \frac{\sigma^2}{2} \langle \xi^2 \rangle \frac{\partial^2}{\partial x^2} P_0 \]
\[ = \frac{1}{2} \left\langle \xi \right\rangle \frac{\partial^2}{\partial x^2} \left( \frac{1}{\sigma^2} \right) \frac{\partial^2}{\partial x^2} P_0 \]
\[ = \frac{1}{\sqrt{2\pi}} \left\langle \xi \right\rangle \frac{\partial^2}{\partial x^2} \left( \frac{1}{\sqrt{2\pi}} \right) \frac{\partial^2}{\partial x^2} P_0 \]

Recall \( \left\langle \xi \right\rangle P_0 = \frac{1}{2} (\langle f(y) \rangle - \xi^2) \frac{\partial^2}{\partial y^2} \]

NS: \( \frac{dV_2}{dt} = \frac{1}{2} \mu y (t, x, y) \frac{\partial V_2}{\partial t} + \frac{1}{n \sigma^2} \frac{\partial^2 V_2}{\partial x^2} \)

\[ L = \frac{1}{2} \mu y (t, x, y) \frac{\partial^2}{\partial y^2} + \frac{1}{28} \sigma^2 (d_1, y) \frac{\partial^2}{\partial y^2} \]

the pricing PDE \( \Rightarrow \) \( \left( \frac{1}{2} L_0 + \frac{1}{n \sigma^2} L_1 + L_2 \right) P^2 = 0 \)

\[ L_0 = \frac{1}{2} \mu y \frac{\partial}{\partial y} + \nu y \frac{\partial}{\partial y} \]
\[ L_1 = \rho \mu x \frac{\partial^2}{\partial x^2} \left( - \sigma y \frac{\partial}{\partial y} \right) \]
\[ L_2 = \left[ \left( \frac{\partial^2}{\partial y^2} \right) f(y) \right] \]
Following the perturbation procedure above, we have:

\[ \Delta \psi(x) \psi_1 = \frac{1}{\sqrt{2}} \left< \Delta \psi(x_2 + \Delta x_2, x_1 + \Delta x_1) \right> \psi_0 \]

\[ = A \psi_0, \quad A = V_2 x_2^2 \frac{\partial^2}{\partial x^2} + V_3 x_3^2 \frac{\partial^2}{\partial x^2} \]

This structure is universal, and detailed expressions don't matter.

For example, in the \( \psi_1 \) procedure:

\[ V_2 = \frac{2}{\sqrt{2} \pi} \frac{1}{C_2 \left< \sqrt{\phi} \right> - \left< \phi \phi \right>}, \quad V_3 = \frac{2}{\sqrt{2} \pi} \left< \sqrt{\phi} \phi \right> \]

However, we are going to use the universal form of the parameterization.

\( \phi \) is the market price of option risk in \( V_2 \).

**Put-Call Parity**

\[ C(t, x) - P(t, x) = x - K e^{-r(t-t)} \]

\( \phi \) is the option price for \( \phi \):

\[ \phi = - (t-t) \left( V_2 x_2^2 \frac{\partial^2}{\partial x^2} + V_3 x_3^2 \frac{\partial^2}{\partial x^2} \right) (C_0 - P_0) \]

\[ = - (t-t) \left( V_2 x_2^2 \frac{\partial^2}{\partial x^2} + V_3 x_3^2 \frac{\partial^2}{\partial x^2} \right) (x - K e^{-r(t-t)}) \]

\[ = 0 \]

\( \therefore \) \( (C_0(x) + \phi(t, x)) - (P_0(x) + \phi(t, x)) = x - K e^{-r(t-t)} \)
The Skew Effect

\[
\Phi(p) = \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial \Phi}{\partial x} P_0
\]

\[
\Rightarrow \Phi(p) = \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial \Phi}{\partial x} P_0
\]

\[
\text{If } p = 0, \text{ then } \frac{\partial \Phi}{\partial x} \text{ term vanishes.}
\]

\[
\int \Phi(p) (p_0 + p) = \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial \Phi}{\partial x} P_0
\]

\[
\Rightarrow \frac{\partial^2 \Phi}{\partial x^2} = \frac{\partial^2 \Phi}{\partial x^2} P_0
\]

\[
\text{This can be absorbed by } \Phi
\]

\[
\text{If } V_2 \text{ is sufficiently small, } \sigma = \sqrt{\sigma^2 - 2V_2}
\]

\[
\Rightarrow \Phi(p) = \frac{\partial^2 \Phi}{\partial x^2} - \frac{\partial \Phi}{\partial x} P_0
\]

\[
\Phi(p) \cong \frac{\partial^2 \Phi}{\partial x^2} P_0
\]

\[
\Phi_T(x) = \Phi(x)
\]

\[
\text{V}_2 \text{ merely corrects the volatility level.}
\]

\[
\text{V}_3 \text{ cannot be absorbed into the original infinitemal operator.}
\]

\[
\Rightarrow \text{ skew, i.e. 3rd moment of stock price return}
\]
\[ \text{if } p = 0. \implies V_b = 0 \]
\[ \therefore \text{spec}(\bar{\sigma}) p = 0 \]

\[ \text{vol. } \sigma, \text{ including the credit} \]
\[ \text{of market price of vol. risk} \]

**Implied Volatility and Calibration**

**Q:** How to link \( V_2, V_3 \) with observed prices or implied volatilities?

Suppose \( h'(x) = (\alpha - K)^+ \)

Recall \( P_0 = C_k = e^{-r(T-t)}N(d_1) - Ke^{-r(T-t)}N(d_2) \)

\[ d_{1,2} = \frac{1}{\sqrt{T-t}} \left[ \log\left( \frac{x}{K} \right) + \left( r \pm \frac{1}{2} \bar{\sigma}^2 \right) (T-t) \right] \]

We can check:

\[ \frac{\partial d_{1,2}}{\partial \alpha} = \frac{1}{x \sqrt{T-t}} \]

\[ N'(d) = \frac{1}{\sqrt{2\pi}} e^{-d^2/2} \]

\[ e^{-d_{1,2}^2/2} = e^{-d_{1,2}^2/2} \left( \frac{x e^{r(T-t)}}{K} \right) \]

and the Delta \( \frac{\partial P_0}{\partial \alpha} = N(d_1) \)

The Gamma \( \frac{\partial^2 P_0}{\partial \alpha^2} = \frac{e^{-d_{1,2}^2/2}}{2 \sqrt{2\pi} (T-t)^{3/2}} \)
\[
\text{"the Equation" } \frac{\partial^2 \sigma}{\partial x^2} = -e^{-\frac{d^2}{2}} \frac{e^{-\frac{d^2}{2}}}{s \sqrt{2\pi \sigma} (T-t)} \left( 1 + \frac{d_1}{\sigma \sqrt{T-t}} \right)
\]

\[
- \quad \mathcal{H}(x) = \frac{e^{-\frac{d^2}{2}}}{\sqrt{2\pi \sigma (T-t)}} \left( V_2 - V_3 \left( 1 + \frac{d_1}{\sigma \sqrt{T-t}} \right) \right)
\]

\[
- \quad \tilde{P}(t,x) = -(T-t) \mathcal{H}(x)
\]

the 2nd connection \quad \frac{e^{-\frac{d^2}{2}}}{\sqrt{2\pi \sigma (T-t)}} \left( V_3 \frac{d_1}{\sigma} + (V_3-V_2) \frac{1}{\sqrt{T-t}} \right)

Q. How can we extract \( V_2, V_3 \) from the market observed prices?

Recall \( C_{BS}(t,x,K,T;I) = C^{BS}(K,T) \)

Expand \( I = \overline{\sigma} + \sqrt{2} I_{1} + ... \)

\[
C_{BS}(t,x;K,T;I) = C_{BS}(t,x;K,T;\overline{\sigma}) + \sqrt{2} I_{1} \frac{\partial C_{BS}}{\partial \overline{\sigma}}(t,x;K,T;\overline{\sigma}) + ...
\]

\[
= \tilde{P}(t,x) + \tilde{P}_{1}(t,x)
\]

\[
\tilde{P}_{1}(t,x) = \sqrt{2} I_{1} \frac{\partial C_{BS}}{\partial \overline{\sigma}}(t,x;K,T;\overline{\sigma}) \bigg|_{\overline{\sigma}}
\]

\[
I = \overline{\sigma} + \tilde{P}_{1}(t,x) \left[ \frac{\partial C_{BS}}{\partial \sigma}(t,x;K,T;\overline{\sigma}) \right]^{-1} + O(\frac{1}{\sigma})
\]

Recall the Vega: \( \frac{\partial C_{BS}}{\partial \sigma} = \frac{ae^{-\frac{d^2}{2}}}{\sigma \sqrt{2\pi \sigma (T-t)}} \sqrt{\frac{T-t}{2\pi \sigma}} \)

\[
I = \overline{\sigma} + \tilde{P}_{1}(t,x) \cdot \frac{1}{a} \cdot e^{\frac{d^2}{2}} \left[ \frac{2\overline{\sigma}}{\sqrt{T-t}} + O(\frac{1}{\sigma}) \right]
\]
\[ I = \bar{\sigma} + \frac{a e^{-\frac{1}{2} \sigma^2}}{\bar{\sigma} \sqrt{2\pi}} \left( \sqrt{\frac{d_1}{\bar{\sigma}} + (V_3 - \bar{\sigma})^2} \right) \cdot \frac{1}{\sqrt{2\pi}} e^{\frac{1}{2} \left( \frac{\log \frac{K}{x}}{T-t} \right)^2 + O(\frac{1}{\sigma^2})} \]

\[ \rho_i(t, x) \]

\[ \Rightarrow I = \bar{\sigma} + \frac{V_3}{\bar{\sigma}^3} \left( \rho_i + \frac{1}{2} \frac{\sigma^2}{\bar{\sigma}^2} \right) - \frac{V_2}{\bar{\sigma}} - \frac{V_3}{\bar{\sigma}} \frac{\log \frac{K}{x}}{(T-t)} + O(\frac{1}{\sigma^2}) \]

\[ \Rightarrow I = A \left[ \frac{\log \text{Strike Price}}{\text{Time to maturity}} \right] + b + O(\frac{1}{\sigma^2}) \]

LMMR

\[ \log \text{Money rate - to - Maturity Ratio} \]

\[ \begin{cases} a = -\frac{V_3}{\bar{\sigma}^3} \\ b = \bar{\sigma} + \frac{V_3}{\bar{\sigma}^3} \left( \rho_i + \frac{1}{2} \frac{\sigma^2}{\bar{\sigma}^2} \right) - \frac{V_2}{\bar{\sigma}} \end{cases} \]

\[ \Rightarrow \begin{cases} V_3 = -a \bar{\sigma}^2 \\ V_2 = \bar{\sigma} \left( (\bar{\sigma} - b) - a \rho_i + \frac{1}{2} \frac{\sigma^2}{\bar{\sigma}^2} \right) \end{cases} \]

**Note:** The implied volatility surface i.e. \[ I = ICK, T-t \] is

\[ I \approx a \cdot \text{LMMR} + b \]

i.e. \[ I \] is a fun of \[ \left( \frac{\log \frac{K}{x}}{T-t} \right) \] combined.

**Note:** If \[ K=x \] then \[ I \approx b \] i.e. \[ b \] is the at-the-money implied vol.
Why don't we go to higher order corrections?

R: Loss of universality, i.e., the concrete calibration procedure depend on specific model of $Y_t$. 
Dividends.

For simplicity, continuous dividend yield \( D_0 \)

\[
dX_t = (\mu - D_0) X_t dt + \sigma X_t dW_t
\]

with the same volatility process.

Recall: the corresponding BS operator:

\[
L^D_{BS}(\sigma) = \frac{\partial}{\partial t} + \frac{\sigma^2 x^2}{2} \frac{\partial^2}{\partial x^2} + (r - D_0) x \frac{\partial}{\partial x} - r
\]

In our calculation:

only change: \( L_2 \Rightarrow L^D_2 = \frac{\partial}{\partial t} + \frac{1}{2} \text{tr}(\Sigma^2) x^2 \frac{\partial^2}{\partial x^2} + (r - D_0) x \frac{\partial}{\partial x} - r
\]

\[
= L^D_{BS}(\Phi_0(x))
\]

For \( P_0 \), we still have

\[
\begin{cases}
L^D_2 P_0 = L^D_{BS}(\Phi_0) P_0 = 0 \\
P_0(0, x) = h(x)
\end{cases}
\]

\[
L^D_{BS}(\Phi_0) P_0 = \frac{1}{\sqrt{2\pi}} \left( \exp \left( \frac{-(\phi_0 - \bar{\phi}_0)^2}{2\sigma^2} \right) \right) P_0 \quad \text{in general}
\]

\[
= \left( V_2 x^2 \frac{\partial^2}{\partial x^2} + V_3 x^3 \frac{\partial^3}{\partial x^3} \right) P_0
\]

\[
(\bar{\phi}_2 - \langle \bar{\phi}_2 \rangle) P_0 = (L^D_{BS} - \langle L^D_{BS} \rangle) P_0: \text{This is no longer dependent on } D_0
\]

NS: \( \hat{P}_1(t, x) = -C(t, \bar{\phi}_0)(V_2 x^2 \frac{\partial^2}{\partial x^2} P_0 + V_3 x^3 \frac{\partial^3}{\partial x^3} P_0) \)

\[
= V_2, V_3 \text{ remain the same (no } D_0 \text{-dependence)}
\]
For a European call,

\[ p_0(t,x) = xe^{-r(t-T)}N(\Phi_d) - Ke^{-(r+\frac{1}{2}\sigma^2)(T-t)}N(\Phi_d^*)] \]

\[ d_1 = \frac{1}{\theta^{\frac{1}{2}}} \left[ \log(\frac{x}{K}) + (r-D_0 + \frac{1}{2}\sigma^2)(T-t) \right] \]

\[ d_2 = d_1 - \theta^{\frac{1}{2}}(T-t) \]

\[ \tilde{p}_1(t,x) = xe^{-r(t-t)} \frac{e^{-d_1/2}}{\theta^{\frac{1}{2}}} \left( V_2 \frac{d_2}{\theta^{\frac{1}{2}}} + (V_2 - V_3) \frac{d_1}{\theta^{\frac{1}{2}}} \right) \]

The calibration formula become,

\[ I = \sigma + V_2 \frac{d_2}{\theta^{\frac{1}{2}}} + \frac{V_2 - V_3}{\sigma} + O(\frac{1}{d}) \]

\[ = \sigma + \frac{V_2}{\theta^{\frac{1}{2}}} \left( r-D_0 + \frac{3}{2}\sigma^2 \right) - \frac{V_3}{\theta^{\frac{1}{2}}} - \frac{V_2}{\theta^{\frac{3}{2}}} \left[ \log \left( \frac{x}{K} \right) \right] + O(\frac{1}{d}) \]

\[ I = a \left[ \frac{\text{log(price)}}{\text{time to maturity}} \right] + b + O(\frac{1}{d}) \]

\[ V_2 = \sigma \left( \sigma - b \right) - a \left( r-D_0 + \frac{3}{2}\sigma^2 \right) \]

\[ V_3 = -a\sigma^3 \]

From calibrated

\[ \Rightarrow a,b \text{ to get } \frac{V_2}{V_3} \]
Implementation

Model & Data

Mean-reverting Stochastic Volatility

\[ dx_t = \mu x_t dt + \sigma f(Y_t) x_t dw_t \]

\[ dy_t = \alpha (m - y_t) dt + \beta (p dt + \sqrt{1-p^2} dz_t) \]

\[ w_t \text{ and } z_t \text{ -- independent.} \]

Q: How to estimate the parameters?

1) The rate of mean reversion \( \alpha \).
2) The long-run mean \( m \) of \( Y_t \).
3) The volatility of volatility \( \sigma \).
4) The correlation coefficient \( p \).

Note: only the stock price \( X_t \) is directly observable.

Discrete Data:

No. Tick-by-tick observations \( \text{OC(30)} \) points per day.

\[ \text{-- unevenly spaced.} \]

Suppose we average over 5 minutes intervals

1 hour = 12 pts.
6-hour trading days = 251 trading days
\[ \Rightarrow 72 \text{ data pts.} \]

\[ \Rightarrow 72 \times 251 = 18,072 \text{ pts} \]
$x_n$ is the 5 min average of the asset price at $t_n = n \Delta t$

Normalized fluctuation of the data:

$$D_n = \frac{2(x_n - x_{n-1})}{\sqrt{\Delta t} (x_n + x_{n-1})} \approx \frac{\text{innocent} (x_n - x_{n-1})}{\text{average} (x_n + x_{n-1})}$$

$\overline{D}_n$ - the observed realization of return

$$\frac{1}{\sqrt{\Delta t}} \frac{\Delta x_n}{x_n} = f(\mu) \frac{\Delta \ln x}{\Delta t} + \mu \sqrt{\Delta t} \text{ (from stochastic model)}$$

NB: $\mu \sqrt{\Delta t}$ is negligible small, $\frac{\Delta \ln x}{\Delta t}$ i.i.d Gaussian rand. var.

$$\overline{x}_n = f(\mu) \overline{\varepsilon}_n \sim \mathcal{N}(0,1)$$

NB: $\frac{\Delta \ln x}{\Delta t} \text{ has var } 1.$

e.g. $f(\mu) = e^\mu$, $\gamma$ - Ornstein.

Then $L_n = \log |\overline{x}_n| = \log f(\mu) + \log |\varepsilon_n|$

log fluctuations $\text{the Orn } \varepsilon_n$.

$$\mathbb{E} [Y_{ij} Y_{jk}] = \frac{b^2}{2d} e^{-2djT}$$
Nugget analysis

Nugget (empirical structure function)

\[ V_j = \frac{1}{N} \sum_{n=1}^{N} (L_{nj} - L_n)^2 \]

\( N \): the total # of pt.

Assume \( p = 0 \) i.e. no correlation (i.e. no volatility and asset prices

then

\[ \mathbb{E} (L_{nj} - L_n)^2 \]

\[ = \mathbb{E} (L_n - L_0)^2 \quad (\text{Stationarity}) \]

\( p=0 \)

\[ = \mathbb{E} (\log f(Y_j) - \log f(Y_0))^2 + \mathbb{E} (\log \epsilon_j - \log \epsilon_0)^2 \]

\[ = 2 \mathbb{E} (\log f(Y_j))^2 - 2 \mathbb{E} (\log f(Y_j) \log f(Y_0)) + 2 \text{Var} (\log \epsilon_1) \]

\[ \approx 2v^2 (1 - e^{-\lambda t}) + 2\sigma^2 \]

\[ \sigma^2 = \text{Var} (\log \epsilon_1) \]

\[ \nu^2 = \text{variance of } Y \]

Note: For more general \( f \), we still have

\[ \mathbb{E} [f(Y_{nj}) f(Y_n)] \sim \nu^2 \ e^{-\nu_1^2} \quad \text{as } x \to \infty \quad \nu_1 \to \text{Var} (\log \epsilon) \]

\[ \therefore \nu^2 \to \nu_1^2 \Rightarrow \mathbb{E} (L_{nj} - L_n)^2 \to 2\sigma^2 (1 - e^{-\lambda t}) + 2\sigma^2 \]
Asymptote \( \Rightarrow 2V_f^2 \)

Intercept \( (j=0) \Rightarrow 2C^2 \) of the log plot

\[ V_j = E(L_{ij} - L_i)^2 \]

\[ V_j^N = 2C^2 + 2V_f^2 (1 - e^{-\lambda j}) \]

\( \lambda \) is the characteristic time of mean reversion

E.g., for S&P500 vol., the characteristic time of mean reversion is 1.5 days

\[ \Rightarrow \text{In terms of calender days, } \frac{1}{\lambda} \approx 0.004. \]

\( \Rightarrow \) establish fast mean-reversion

Now let's turn to the calibration of the price:

1. Estimate \( \sigma \) (effective historical volatility)

   from stock-price returns

2. Use variogram analysis of historical stock-price returns to establish that vol. is fast mean-reverting.
Fit \[ I = a \left( \frac{\text{stock price}}{\text{time to maturity}} \right) + b \]

to the implied volatility surface \( I \) across strikes and maturities for liquid options.

\[ \Rightarrow \] estimation of \( a, b \),

\[ (\text{a}) \quad V_2 = \bar{\sigma} (\bar{\sigma} - b) - a \left( r + \frac{3}{2} \bar{\sigma}^2 \right) \]

\[ V_3 = -a \bar{\sigma}^3 \]

\( V_2, V_3 \) correct for European options w/ payoff \( \phi(X_t) \) by

\[ P_0 - (T - t) \left( V_2 x^2 \frac{\partial P_0}{\partial x^2} + V_3 x^3 \frac{\partial^2 P_0}{\partial x^3} \right) \]

\( P_0 \) – BS price w/ constant \( \bar{\sigma} \) and interest rate \( r \)

\( V_2, V_3 \) – market constants (arising from stochastic volatility)

\( P_0 \) due to stochastic volatility.

Note: \( V_2, V_3 \) – market constants (arising from stochastic volatility)

Can be used to price other types of derivatives

e.g. Asian, Barrier, American etc., and for hedging.
Features of this approach

1°. Model independence.

i.e. no specific stochastic volatility model is assumed except: fast mean reversion.

2°. Parsimony of Parameters

16. Model parameters:

\[
\begin{align*}
\text{mean level of vol.} \\
\text{variance of vol.} \\
\text{rate of mean reversion of vol.} \\
\text{correlation } \rho \\
\text{volatility risk premium (NB: the difficulty of estimating)}
\end{align*}
\]

\{ at least 5 parameters \}

But only mean level of vol \( \bar{\sigma} \) and 3 parameters \( a, b \) need to price.


\( \sigma \) parameter, through daily fitting, say. \( \Rightarrow \) stable.

\( S & P 500 \) Eurodollar cells \( a = -0.154 \pm 0.032 \quad b = 0.149 \pm 0.007 \)

\( \bar{\sigma} = 0.1, \quad \rho = 0.82 \)

\( V_2 = -0.0044, \quad V_3 = 0.000154 \)
Hedging Strategies

In an incomplete market, perfect hedge — not possible.

Aim of hedge:
- find a reasonable trade-off $\Delta$ with:
  - the risk of a failed hedge and
  - the cost of implementing hedge.

Review BS Delta Hedging

Under a constant volatility $\sigma$,
\[
\Delta t = \mu \Delta t + \sigma \Delta W_t
\]
\[
\Delta = \frac{\partial P_0}{\partial x}
\]

Hedged portfolio: $\Delta$ units of the risky asset
\[
\sum_t e^{rt} (P_t - X_t \Delta)
\]
units of risk-free asset.

As $t \rightarrow T$: this portfolio $\equiv P_0$

As $t \rightarrow T$: it replicates $h(x,t)$ at maturity.

\[
\Delta P_0 = \frac{\partial P_0}{\partial x} + \frac{1}{2} \sigma^2 x^2 \frac{\partial^2 P_0}{\partial x^2} + \mu x \frac{\partial P_0}{\partial x} \Delta t + \frac{\partial P_0}{\partial x} \sigma \Delta W_t
\]

BS Formula
\[
\Delta P_0 = \frac{\partial P_0}{\partial x} + \frac{\partial P_0}{\partial x} = \frac{\partial P_0}{\partial x} \Delta x + r (P_0 - x \frac{\partial P_0}{\partial x}) dt = \Delta dx + r (P_0 - x \Delta) dt
\]
i.e. while maintaining the variation due to market (self-financing).
The strategy and its cost

Suppose we follow the same BS strategy in a stochastic volatility environment. \( \text{df}_t = \mu f_t \text{d}t + \sigma f_t \text{d}W_t \)
\[ \text{df}_t = \sigma f_t \text{d}W_t + \mu f_t \text{d}t \]
\[ \text{dy}_t = \alpha (m - y_t) \text{d}t + \beta \text{d}W_t \]

The portfolio has
\[ a_t = \frac{\partial P_0}{\partial x} \text{ stocks} \]
\[ b_t = e^{rt} (P_0(x_t) - x_t \frac{\partial P_0}{\partial x}(x_t, x_t)) \text{ bonds} \]
at time \( t \).

Its value at time \( t \):
\[ a_t x_t + b_t e^{rt} = P_0(x_t, x_t) \]
at time \( T \).
\[ P_0(T, x_t) = h(x_t) \]

The strategy replicates the derivative at maturity.

Is it self-financing?

\[ \text{d} P_0(t, x_t) = \left( \frac{\partial P_0}{\partial t} + \frac{1}{2} \sigma^2 f_t \frac{\partial^2 P_0}{\partial x^2} f_t \right) \text{d}t + \frac{\partial P_0}{\partial x}(x_t, x_t) \text{d}W_t \]

But the change due to the market is
\[ a_t \text{d}x_t + b_t e^{rt} \text{d}W_t \]
(c.f. the self-financing part)
The infinitesimal cost of the strategy is
\[
dP(t, x_t) - (\alpha dX_t + \beta X_t e^{rt} dt).
\]
\[
= \left( \frac{\partial P}{\partial t} + \frac{1}{2} \Sigma_x \Sigma_t (x^2 \frac{\partial^2 P}{\partial x^2}) \right) dt + \alpha dX_t - \left( \alpha dX_t + \beta X_t e^{rt} dt \right).
\]
\[\text{Note: } P_{t=0} = 0 \]
\[
= \frac{1}{2} \left( f(x, y(t)) - \sigma^2 \right) x_t \frac{\partial^2 P}{\partial x^2} dt.
\]

The cumulative cost up to time \( t \) is
\[
E_0(t) = \frac{1}{2} \int_0^t \left( f(x, y(s)) - \sigma^2 \right) x_s \frac{\partial^2 P}{\partial x^2} (s, x_s) ds.
\]

In addition to the initial cost \( P_0(c, x_0) \), the additional total cost is
\[
E_0(T) = \frac{1}{2} \int_0^T \left( f(x, y(s)) - \sigma^2 \right) x_s \frac{\partial^2 P}{\partial x^2} (s, x_s) ds.
\]
\[E_0(T) \text{ — further financed.}\]

Averaging effect
\[
\text{\( x_t \) is running on the fast time scale (i.e. } \alpha x_1 \)
\]
\[
\frac{1}{\sigma^2} \int_0^T \frac{\partial^2 P}{\partial x^2} (s, x_s) ds \approx \frac{1}{\sigma^2} \int_0^T x_s \frac{\partial^2 P}{\partial x^2} (s, x_s) ds
\]
\[
= \frac{1}{2}\sigma^2 \int_0^T x_s \frac{\partial^2 P}{\partial x^2} (s, x_s) ds
\]
\[
E_0(T) \text{ will be small.}\]
Q: How to evaluate this small cost?

Recall the Poisson eqn \( L_0 \Phi = f(y) - \Phi \)

\[
\therefore \quad f(y) - \Phi^2 = (L_0 \Phi)(y)
\]

\( L_0 \) - infinitesimal generator of \( \Phi \).

**MFG formula:**

\[
\Phi(x) = \chi(L_0 \Phi)(y) ds + \Phi(y) \delta x \frac{d}{dx} x
\]

\[
\frac{\gamma^2 - k}{2a}
\]

\[
\frac{\gamma}{2a}
\]

\[
\frac{\gamma}{2a} [d \Phi(x) - \sqrt{2a} \Phi' d \Phi(x) d \Phi(x)]
\]

The cumulative cost:

\[
E_0(t) = \frac{1}{2a} \int_0^t x^2 \frac{d}{dx} \frac{d}{dx'} (s,x,s)^{L_0 \Phi(x)} ds + \chi(\Phi(x) d(x^2 \frac{\delta y}{\delta x} (s,x,s)))
\]

\[
= x^2 \frac{\delta y}{\delta x} (s,x,s) d \Phi(x) + \chi(\Phi(x) d(x^2 \frac{\delta y}{\delta x} (s,x,s)))
\]

\[
+ (x^2 \frac{\delta y}{\delta x} (s,x,s) + 2x \frac{\delta y}{\delta x} (s,x,s)) \Phi(x) \Phi(x) \delta y d t + \text{correlation term} \quad \beta = \gamma \sqrt{2a}
\]
\[ E(x) = \frac{1}{2} \left[ x^2 \frac{\partial^2 \phi}{\partial x^2} (t, x(t), \phi(x(t))) - x^2 \frac{\partial^2 \phi}{\partial x^2} (t, x(t), \phi(x(t))) \right] \]

\[ \int_0^t \phi(x(s)) \left( X_s^2 \frac{\partial^2 \phi}{\partial x^2} \right) ds \]

\[ \frac{v}{\sqrt{2\alpha}} \int_0^t f(x(s), \phi(x(s))) \left( 2X_s \frac{\partial \phi}{\partial x} + X_s^2 \frac{\partial^2 \phi}{\partial x^2} \right) ds \]

\[ \frac{v}{\sqrt{2\alpha}} \int_0^t \frac{X_s^2 \frac{\partial^2 \phi}{\partial x^2} \delta(x(s))}{dx} \]

\[ = \mathcal{O}(\frac{1}{\sqrt{\alpha}}) \]

\[ E(x) \text{ has the form } \]

\[ \frac{1}{\sqrt{\alpha}} (B_t + M_t) + \mathcal{O}(\frac{1}{\sqrt{\alpha}}) \]

\[ B_t = -\frac{v}{\sqrt{2\alpha}} \int_0^t f(x(s), \phi(x(s))) \left( 2X_s \frac{\partial \phi}{\partial x} + X_s^2 \frac{\partial^2 \phi}{\partial x^2} \right) ds \]

\[ M_t = -\frac{v}{\sqrt{2\alpha}} \int_0^t \frac{X_s^2 \frac{\partial^2 \phi}{\partial x^2} \delta(x(s))}{dx} \]

\[ \text{The BS strategy } \Rightarrow \text{ not mean self-financing to } \sigma^2 \]

\[ \text{c.f. } B^B_\infty = 0 \]

\[ \text{NB: a better hedging strategy should have the mean } - \text{variance.} \]
Mean Self-financing Hedging Strategy

Q: How to finish \( B_t \) to the next order to achieve \( \alpha \)-menu?

Consider
\[
\begin{align*}
A_t &= \frac{\phi(x_t; \tilde{m} = \frac{\delta}{\sigma^2})}{\sigma^2}(\tilde{m}, \tilde{x}_t) \quad \text{phase of risky asset:} \\
B_t &= e^{-\frac{\sigma^2}{2}} \left( \phi(x_t; \tilde{m}) + \phi(x_t; \tilde{m}) - \phi(x_t; \tilde{m}) \right) + \phi(x_t; \tilde{m}) \quad \text{bank}
\end{align*}
\]

Recall
\[
V_3 = \frac{\rho V}{\mu^{2/3}} <f(x)>
\]

\[
B_+ = -\frac{\rho V}{\mu^{1/3}} \int_0^+ \frac{\phi(x_s; \tilde{m}) \phi(x_s; \tilde{m})}{x_s^{2/3}} \left( 2x_s^{1/3} \sigma^2 \phi(x_s; \tilde{m}) + x_s^{2/3} \phi(x_s; \tilde{m}) \right) dx_s
\]

Try
\[
\begin{align*}
\bar{x}_s(x, \tilde{T}) &= V_3 \left( 2x^{2/3} \sigma^2 \phi(x_s; \tilde{m}) + x^{2/3} \phi(x_s; \tilde{m}) \right) \\
\tilde{x}(\tilde{T}, x) &= 0 \quad \text{replication at maturity.}
\end{align*}
\]

\[
\begin{align*}
\bar{x}(\tilde{T}, x) &= -(\tilde{T} - t) V_3 \left( 2x^{2/3} \sigma^2 \phi(x_s; \tilde{m}) + x^{2/3} \phi(x_s; \tilde{m}) \right) \\
\tilde{x}(\tilde{T}, x) &= -(\tilde{T} - t) V_3 \left( 2x^{2/3} \sigma^2 \phi(x_s; \tilde{m}) + x^{2/3} \phi(x_s; \tilde{m}) \right)
\end{align*}
\]

NB: \( \bar{x} \sim \mathcal{N}(\frac{-1}{\alpha}) \)

\[
A_t = \frac{\partial B}{\partial x} - \frac{\partial (\tilde{B}_- T - \phi(x_t; \tilde{m}))}{\partial \tilde{m}} \left( \frac{\partial^2 \phi(x_t; \tilde{m})}{\partial x^2} + \frac{5x^2}{\phi(x_t; \tilde{m})} \frac{\partial \phi(x_t; \tilde{m})}{\partial x} + x^2 \frac{\partial^2 \phi(x_t; \tilde{m})}{\partial x^2} \right)
\]

\[
\begin{align*}
\text{Gamma} & = \tilde{\epsilon} \left( \frac{\sigma^2 \phi(x_t; \tilde{m})}{\sigma^2} + 5x^2 \frac{\phi(x_t; \tilde{m})}{\sigma^2} + x^4 \frac{\partial^2 \phi(x_t; \tilde{m})}{\partial x^2} \right) \\
\text{Sipol} & = \frac{\phi(x_t; \tilde{m})}{\sigma^2} \\
\text{Kapp} & = \frac{2x^2 \phi(x_t; \tilde{m})}{\sigma^2}
\end{align*}
\]
\[ b_t = e^{-\gamma t} ( P_0 - A_t x_t) \]

\[ \Delta E_t (W) = \frac{1}{2} \int_0^T (f(\gamma y) - \sigma^2) X_t^2 \frac{\partial^2 f}{\partial \gamma^2} \, dt + \frac{1}{2} \int_0^T (f(\gamma y) - \sigma^2) X_t^2 \frac{\partial^2 f}{\partial \gamma \partial x} \, dt + \int_0^T V_2 \left( 2 X_t^2 \frac{\partial^2 f}{\partial x^2} + X_t^3 \frac{\partial^2 f}{\partial x^3} \right) dt \]

\[ \text{(I)} \]

\[ \text{NB: the 1st term is the original } E_t (X) \text{ (under BS strategy)} \]

\[ \text{which is } \quad E_t (X) = \frac{1}{\alpha^2} (B_t + M_t) + O (\sqrt{t}) \]

\[ \text{NB: } \quad \frac{1}{\alpha^2} \text{P}_t = -\frac{\alpha^2}{\alpha^2} \int_0^T \bar{X}_t (\gamma) \bar{X}_t (\gamma) \left( 2 X_t^2 \frac{\partial^2 f}{\partial x^2} + X_t^3 \frac{\partial^2 f}{\partial x^3} \right) \, ds \]

\[ \text{NB: the } B_t \text{-term combine with term } (\text{III}) \text{ to give} \]

\[ \frac{1}{\alpha^2} \int_0^T \left[ 2 X_t^2 \frac{\partial^2 f}{\partial x^2} + X_t^3 \frac{\partial^2 f}{\partial x^3} \right] \left[ \langle \mathbf{f} \mathbf{y} \rangle - \bar{f}(\gamma) \bar{X}_t (\gamma) \right] \, ds \quad \\
\text{[which, integral } O (\frac{1}{\alpha^2}) \text{] Recall } V_2 = \frac{\alpha^2}{\alpha^2} \langle \mathbf{f} \mathbf{y} \rangle \]

\[ \text{is Again a form of averaging} \]
\[ F_0(T) = \frac{1}{\sqrt{2\pi}} \left( M_T + O(\varepsilon^1) \right) \]

\[ = \frac{1}{\sqrt{2\pi}} \int_0^T \frac{\gamma^2}{\sqrt{\text{var}}} \frac{\partial^2}{\partial X^2} \delta(X_\tau) d\tau + O(\varepsilon^1) \]

zero mean (i.e. the bias is removed in hedging)

nb: \( \frac{1}{\sqrt{2\pi}} \text{Mt} - \text{non-hedgeable, due to stochastic volatility} \)

nb: Now the new strategy is mean-self-financing to order \( O(\varepsilon^1) \).

nb: Implementation of this hedging strategy need only \( V_3 \) and \( \bar{F} \).