RATIONALITY OF MODULI OF ELLIPTIC FIBRATIONS WITH FIXED MONODROMY

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Abstract. We prove rationality results for moduli spaces of elliptic K3 surfaces and elliptic rational surfaces with fixed monodromy groups.

0. Introduction

We will study moduli spaces of smooth non-isotrivial Jacobian elliptic fibrations over curves

$$\pi : \mathcal{E} \to C$$

with fixed global monodromy group $\tilde{\Gamma} = \tilde{\Gamma}(\mathcal{E}) \subset \mathrm{SL}_2(\mathbb{Z})$. In [4] we developed techniques aimed at the classification of possible global monodromies $\tilde{\Gamma}$. The present paper gives a natural application of these techniques.

We work over $\mathbb{C}$. Let $\mathcal{B}$ be an irreducible algebraic family of Jacobian elliptic surfaces. Then the set of subgroups $\tilde{\Gamma} \subset \mathrm{SL}_2(\mathbb{Z})$ such that $\tilde{\Gamma}$ is the (global) monodromy group of some $\mathcal{E}$ in this family is finite. Moreover, for every such group $\tilde{\Gamma}$ the subset of fibrations with this monodromy

$$\mathcal{B}_{\tilde{\Gamma}} := \{b \in \mathcal{B} \mid \tilde{\Gamma}(\mathcal{E}_b) = \tilde{\Gamma}\}$$

is an algebraic (not necessarily closed) subvariety of $\mathcal{B}$.

Generalizing this observation, we introduce (maximal) *parameter spaces* $\mathcal{F}_{\tilde{\Gamma}}$ of elliptic fibrations with fixed global monodromy $\tilde{\Gamma}$ (considered up to fiberwise birational transformations acting trivially on the base of the elliptic fibration). These parameter spaces can be represented as quotients of quasi-projective varieties by algebraic groups. In particular, we can consider *irreducible connected components* of the

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parameter space $\mathcal{F}_{\tilde{\Gamma}}$, which we call *moduli spaces.* Even though these moduli spaces need not be algebraic varieties, we can still make sense of their birational type.

Theorem. *Let $\tilde{\Gamma} \subset \mathrm{SL}_2(\mathbb{Z})$ be a proper subgroup of finite index. Then all moduli spaces of elliptic rational or Jacobian elliptic K3 surfaces with global monodromy $\tilde{\Gamma}$ are rational.*

Corollary. *For all $\tilde{\Gamma}$ with moduli $\mathcal{F}_{\tilde{\Gamma}}$ of dimension > 0 there exists a number field K such that there are infinitely many elliptic K3 surfaces over K with global monodromy $\tilde{\Gamma}$.*

We proceed to give a more detailed description of our approach. First of all, we can work not with the monodromy group $\tilde{\Gamma}$ itself but rather with its image

$$\Gamma \subset \mathrm{PSL}_2(\mathbb{Z})$$

under the natural projection $\mathrm{SL}_2(\mathbb{Z}) \twoheadrightarrow \mathrm{PSL}_2(\mathbb{Z})$. Denote by

$$\mathcal{H} = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$$

the upper half-plane and by

$$\overline{\mathcal{H}} = \mathcal{H} \cup \mathbb{Q} \cup \{\infty\}.$$

The natural j-map

$$j \,:\, C \to \mathbb{P}^1 = \overline{\mathcal{H}}/\mathrm{PSL}_2(\mathbb{Z})$$

decomposes as

$$j = j_\Gamma \circ j_{\mathcal{E}},$$

where

$$\begin{array}{rcccc} j_{\mathcal{E}}: & C & \to & M_\Gamma & = & \overline{\mathcal{H}}/\Gamma \\ j_\Gamma: & M_\Gamma & \to & \mathbb{P}^1 & = & \overline{\mathcal{H}}/\mathrm{PSL}_2(\mathbb{Z}). \end{array}$$

Here we denoted by M_Γ the *j-modular curve* corresponding to Γ; it is equipped with a special triangulation, obtained as the pullback of the standard triangulation of $S^2 = \mathbb{P}^1(\mathbb{C})$ (by two triangles with vertices at $0, 1$ and ∞) under the map j_Γ (which ramifies only over $0, 1$ and ∞). We call the obtained triangulation of M_Γ a j_Γ-triangulation. Denote by T_Γ the preimage in M_Γ of the closed interval $[0, 1] \subset \mathbb{P}^1$. The graph T_Γ is our main tool in the combinatorial analysis of Γ.

Denote by $\chi(\mathcal{E})$ the Euler characteristic of $\mathcal{E}$. It splits equivalence classes of Jacobian elliptic surfaces (modulo fiberwise birational transformations) into *algebraic families.* In particular, if $C = \mathbb{P}^1$ then the *algebraic variety* $\mathcal{F}_r$ parametrizing (equivalence classes of) Jacobian elliptic surfaces with given $\chi(\mathcal{E})$ is irreducible; here we put $r = \chi(\mathcal{E})/12$. Our goal is to analyze the birational type of (irreducible components)

$$\mathcal{F}_{r,\tilde{\Gamma}} \subset \mathcal{F}_r$$

parametrizing fibrations with fixed monodromy group $\tilde{\Gamma}$. It suffices to study parameter spaces $\mathcal{F}_{r,\Gamma}$ corresponding to $\Gamma \subset \mathrm{PSL}_2(\mathbb{Z})$, since every irreducible component of $\mathcal{F}_{r,\tilde{\Gamma}}$ coincides with a component of $\mathcal{F}_{r,\Gamma}$.

From now on we assume that $C = M_\Gamma = \mathbb{P}^1$. Denote by $\mathcal{R}_{d,\Gamma}$ the space of rational maps $\mathbb{P}^1 \to \mathbb{P}^1$ (of degree d) with prescribed ramification (encoded in T_Γ). The spaces $\mathcal{F}_{r,\Gamma}$ are quotients by the action of $\mathrm{PGL}_2 \times H_\Gamma$ of fibrations over $\mathcal{R}_{d,\Gamma}$ with fibers (Zariski open subsets of) $\mathrm{Sym}^\ell(\mathbb{P}^1)$ (for appropriate d and ℓ). Here PGL_2 acts (on the left) by changing the parameter on the base $C = \mathbb{P}^1$ and H_Γ is the group of automorphisms of $M_\Gamma = \mathbb{P}^1$ stabilizing the embedded graph T_Γ (acting on the right). The nontriviality of H_Γ means that there is a $\Gamma' \subset \mathrm{PSL}_2(\mathbb{Z})$ containing Γ as a normal subgroup with $H_\Gamma = \Gamma'/\Gamma$. So in most cases in order to prove rationality of $\mathcal{F}_{r,\Gamma}$ it is sufficient to establish it for $\mathrm{PGL}_2\backslash\mathcal{R}_{d,\Gamma}$, which can be deduced from general rationality results for PGL_2-quotients (see [5], [7]). To cover *all* cases we need to set up a rather extensive combinatorial analysis.

Here is a roadmap of the paper. In Section 1 we discuss finite covers $M_\Gamma \to \mathbb{P}^1$ in the spirit of Grothendieck's "Dessins d'Enfants" program (see [9],[10] and the references therein) and introduce the invariants $\mathrm{GD}(\Gamma), \mathrm{RD}(\Gamma)$ and $\mathrm{ET}(\Gamma)$. In "ideal" cases $\mathrm{ET}(\Gamma)$ coincides with the number of triangles in the j_Γ-triangulation of M_Γ (the notation $\mathrm{ET}(\Gamma)$ stands for "*Effective Triangles*"). In Section 2 we recall basic facts about elliptic fibrations and introduce the invariant $\mathrm{ET}(\mathcal{E})$. For an "ideal" elliptic fibration one has $\mathrm{ET}(\Gamma) = \mathrm{ET}(\mathcal{E})$. In Section 3 we discuss *moduli* of elliptic fibrations with fixed monodromy. In Sections 4 and 5 we formulate and prove several rationality results for PGL_2 and related quotients. In Section 7 we classify families of rational elliptic surfaces and elliptic K3 surfaces with different monodromy groups. In Section 6, we study relations between the combinatoric of the graph Γ and the topology of $\mathcal{E}$. And finally, in Sections 10 and 8 we list (certain) relevant subgroups $\Gamma \subset \mathrm{PSL}_2(\mathbb{Z})$ (represented by trivalent graphs T_Γ). There are too many monodromy groups of elliptic K3 surfaces to be drawn on paper, but we show how to obtain them from our list by simple operations.

1. Finite covers

We keep the notations of the introduction. Let Γ be a subgroup of finite index in $\mathrm{PSL}_2(\mathbb{Z})$. Consider the map

$$\overline{\mathcal{H}}/\Gamma = M_\Gamma \xrightarrow{j_\Gamma} \mathbb{P}^1 = \overline{\mathcal{H}}/\mathrm{PSL}_2(\mathbb{Z}),$$

ramified over the points $0, 1, \infty \in \mathbb{P}^1$. Denote their preimages in M_Γ by A, B and I, respectively. The possible ramification orders are 3 or 1 for A-points, 2 or 1 for B-points and arbitrary for I-points. Joining points $0, 1$ and ∞ by three non-intersecting intervals in $\mathbb{P}^1(\mathbb{C})$ we obtain a decomposition of $S^2 = \mathbb{P}^1(\mathbb{C})$ into two triangles. It induces a special triangulation of M_Γ with vertices in A, B and I-points which we call the *j_Γ-triangulation.* The preimage of the segment $[0,1] \subset \mathbb{P}^1$ defines a graph T_Γ which determines the j_Γ-triangulation uniquely. Interior vertices of T_Γ are marked by A_6 and ends are marked by either A_2 or B_2.

Notations 1.1. The *graph datum* $\mathrm{GD}(\Gamma)$ of T_Γ is the formal sum

$$\mathrm{GD}(\Gamma) := [a_6A_6 + a_2A_2 + b_2B_2],$$

where a_i $(i = 6, 2)$ is the number of A_i vertices and b_2 is the number of B_2-vertices. Denote by $\tau^0 = \tau^0(\Gamma)$ the number of vertices of T_Γ (including the ends), by $\tau^1 = \tau^1(\Gamma)$ the number of edges and by $\tau^2 = \tau^2(\Gamma) = \pi_0(M_\Gamma \setminus T_\Gamma)$.

Remark 1.2. For given a_2, b_2 there is a unique group with

$$\mathrm{GD}(\Gamma) = [A_6 + a_2A_2 + b_2B_2].$$

Forgetting the markings of T_Γ we obtain a connected unmarked *topological* graph T_Γ^u with (possibly some) ends and all interior vertices of valency 3 — a *trivalent graph.*

Lemma 1.3. *Let X be a compact orientable Riemann surface of genus $g(X)$ and $T^u \subset X$ an embedding of a connected trivalent graph such that*

- *the set $X \setminus T^u$ is a disjoint union of topological cells;*
- *all interior vertices of T^u are trivalent;*
- *the ends of T^u are arbitrarily marked by two colors A_2 and B_2.*

Then there exist a subgroup $\Gamma \subset \mathrm{PSL}_2(\mathbb{Z})$ and a unique complex structure on X such that $X = M_\Gamma$ and $T^u = T_\Gamma^u$.

Proof. Assume that we have an embedded graph $T^u \subset X$ satisfying the conditions above. Mark by A all trivalent vertices and enlarge the graph T^u by putting a B-vertex in the middle of any edge bounded by two A-vertices. Put one I-vertex into every connected component of $X \setminus T^u$ and connect all I-vertices with A and B-vertices at the boundary of the corresponding domain. By assumption, every connected component of $X \setminus T^u$ is contractible. Consider the boundary of the individual cell. Every A-vertex of the boundary is connected by edges to B-vertices only. Similarly, the B-vertices are connected by edges only to A-vertices. Hence every triangle of the induced triangulation has vertices colored by three colors: A, B and I. This gives a j_Γ-triangulation of X. Following Alexander [1], we observe that a j_Γ-triangulation defines a map

$$h : X \to \mathbb{P}^1$$

which is cyclically ramified over A, B and I (see [4]). The trivalence of T^u implies that h has only 3 or 1-ramifications over $0 \in \mathbb{P}^1$ and only 2 or 1-ramifications over $1 \in \mathbb{P}^1$. Since $\mathrm{PSL}_2(\mathbb{Z}) = \mathbb{Z}_3 * \mathbb{Z}_2$ there is exactly one subgroup $\Gamma \subset \mathrm{PSL}_2(\mathbb{Z})$ (of finite index) which corresponds to the covering $X \to \mathbb{P}^1$. Any graph T^u_Γ constructed via a subgroup $\Gamma \subset \mathrm{PSL}_2(\mathbb{Z})$ satisfies the conditions above. Indeed, we have already described the j_Γ-triangulation on M_Γ. Triangles adjacent to a given I-vertex constitute a contractible cell. Thus the division of M_Γ into neighborhoods of I-vertices is a cellular decomposition of M_Γ. Hence after removing I-vertices with open edges from them we obtain the preimage of $[0, 1]$. If we forget the B-vertices which lie between two A-vertices we obtain the graph T^u_Γ. Thus $T^u_\Gamma \subset X = M_\Gamma$ is the boundary of this cellular decomposition and T_Γ is simply T^u_Γ with an A, B-marking of the ends. $\square$

Remark 1.4. Graphs which are isotopic in X (modulo diffeomorphisms of X of degree 1) define conjugated subgroups of $\mathrm{PSL}_2(\mathbb{Z})$.

Remark 1.5. Even if we omit the condition of compactness of X we still get a bijection between all subgroups of finite index of $\mathrm{PSL}_2(\mathbb{Z})$ and embedded trivalent graphs with marked ends.

Remark 1.6. The topology of X restricts the topology of T^u_Γ. The graph T^u_Γ must contain some 1-skeleton of X. Hence T^u_Γ can be a tree only if $X = S^2$. In particular, the map $\pi_1(T^u_\Gamma) \to \pi_1(X)$ is surjective.

For $X = \mathbb{P}^1$ the connectedness of T^u guarantees that all the components of $X \setminus T^u$ are contractible. Hence we can classify graphs in $X = \mathbb{P}^1$ by drawing them on the plane. In general, connectedness of T^u is necessary but not sufficient.

Definition 1.7. Define

$$\begin{aligned} \mathrm{ET}(\Gamma) &:= 6\tau^0 = 6(a_6 + a_2 + b_2) \\ \Delta(\Gamma) &:= 6a_6 + 2a_2 \end{aligned}$$

Thus both $\mathrm{ET}(\Gamma)$ and $\Delta(\Gamma)$ depend only on the marking of the ends but not on the embedding of the graph. Observe that $\Delta(\Gamma)$ is the *number of triangles* in the corresponding j_Γ-triangulation of M_Γ and that

$$[\mathrm{PSL}_2(\mathbb{Z}) : \Gamma] = \Delta(\Gamma)/2.$$

Remark 1.8. If M_Γ arises from an elliptic fibration as in the Introduction then $\Delta(\Gamma)/2$ equals the number of *Dehn twists* in Γ around the multiplicative singular fibers.

Notations 1.9. Let $f : C \to \mathbb{P}^1$ be a cover of degree d and $p \in \mathbb{P}^1$ a ramification point of f. The *local ramification datum* is an $\mathbb{N}$-valued vector $v = (v_k)$, $(\sum v_k = d)$, where v_k is the order of ramification of f at a point $c_k \in f^{-1}(p)$. A *reduced local ramification datum* is a vector $\overline{v}$ obtained from v by omitting all entries $v_k = 1$. The vector v is defined up to permutation of the entries.

For $f = j_{\mathcal{E}} : C \to M_\Gamma = \mathbb{P}^1$ we have distinguished ramification points, namely those over A- and B-vertices of the graph $T_\Gamma \subset M_\Gamma$. The (global) $j_{\mathcal{E}}$-ramification datum is the vector

$$\mathrm{RD}(j_{\mathcal{E}}) := [v_{1,A}, \ldots, v_{n,A}, v_{n+1,B}, \ldots, v_{n',B}, \overline{v}_{n'+1}, \ldots, \overline{v}_{n''}],$$

where the $v_{i,A}$ are local ramification data over A-vertices for $i = 1, \ldots, n$, (resp. $v_{i,B}$ for B-vertices, $i = n+1, \ldots, n'$) and $\overline{v}_i$ are *reduced* local ramification data for unspecified other points in M_Γ for $i > n'$ (distinct from A- and B-vertices of M_Γ).

For $f = j_\Gamma : M_\Gamma \to \mathbb{P}^1$ the distinguished (and the only) ramification points are $0, 1, \infty$. We write

$$\mathrm{RD}(j_\Gamma) := [v_0, v_1, v_\infty]$$

for the global j_Γ-ramification datum.

Example 1.10. Assume that $\mathrm{GD}(\Gamma) = [nA_6 + A_2 + B_2]$ is the graph datum of $T_\Gamma \subset M_\Gamma$ and let $j_{\mathcal{E}} : \mathbb{P}^1 \to M_\Gamma = \mathbb{P}^1$ be a finite cover. Then the $j_{\mathcal{E}}$-ramification datum

$$\mathrm{RD}(j_{\mathcal{E}}) = [(2,3)_A, (2,2,1)_B, (2), (2)]$$

means that $\deg(j_{\mathcal{E}}) = 5$, that $j_{\mathcal{E}}$ has ramification points of order 2 and 3 over one point $A_2 \in T_\Gamma$ and $(2,2,1)$ over one B_2-point and ramifications of order 2 over two other unspecified points in M_Γ.

2. Elliptic fibrations

In this section we briefly recall some basic facts of Kodaira's theory [8] of elliptic fibrations. For more details we refer to [2], [6] and [11]. Let

$$\pi\colon \mathcal{E} \to C$$

be a smooth non-isotrivial relatively minimal Jacobian elliptic fibration over a smooth projective curve C. This means that:

- $\mathcal{E}$ is a smooth compact complex projective surface and π is a proper holomorphic map;
- the generic fiber of π is a smooth curve of genus 1;
- the fibers of $\mathcal{E}$ do not contain exceptional curves of the first kind, i.e., rational curves F such that $(F^2) = -1$ (*relative minimality*);
- there exists a (global) zero section $s : C \to \mathcal{E}$ (*Jacobian elliptic fibration*);
- the j-function which assigns to each smooth fiber $\pi^{-1}(p) = \mathcal{E}_p \subset \mathcal{E}$ its j-invariant is a non-constant rational function on C (*non-isotriviality*).

It is well known that $s^2 < 0$. We define

$$\mathrm{ET}(\mathcal{E}) := -24s^2.$$

Lemma 2.1. *We have*

$$\mathrm{ET}(\mathcal{E})/2 = -12s^2 = \chi(\mathcal{E}) = c_2(\mathcal{E}).$$

Proof. Well known, but we decided to include an argument. Since $\mathcal{E}$ is smooth and relatively minimal the canonical bundle $K_{\mathcal{E}}$ of $\mathcal{E}$ is induced from a one-dimensional bundle K on the base C. The sheaf $\pi^*K(C)$ is a subsheaf of $K_{\mathcal{E}}$. Since there are singular fibers we have the following equality

$$h^0(\mathcal{E}, \Omega^1) = h^1(\mathcal{E}, \mathcal{O}) = g$$

where g is the genus of C. By Riemann-Roch we obtain

$$\chi(\mathcal{O}) = 1 - g + h^0(\mathcal{E}, K_{\mathcal{E}}) = \chi(\mathcal{E})/12.$$

We also know that $s^2 + sK_{\mathcal{E}} - 2g + 2 = 0$ (genus formula). Therefore,

$$1 - g + h^0(\mathcal{E}, K_{\mathcal{E}}) = \deg(K) - 2g + 2 = \chi(\mathcal{E})/12$$

since $\deg(K) > 2g - 2$ and hence $h^1(C, K) = 0$. Further,

$$sK_{\mathcal{E}} = \deg(K).$$

Thus $s^2 + sK_{\mathcal{E}} - 2g + 2 = 0$ transforms to $s^2 + \chi(\mathcal{E})/12 = 0$. □

Let $C^{\rm sing} = \{p_1, \dots, p_k\} \subset C$ be the set of points on the base corresponding to singular fibers. The topological Euler characteristic $\chi(\mathcal{E}) = c_2(\mathcal{E})$ is equal to the sum of Euler characteristics of the singular fibers $\mathcal{E}_{p_i} = \pi^{-1}(p_i)$ (since every generic fiber has Euler characteristic equal to 0). Therefore,

$$\mathrm{ET}(\mathcal{E}) = \sum_{p_i \in C^{\rm sing}} \mathrm{ET}(\mathcal{E}_{p_i}),$$

where the summation runs over all singular fibers of $\mathcal{E}$ and $\mathrm{ET}(\mathcal{E}_{p_i})$ is the contribution from the corresponding singular fiber. Since the fibration is Jacobian every singular fiber has a unique representative from Kodaira's list and it is defined by the local monodromy. The possible types of singular fibers and their ET-contributions are:

	ET		ET
I_0		I_0^*	12
I_n	$2n$	I_n^*	$2n+12$
II	4	IV^*	16
III	6	III^*	18
IV	8	II^*	20

Here I_0 is a smooth fiber, I_n is a multiplicative fiber with n-irreducible components. The types II, III and IV correspond to the case of potentially good reduction. More precisely, the neighborhood of such a fiber is a (desingularization of a) quotient of a local fibration with smooth fibers by an automorphism of finite order. The corresponding order is 4 for the case III and 3 in the cases II, IV. The fibers of type I_0^*, (resp. $\mathrm{I}_n^*, \mathrm{II}^*, \mathrm{III}^*, \mathrm{IV}^*$) are obtained from fibers I_0 (resp. $\mathrm{I}_n, \mathrm{IV}, \mathrm{III}, \mathrm{II}$) (after changing the local automorphism by the involution $x \mapsto -x$ in the local group structure of the fibration). We shall call them $*$-*fibers* in the sequel.

Remark 2.2. The local invariant $\mathrm{ET}(\mathcal{E}_p)$ has a monodromy interpretation. Namely, any element of a local monodromy at $p \in C^{\rm sing}$ has a minimal representation as a product of elements conjugated to $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ in $\mathrm{SL}_2(\mathbb{Z})$. The length of this representation equals $\mathrm{ET}(\mathcal{E}_p)/2$. This explains the equality $\mathrm{ET}(\mathcal{E}_p^*) = \mathrm{ET}(\mathcal{E}_p) + 12$ — the element $\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ is a product of 6 elements conjugated to $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ (elementary Dehn twists).

3. Moduli spaces

Every Jacobian elliptic fibration $\mathcal{E} \to \mathbb{P}^1$ admits a *Weierstrass model* $\bar{\mathcal{E}}$ (see [6], for example). Its geometric realization is given as follows:

there exists a pair of sections

$$\begin{aligned} g_2 &\in H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(4r)), \\ g_3 &\in H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(6r)) \end{aligned}$$

such that $\mathcal{E}$ is given by

$$y^2 z = 4x^3 - g_2 x z^2 - g_3 z^3, \tag{3.1}$$

subject to conditions

- the discriminant $\Delta = g_2^3 - 27g_3^2$ is not identically 0;
- for every point $p \in \mathbb{P}^1$ we have

$$\min(3\nu_p(g_2), 2\nu_p(g_3)) < 12, \tag{3.2}$$

  where ν_p is the valuation corresponding to $p \in \mathbb{P}^1$.

Two pairs (g_2, g_3) and (g_2', g_3') define isomorphic Jacobian elliptic surfaces $(\mathcal{E}, s)$ and $(\mathcal{E}', s')$ iff there exits an $h \in \mathrm{GL}_2(\mathbb{C})$ transforming (g_2, g_3) into (g_2', g_3') under the natural action of GL_2 on (the GL_2-linearized) $\mathcal{O}_{\mathbb{P}^1}(r)$. We define $\mathcal{F}_r$ as the set of isomorphism classes of pairs (g_2, g_3) subject to the conditions above.

The parameter space $\mathcal{F}_r$ has a natural structure of a (categorical) quotient of some open subvariety U_r of the sum of two linear GL_2-representations

$$H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(4r)) \oplus H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(6r))$$

by the action of GL_2. Equivalently, $\mathcal{F}_r$ is a (categorical) quotient of the open subvariety $U_r' = U_r/\mathrm{G}_m$ of the weighted projective space

$$\mathbb{P}_{4r,6r}(4r+1, 6r+1)$$

by the action of PGL_2.

Lemma 3.1. *The variety U_r' is a disjoint union of locally closed subvarieties $U'_{r,\tilde{\Gamma}}$, each preserved under the action of PGL_2, such that for every $u \in U'_{r,\tilde{\Gamma}}$ one has $\tilde{\Gamma}(\mathcal{E}_u) = \tilde{\Gamma}$.*

Proof. The set of (g_2, g_3) (of bounded degree) such that the j-map decomposes as $j_\Gamma \circ j_{\mathcal{E}}$ (with fixed Γ) is an algebraic variety. It contains a *finite* number of proper subvarieties corresponding to proper finite index subgroups of Γ (their number is bounded as a function of r). The complement to these subvarieties consists of finitely many irreducible components, each preserved under the action of PGL_2. For u in any of these components we have $\tilde{\Gamma}(\mathcal{E}_u) = \tilde{\Gamma}$. □

The unstable points of the PGL_2-action on the weighted projective space correspond to sections g_2, g_3 with high order of vanishing at some point p. Namely $\nu_p(g_2) > 2r, \nu_p(g_3) > 3r$. However, the inequality (3.2) implies that $6r < 12$. Thus, for $r \geq 2$, $\mathcal{F}_r$ is a PGL_2-quotient of some open subvariety of the semistable locus

$$\mathbb{P}^{\mathrm{ss}}_{4r,6r}(4r+1, 6r+1) \subset \mathbb{P}_{4r,6r}(4r+1, 6r+1).$$

It follows that $\mathcal{F}_r$ is a quasi-projective algebraic variety. This variety is clearly unirational and in fact rational by [7].

Moreover, for $r \geq 2$ we can define a set of *subvarieties* $\mathcal{F}_{r,\tilde{\Gamma}} \subset \mathcal{F}_r$ such that for every $b \in \mathcal{F}_{r,\tilde{\Gamma}}$ the corresponding Jacobian elliptic surface $(\mathcal{E}_b, s)$ has global monodromy group $\tilde{\Gamma}$.

Remark 3.2. Notice that the maps $j_{\mathcal{E}}$ for elliptic fibrations corresponding to different points of the same irreducible component of $\mathcal{F}_{r,\tilde{\Gamma}}$ can have different $\mathrm{RD}(j_{\mathcal{E}})$, even over the A_2 or B_2-ends of $T_\Gamma \subset M_\Gamma$. Thus, for a given irreducible component, we have the notion of a *generic* ramification datum $\mathrm{RD}(j_{\mathcal{E}})$ and its *degenerations*.

The case $r = 1$, corresponding to rational elliptic surfaces, is more subtle - the subvariety U_1' contains unstable points. The quasi-projective locus of semistable points $U_r^{\mathrm{ss}\prime}$ is a disjoint union of locally closed PGL_2-semistable subsets $U^{\mathrm{ss}}_{r,\tilde{\Gamma}}$; taking quotients we obtain varieties $\mathcal{F}_{1,\tilde{\Gamma}}$ parametrizing rational elliptic fibrations with global monodromy $\tilde{\Gamma}$.

Let $W_1' = U_1' - U_1^{\mathrm{ss}\prime}$ be the complement. It consists of pairs (g_2, g_3) with

$$g_2 = l^3 f_2, \quad g_3 = l^4 f_3,$$

where l is a linear form (vanishing at a point p and) coprime to f_2, f_3 and $\deg(f_2) = 1, \deg(f_3) = 2$. For $w \in W_1'$ we have $\deg(j) \leq 4$. The case of $\tilde{\Gamma} \neq \mathrm{SL}_2(\mathbb{Z})$ corresponds to $\deg(j_\Gamma) \geq 2$. Thus we have to consider two cases:

- $\deg(j_\Gamma) = \deg(j_{\mathcal{E}}) = 2$;
- $\deg(j_\Gamma) \leq 4, \deg(j_{\mathcal{E}}) = 1$.

The first case does not occur since $j^{-1}(0)$ has ramification of type $(3,1)$ (by assumption that f_2 is coprime to l and that $3\nu_p(g_2) < 12$). Thus the j-map cannot be decomposed even locally into a product of two maps. The second case leads to

Lemma 3.3. *If $w \in W_1'$ and $\tilde{\Gamma}(\mathcal{E}_w) \neq \mathrm{SL}_2(\mathbb{Z})$ then $\deg(j_{\mathcal{E}}) = 1$ and one has one of the following graph and ramification data:*

$\mathrm{GD}(\Gamma)$	$\mathrm{RD}(j_\Gamma)$
$[A_6 + A_2]$	$[(3,1)_0, (2,2)_1, (3,1)_\infty]$
$[A_6 + A_2 + 2B_2]$	$[(3,1)_0, (2,1,1)_1, (4)_\infty]$
$[A_6 + 3B_2]$	$[(3)_0, (1,1,1)_1, (3)_\infty]$
$[A_6 + B_2]$	$[(3)_0, (2,1)_1, (2,1)_\infty]$

Proof. The formula $j = lf_2^3/(lf_2^3 - f_3^2)$ shows that j_Γ has a point with local ramification datum $(3,1)$ or (3), corresponding to

$$[\mathrm{PSL}_2(\mathbb{Z}) : \Gamma(\mathcal{E})] = 4 \text{ or } 3.$$

Since only two more branch points are allowed and one of them is 1 (with local ramifications 1 or 2), the Euler characteristic computation gives the ramification data listed in the statement plus one more:

$$[(3,1)_0, (2,2)_1, (2,2)_\infty].$$

However, this datum is impossible for topological reasons (the only possible graph datum is $[A_6 + A_2]$ and there is a unique embedded graph T_Γ with this datum).

If $\deg(j) = 3$ then one has a cyclic point of order 3, leading to the data above. ☐

Corollary 3.4. *Every irreducible component $W'_{1,\tilde{\Gamma}} \subset W_1'$ such that $\tilde{\Gamma}(\mathcal{E}_w) \neq \mathrm{SL}_2(\mathbb{Z})$ for $w \in W'_{1,\tilde{\Gamma}}$ is rational.*

Consider an irreducible component $\mathcal{F}_{r,\tilde{\Gamma}}$ and the corresponding decomposition $j = j_\Gamma \circ j_{\mathcal{E}}$. Here

$$j_{\mathcal{E}} = (j_{\mathcal{E},2}, j_{\mathcal{E},3}) \,:\, \mathbb{P}^1 \to M_\Gamma = \mathbb{P}^1$$

is a pair of homogeneous polynomials in 2 variables. Let

$$\mathcal{G} = \{(g_2, g_3)\} \subset H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(4r)) \oplus H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(6r))$$

be the subset corresponding to smooth elliptic fibrations. Put

$$\mathcal{J}_\Gamma := \{j \mid \exists j_{\mathcal{E}} \,:\, \mathbb{P}^1 \to M_\Gamma \; s.t. \; j = j_\Gamma \circ j_{\mathcal{E}}\}.$$

Lemma 3.5. *If $j \in \mathcal{J}_\Gamma \cap \mathcal{J}_{\Gamma'}$ with $\Gamma \neq \Gamma'$ then there exist an $h \in \mathrm{PSL}_2(\mathbb{Z})$, a group $\Gamma'' \subset \Gamma \cap h\Gamma' h^{-1}$ and a map $j''_{\mathcal{E}} \,:\, \mathbb{P}^1 \to M_{\Gamma''}$ such that $j = j_{\mathcal{E}''} \circ j_{\Gamma''}$.*

Proof. The monodromy group and its image in $\mathrm{PSL}_2(\mathbb{Z})$ are uniquely determined by the smooth part of the elliptic fibration. Therefore, in any smooth family of elliptic surfaces

$$\Gamma(\text{generic fiber}) \supseteq \Gamma(\text{special fiber}).$$

Since Γ is defined modulo conjugation by elements in $\mathrm{SL}_2(\mathbb{Z})$ the claim follows. $\square$

Corollary 3.6. *We have a decomposition $\mathcal{G} = \bigsqcup \mathcal{G}_\Gamma$ into a finite (disjoint) union of algebraic GL_2-stable subvarieties such that for all $g = (g_2, g_3) \in \mathcal{G}_\Gamma$ the monodromy group $\tilde{\Gamma}(\mathcal{E}_g) \subset \mathrm{SL}_2(\mathbb{Z})$ is a subgroup of a central $\mathbb{Z}/2$-extension of Γ.*

Remark 3.7. For a given $g \in \mathcal{G}_\Gamma$ the map $j_{\mathcal{E}}$ is not unique. Let $j_{\mathcal{E}}$ and $j'_{\mathcal{E}}$ be two such maps. Then $j_{\mathcal{E}} = h_\Gamma \circ j'_{\mathcal{E}}$ where $h_\Gamma \in \mathrm{Aut}(T_\Gamma)$ is an automorphism of M_Γ, preserving T_Γ.

Lemma 3.8. *We have a decomposition*

$$\mathcal{G}_\Gamma = \bigsqcup_k \mathcal{G}_{\tilde{\Gamma},k}$$

into a finite union of algebraic irreducible *GL_2-stable subvarieties such that $\tilde{\Gamma}(\mathcal{E}_g) = \tilde{\Gamma}$ for all $g \in \mathcal{G}_{\tilde{\Gamma},k}$.*

Proof. Assume that some $g \in \mathcal{G}_\Gamma$ belongs to $\mathcal{G}_{\tilde{\Gamma},1} \cap \mathcal{G}_{\tilde{\Gamma},2}$, where $\mathcal{G}_{\tilde{\Gamma},1}, \mathcal{G}_{\tilde{\Gamma},2}$ are different (non-conjugated) lifts of Γ into $\mathrm{SL}_2(\mathbb{Z})$. Lemma 3.5 implies that there exists a *proper* subgroup $\Gamma'' \subset \Gamma$ such that g belongs to $\mathcal{G}_{\Gamma''}$, contradiction. $\square$

Let $\mathcal{G}_{\tilde{\Gamma}} = \mathcal{G}_{\tilde{\Gamma},k}$ be an irreducible component of $\mathcal{G}_\Gamma$ as in Lemma 3.8 and $g \in \mathcal{G}_{\tilde{\Gamma}}$ its generic point. It determines a set of $*$-fibers on the base $\mathbb{P}^1$. We denote their number by ℓ. Choose (one of) the $j_{\mathcal{E}_g}$, with ramification datum $\mathrm{RD} = \mathrm{RD}(j_{\mathcal{E}_g})$. We get a map

$$\phi_{\mathcal{U}} \ : \ \mathcal{U}_g \to \mathcal{U}_{j_g} \times (\mathbb{P}^1)^\ell,$$

where $\mathcal{U}_g \subset \mathcal{G}_{\tilde{\Gamma}}$ is a neighborhood of g and $\mathcal{U}_{j_g} \subset \mathcal{R}(\mathrm{RD})$ is a neighborhood of the map $j_g = j_{\mathcal{E}_g}$ in the space

$$\mathcal{R}(\mathrm{RD}) := \{ j \ : \ \mathbb{P}^1 \to \mathbb{P}^1 \mid \mathrm{RD}(j) = \mathrm{RD} \}$$

of rational maps with ramification datum RD.

Lemma 3.9. *The map $\phi_{\mathcal{U}}$ is a local (complex analytic) surjection.*

Proof. First observe that on M_Γ there is a projective local system $(\mathbb{Z} \oplus \mathbb{Z})/(\mathbb{Z}/2)$ which induces a projective local system on an open part of $\mathbb{P}^1$. The obstruction to the extension of this system to a linear $\mathbb{Z} \oplus \mathbb{Z}$-system is an integer modulo 2 which depends only on the topological type of the projective system. Therefore, it doesn't change under a small variation of maps j with fixed RD. Now it suffices to apply Kodaira's main theorem which guarantees the existence and uniqueness of an elliptic fibration with a given linear $\mathbb{Z} \oplus \mathbb{Z}$-system. $\square$

Corollary 3.10. *Let $\mathcal{F}'_{r,\tilde{\Gamma}} \subset \mathcal{F}_{r,\tilde{\Gamma}}$ be an (irreducible) component with generic ramification datum* RD. *Then $\mathcal{F}'_{r,\tilde{\Gamma}}$ surjects (rationally) onto the quotient of the variety of rational maps $\mathcal{R}(\mathrm{RD})$ by H_Γ.*

Proof. Since both $\mathcal{F}'_{r,\tilde{\Gamma}}$ and $\mathcal{R}(\mathrm{RD})$ are algebraic varieties the local complex analytic surjection from Lemma 3.9 extends to an algebraic correspondence. Moreover, two decompositions of the map j as $j = j_\Gamma \circ j_\mathcal{E}$ differ by an element in H_Γ. This gives a map to the quotient space, which is a (global) rational surjection. □

Lemma 3.11. *Every irreducible component $\mathcal{F}_{r,\tilde{\Gamma}}$ contains an open part $\mathcal{F}'_{r,\tilde{\Gamma}}$ with the following properties:*

- *$\mathcal{F}'_{r,\tilde{\Gamma}}$ is a quotient of an algebraic variety $U'_\ell := U'_{r,\tilde{\Gamma},\ell}$ by the (left) action of* GL_2 *and (right) action of a subgroup H_Γ of* $\mathrm{Aut}(T_\Gamma)$;
- *U'_ℓ admits a fibration with fiber (a Zariski open subset of)* $\mathrm{Sym}^\ell(\mathbb{P}^1)$ *and base the variety of maps $f : \mathbb{P}^1 \to M_\Gamma$ with fixed local ramification data over A_2 and B_2-points of $T_\Gamma \subset M_\Gamma$;*
- *the action of* PGL_2 *on U'_ℓ is induced from the standard* PGL_2*-action on* $\mathbb{P}^1$;
- *the group* $\mathrm{Aut}(T_\Gamma)$ *is a subgroup of* $\mathrm{PGL}_2(\mathbb{C})$ *(acting on M_Γ).*

Proof. Elliptic surfaces parametrized by a smooth irreducible variety have the same $\mathrm{ET}(\mathcal{E})$, which depends on the number ℓ of $*$-fibers in $\mathcal{E}$, on the degree of $j_\mathcal{E}$ and on the ramification properties over the ends of T_Γ. Once ℓ is fixed, for any given $j_\mathcal{E}$, the $*$-mark can be placed over arbitrary ℓ-points of $\mathbb{P}^1$. Their position defines a unique surface $\mathcal{E}$. This implies that U'_ℓ is fibered with fibers (birationally) isomorphic to $\mathrm{Sym}^\ell(\mathbb{P}^1) = \mathbb{P}^\ell$. The ramification properties of $j_\mathcal{E}$ remain the same on the open part of U'_ℓ (since the number of $*$-fibers remains the same). Thus the base of the above fibration is the space of rational maps $f : \mathbb{P}^1 \to \mathbb{P}^1 = M_\Gamma$ with fixed ramification locus. Any such map defines an elliptic surface $\mathcal{E}$ with given Γ (see [4]). The PGL_2-action on $U'_{r,\tilde{\Gamma},\ell}$ identifies points corresponding to isomorphic surfaces $\mathcal{E}$. Additional nontrivial isomorphisms correspond to exterior automorphisms of Γ, coming from the action on M_Γ, i.e., automorphisms of the graph T_Γ. □

Remark 3.12. If the $\mathrm{PGL}_2 \times \mathrm{Aut}(T_\Gamma)$-action on U'_1 is generically free then the rationality of $\mathrm{PGL}_2 \backslash U'_1 / \mathrm{Aut}(T_\Gamma)$ implies the rationality the corresponding quotients for all ℓ. When the action is generically nonfree the degrees of $j_\mathcal{E}$ are small and all cases can be handled separately.

Most of the graphs T_Γ have trivial automorphisms. In particular, any nontrivial automorphism acts on the ends of the graph. In general, automorphisms of the pair (M_Γ, T_Γ) correspond to elements of Γ'/Γ where $\Gamma' \subset \mathrm{PSL}_2(\mathbb{Z})$ is a maximal subgroup with the property that Γ is a normal subgroup of Γ'.

Lemma 3.13. *The group* $\mathrm{Aut}(T_\Gamma)$ *acts freely on the set of ends and end-loops.*

Proof. Consider $j_\Gamma : M_\Gamma \to \mathbb{P}^1$. Then $g \in \mathrm{Aut}(T_\Gamma)$ is any element in $\mathrm{PGL}_2(\mathbb{C})$ such that $j_\Gamma(gz) = j_\Gamma(z)$ for all $z \in T_\Gamma \subset M_\Gamma$. If g stabilizes an end or an end-loop of T_Γ then it stabilizes the unique adjacent vertex and its other end. Any element of $\mathrm{PGL}_2(\mathbb{C})$ preserving a closed interval is the identity. □

Lemma 3.14. *Let*

$$\mathcal{R} := \{f : \mathbb{P}^1 \to \mathbb{P}^1\}$$

be the space of rational maps with ramifications over exactly 0 *and* ∞*. Then* $\mathcal{R}$ *is a* G_m*-fibration over the product of symmetric spaces* $\mathrm{Sym}^{m_i}(\mathbb{P}^1)$*.*

Proof. Indeed any two cycles c_1 and c_2 of fixed degree are equivalent on $\mathbb{P}^1$. Therefore, there is a rational function f on $\mathbb{P}^1$ with $c_1 = f^{-1}(0)$ and $c_2 = f^{-1}(\infty)$. If c_1, c_2 do not intersect then $\deg(f) = \deg(c_1) = \deg(c_2)$. The function f is defined modulo multiplication by a constant. The space of cycles $c_1 = \sum_i n_i p_i$ is a product of symmetric powers $\mathrm{Sym}^m(\mathbb{P}^1)$ where m is the number of equal n_i. □

4. General rationality results

Let X be an algebraic variety of dimension n over $\mathbb{C}$. We say that X is rational if its function field $\mathbb{C}(X)$ is isomorphic to $\mathbb{C}(x_1, \dots, x_n)$. We say that X is stably rational if $\mathbb{C}(X)(x_1, \dots, x_k)$ is isomorphic to $\mathbb{C}(x_1, \dots, x_{n+k})$ (for some $k \in \mathbb{N}$).

Consider the space

$$\mathcal{R}_\ell := \{f_1/f_2 \mid f_1, f_2 \in \mathrm{Sym}^\ell(V),\ \gcd(f_1, f_2) = 1\},$$

of rational functions on $\mathbb{P}^1$ of degree ℓ (here $V = \langle x, y \rangle$ is a vector space over $\mathbb{C}$). The polynomials f_1, f_2 are well defined modulo (simultaneous) multiplication by a constant. Thus

$$\mathcal{R}_\ell \subset (\mathrm{Sym}^\ell(V) \oplus \mathrm{Sym}^\ell(V))/\mathrm{G}_m$$

(with G_m acting diagonally). The space $\mathrm{Sym}^\ell(V) \oplus \mathrm{Sym}^\ell(V)$ carries a natural GL_2-action on the left and a commuting action of GL_2 on the right: the left action corresponds to changes of coordinates in V and the right action corresponds to the action on the target space $\mathbb{P}^1 = M_\Gamma$. This gives an irreducible representation of $\mathrm{GL}_2 \times \mathrm{GL}_2$.

We will also consider the spaces

$$\mathcal{R}_\ell(r_1, r_2) \subset (\mathrm{Sym}^\ell(V) \oplus \mathrm{Sym}^\ell(V))/\mathrm{G}_m$$

of those pairs of polynomials (f_1, f_2) such that

$$f_i = \prod_k (\lambda_k x - \mu_k y)^{r_{ik}}$$

with $r_i = (r_{ik})$ (and $r_{ik} \in \mathbb{N}$). By convention, $\mathcal{R}_\ell = \mathcal{R}_\ell^r(0,0)$. Notice that for $(r_1, r_2) \neq (0,0)$ the spaces $\mathcal{R}_\ell^r(r_1, r_2)$ do not carry a right action of GL_2 (but do carry the "antidiagonal" action of G_m). For $r_1 = r_2$ they carry a right action of $\mathbb{Z}/2 \times \mathrm{G}_m$.

Proposition 4.1. *The quotient of (the semi-stable locus of) $\mathcal{R}_\ell(r_1, r_2)$ by the left GL_2-action is rational.*

Proof. Follows from [5] and [7]. □

Proposition 4.2. *Let H be a finite cyclic subgroup of $\mathrm{SL}_2(\mathbb{C})$. The quotient of (the semi-stable locus of) $\mathcal{R}_\ell$ by the left GL_2 and right H-action is rational.*

Proof. Observe that we have a decomposition of $\mathrm{GL}_2 \times H$-modules

$$V \oplus V = (\mathbb{C}[\lambda] \oplus \mathbb{C}[\lambda^{-1}]) \otimes V.$$

Therefore, the quotient

$$\mathrm{GL}_2 \backslash \mathcal{R}_\ell / H$$

is birational to the quotient of a line bundle $\mathcal{O}_{\mathcal{P}}(1,1)$, where

$$\mathcal{P} := \mathbb{P}(\mathrm{Sym}(V)) \times \mathbb{P}(\mathrm{Sym}(V)),$$

which carries the obvious PGL_2-action on the left and a fiberwise linear action of H on the right. The quotient

$$\mathrm{PGL}_2 \backslash \mathcal{O}_{\mathcal{P}}(1,1)$$

is a line bundle over

$$\mathrm{PGL}_2 \backslash \mathbb{P}(\mathrm{Sym}(V)) \times \mathbb{P}(\mathrm{Sym}(V)),$$

and, therefore, rational by [7]. The quotient of (the total space of) a line bundle on a variety by a fiberwise linear action of a cyclic group is again a line bundle on this variety, hence rational in this case. □

Proposition 4.3. *The quotient of (the semi-stable locus of) $\mathcal{R}_\ell(r, r)$ by the left GL_2-action and right $\mathbb{Z}/2$-action is rational.*

Proof. Denote by

$$\mathcal{P} = \mathcal{P}_\ell(r,r) = \prod_j \mathbb{P}(\mathrm{Sym}^{r_j}(V)) \times \prod_j \mathbb{P}(\mathrm{Sym}^{r_j}(V))$$

and by

$$\mathcal{O} = \mathcal{O}_{\mathcal{P}}^*.$$

Then $\mathcal{R} = \mathcal{R}_\ell(r,r)$ is equal to

$$\mathrm{PGL}_2 \backslash \mathcal{R}/(\mathbb{Z}/2).$$

Assume that the PGL_2-action on $\prod_j \mathbb{P}(\mathrm{Sym}^{r_j}(V))$ has a (rational) section. Then, birationally,

$$\mathcal{P} = S \times \mathrm{PGL}_2 \times S \times \mathrm{PGL}_2 .$$

The line bundle $\mathcal{O}$ is trivial on (some Zariski open subset of) $S \times S$. Therefore, $\mathrm{PGL}_2 \backslash \mathcal{O}$ is birational to

$$(\mathrm{PGL}_2 \backslash \mathcal{O}') \times S \times S.$$

where

$$\mathcal{O}' := \mathcal{O}|_{\mathrm{PGL}_2 \times \mathrm{PGL}_2}.$$

Since S is birational to $\mathbb{C}^n$ (for some n) and the $\mathbb{Z}/2$-action on $\mathbb{C}^n$ is linear, we find that $\mathcal{R}$ is birational to a vector bundle over

$$\mathrm{PGL}_2 \backslash \mathcal{O}'.$$

The bundle $\mathcal{O}'$ is invariant with respect to the $\mathbb{Z}/2$-action exchanging the PGL_2-components. Thus its restriction to each component is either equal to $\mathcal{O}_{\mathrm{PGL}_2}$ or to $\mathcal{O}_{\mathrm{PGL}_2}(1)$. In both cases its restriction to (any) PGL_2-orbit (under the diagonal action of PGL_2) is equal to the trivial line bundle. Therefore it descends to a line bundle on the quotient

$$X := \mathrm{PGL}_2 \backslash (\mathrm{PGL}_2 \times \mathrm{PGL}_2).$$

Of course, $X = \mathrm{PGL}_2$ with $\mathbb{Z}/2$ acting as $x \mapsto x^{-1}$. If the quotient line bundle $\mathcal{Q}_X$ on X is trivial then $\mathcal{Q}_X/(\mathbb{Z}/2)$ is birational to $(X/(\mathbb{Z}/2)) \times \mathrm{G}_m$ and hence rational (since the involution is induced from a projective involution on $\mathbb{P}^3$, containing PGL_2 as an open subset).

If $\mathcal{Q}_X$ is nontrivial then it is birational to GL_2 (with $\mathbb{Z}/2$ acting as $x \mapsto x^{-1}$). The rationality of the quotient is proved in Lemma 4.4 $\square$

Lemma 4.4. *The quotient (of the function field) of* GL_2 *by the involution* $i : x \mapsto x^{-1}$ *is rational.*

Proof. The involution i can be decomposed as a product of two commuting involutions $i = i_1 \circ i_2$, where

$$i_1 : x := \begin{pmatrix} a & b \\ c & -a+d \end{pmatrix} \mapsto \begin{pmatrix} -a+d & -b \\ -c & a \end{pmatrix}$$

and

$$i_2 : y \mapsto y \cdot \det(y)^{-1}.$$

Remark that we have another set of independent generators of the field $\mathbb{C}(a,b,c,d)$, namely $(a,b,c,\det(x))$ (write $d = (\det(x) + bc + a^2)/a)$. In the coordinates $(a,b,c,\det(x))$ both involutions are standard

$$i_1 : (a,b,c) \to (-a,-b,-c)$$

and

$$i_2 : \det(x) \to \det(x)^{-1}.$$

It is easy to write down a set of independent generators of the field of invariants. If

$$D := \frac{\det(x)+1}{\det(x)-1}$$

then

$$\begin{array}{rccc} i_2 : & D & \mapsto & -D \\ i_1 : & (a,b,c,D) & \mapsto & (-a,-b,-c,-D). \end{array}$$

This finishes the proof. □

Lemma 4.5. *Let X be a rational surface and H a finite subgroup of the group of (birational) automorphisms of X. Then X/H is rational.*

Proof. A unirational surface (over $\mathbb{C}$) is rational. □

5. Special rationality results

In this section we collect rationality results for spaces of rational maps $\mathbb{P}^1 \to \mathbb{P}^1$ with prescribed (special) ramification over exactly three distinguished points $(0,1,\infty)$ and unspecified ramifications over other points.

Let $\mathcal{R}(r_0,r_1,r_\infty)$ be the space of rational maps $f : \mathbb{P}^1 \to \mathbb{P}^1$ with local ramification data (vectors) r_0,r_1,r_∞ over the points $0,1,\infty$.

Proposition 5.1. *Assume that (r_0,r_1,r_∞) satisfies one of the following:*

- *all entries of the vectors r_0,r_∞ are even and some fixed number of entries of r_1 is even;*
- *all entries of the vectors r_0,r_∞ are even and a fixed number of entries of r_1 is divisible by 3;*
- *all entries of the vectors r_0,r_∞ are divisible by 3 and all entries of r_1 are even.*

Then $\mathcal{R}(r_0,r_1,r_\infty)$ is a finite union of irreducible rational varieties.

Proof. In these cases the map $f = f_0/f_\infty$ is given by coprime polynomials satisfying the equations:

- $f_0^2 - f_\infty^2 = g_1^2 g_1'$;
- $f_0^2 - f_\infty^2 = g_1^3 g_1'$;
- $f_0^3 - f_\infty^3 = g_1^2 g_1'$,

where g_1' is an arbitrary polynomial. The first equation leads to

$$(f_0 - f_\infty)(f_0 + f_\infty) = g_1^2 g_1'$$

and, by coprimality, to

$$\begin{array}{rcl} f_0 - f_\infty & = & g_{11}^2 g_{11}', \\ f_0 + f_\infty & = & g_{12}^2 g_{12}', \end{array}$$

with arbitrary $g_{11}, g_{11}', g_{12}, g_{12}'$ (satisfying the obvious degree conditions) — a union of rational varieties.

The second case is analogous. Consider the third case: since $f_0^3 - f_\infty^3$ is a square we obtain

$$\begin{array}{rcl} f_0 - f_\infty & = & g_1^2 \\ f_0 - \zeta f_\infty & = & g_2^2 \\ f_0 - \zeta^2 f_\infty & = & g_3^2 \end{array}$$

(where $\zeta^3 = 1$) and we need to solve

$$g_1^2 - g_2^2 = g_3^2.$$

Now we apply the parametrization as above. □

Corollary 5.2. *Let $\mathcal{R}(r_0, r_1, r_\infty)$ be as in 5.1. Then*

$$\mathrm{PGL}_2 \backslash \mathcal{R}(r_0, r_1, r_\infty)$$

is rational.

Proof. We have established an explicit parametrization of $\mathcal{R}(r_0, r_1, r_\infty)$ as a direct sum of spaces of polynomials (with different weights as irreducible GL_2-representations). By a result of Katsylo the corresponding quotients are rational. □

Remark 5.3. Only the first case with $g_1' = 1$ can admit a nontrivial action of H_Γ (which necessarily is $\mathbb{Z}/3$). But even in this case the action of $\mathbb{Z}/3$ is linear and it commutes with the action of GL_2 on pairs of polynomials. Proposition 4.2 implies rationality.

Lemma 5.4. *Every irreducible component of the variety $\mathcal{R}$ of rational maps $f : \mathbb{P}^1 \to \mathbb{P}^1$ of degree 5 and prescribed global ramification datum*

$$\mathrm{RD}(f) = [(2,2,1)_0, (2,2,1)_1, (2,2,1)_\infty, (2), (2)]$$

is rational.

Proof. Changing the variables (fixing two ramification points over $1 \in \mathbb{P}^1$ as $0, \infty$), we can write $f = F_1/F_2$ where

$$\begin{aligned} F_1(x) &= \hat{f}_1(x)^2 \hat{a}_1(x)^2 \hat{b}_1(x) \\ F_2(x) &= \hat{f}_2(x)^2 \hat{a}_2(x)^2 \hat{b}_2(x) \end{aligned}$$

where $\hat{f}_1, \hat{f}_2, \hat{a}_1, \hat{a}_2, \hat{b}_1, \hat{b}_2$ are linear forms in x. Since the leading coefficients of F_1 and F_2 are equal we can assume that they are both equal to 1 and write $\hat{f}_1(x) = x + f_1, \ldots, \hat{b}_2(x) = x + b_2$, with some nonzero constants $f_1, \ldots, b_2$. Since we have one free parameter (under the action of PGL_2) we can assume that $b_1 = 1$. Thus

$$\hat{f}_1(x)^2 \hat{a}_1(x)^2 \hat{a}_2(x) - \hat{f}_2(x)^2 \hat{b}_1(x)^2 \hat{b}_2(x) = \sum_i g_i x^i = c_1 x^2 (x + c_2)$$

with arbitrary constants c_1, c_2. We get a system of equations on the coefficients g_j:

$$g_4 = 0, g_1 = 0, g_0 = 0.$$

Remark that the coefficients of g are symmetric functions on pairs (f_1, a_1) and (f_2, a_2). To parametrize $\mathcal{R}$ we introduce the following variables:

$$X_1 = a_1 + f_1,\ Y_1 = a_1 f_1,\ X_2 = f_2 + a_2,\ Y_2 = f_2 a_2,\ b_1,\ b_2.$$

Write the equations on the coefficients g_j as

$$\begin{aligned} 2X_1 + b_1 &= 2X_2 + b_2 \\ Y_1^2 b_1 &= Y_2^2 b_2 \\ Y_1^2 + 2X_1 Y_1 b_1 &= Y_2^2 + 2X_2 Y_2 b_2. \end{aligned}$$

Since $b_1 = 1$, for a fixed b_2 we get

$$\begin{aligned} 2X_1 + 1 &= 2X_2 + b_2 \\ Y_1 &= \pm\sqrt{b_2} Y_2 \\ b_2 Y_2 + 2\sqrt{b_2} X_1 &= Y_2 + 2X_2 b_2. \end{aligned}$$

This is a union of two (affine) lines. After a rational covering ($\sqrt{b_2}$) our surface is (rationally) a $\mathbb{P}^1$-bundle over $\mathbb{P}^1$, a rational surface. □

Lemma 5.5. *Every irreducible component of the variety $\mathcal{R}$ of rational maps $f : \mathbb{P}^1 \to \mathbb{P}^1$ of degree 4 and ramification datum*

$$\mathrm{RD}(f) = [(2,2)_0, (2,1,1)_1, (2,1,1)_\infty]$$

is a rational surface.

Proof. Using the PGL_2-action on the preimage $\mathbb{P}^1$ we can assume that the points $(2,2)$ are $+1, -1$, respectively, and that the point of degree

2 (in the local ramification datum $(2,1,1)$) over 0 is ∞. Thus we can write

$$(x^2-1)^2 - c(x+c_1)(x+c_2)(x+c_3)^2 = g_2(x),$$

where g_2 is an arbitrary polynomial of degree 2 and c is some constant. We get two equations

$$c = 1,$$

$$c_1 + c_2 + 2c_3 = 0.$$

Thus we have a (rational) surjection of $\mathbb{P}^2$ onto $\mathcal{R}$. □

Lemma 5.6. *Every irreducible component of the variety $\mathcal{R}$ of rational maps $f : \mathbb{P}^1 \to \mathbb{P}^1$ of degree 4 with ramification datum*

$$\mathrm{RD}(f) = [(2,2)_0, (3,1)_1, (2,1,1)_\infty, (2), (2)]$$

is a rational curve.

Proof. A generic map with this ramification datum is given by the equation $f = f_1/f_2$, where

$$f_1 = (x^2-1)^2, \ \ f_2 = (x+c_1)(x+c_2)(x+c_3)^2$$

and

$$f_1 - f_2 = (x^2-1)^2 - c(x+c_1)(x+c_2)(x+c_3)^2 = g_1(x),$$

where $g_1(x)$ is linear. Thus $c = 1$ and

$$c_1 + c_2 + 2c_3 = 0,$$

$$c_1c_2 + 2c_1c_3 + 2c_2c_3 + c_3^2 = 0,$$

clearly rational. □

6. Combinatorics

In this section we investigate relations between $\mathrm{ET}(\mathcal{E})$ and $\mathrm{ET}(\Gamma)$. We keep the notations of the previous sections.

Lemma 6.1. *Let $j : \mathcal{E} \to C$ be an elliptic fibration. Then*

$$\mathrm{ET}(\mathcal{E}) = \deg(j_{\mathcal{E}})\Delta(\Gamma) + 8\alpha_2 + 4\alpha_1 + 6\beta_1 + 12\ell. \tag{6.1}$$

Here α_1 and α_2 equal the number of points over A_2-ends of T_Γ with ramification multiplicity 1 $\pmod 3$ *and* 2 $\pmod 3$, *respectively, β_1 is the number of odd ramification points over the B_2-ends and ℓ is the number of $*$-fibers of $\mathcal{E}$.*

Proof. The summand $\deg(j_{\mathcal{E}})\Delta(\Gamma)$ corresponds to multiplicative fibers of $\mathcal{E}$. The next summands are the contributions of those singular fibers of $\mathcal{E}$ which are in the preimage of A_2 or B_2-ends of T_Γ. If the ramification order at a point p over a B_2-end is even then the corresponding fiber with minimal ET is smooth and hence does not contribute to $\mathrm{ET}(\mathcal{E})$. If it is odd then the fiber with minimal ET is of type III and we have to add $6\beta_1$. Similarly, for the preimages of A_2-ends and $*$-twists. □

Corollary 6.2. *In particular,*

$$\mathrm{ET}(\mathcal{E}) \leq \deg(j_{\mathcal{E}})\,\mathrm{ET}(\Gamma) + 12\ell,$$

with equality if

$$\begin{array}{rcl} 2\alpha_2 + \alpha_1 & = & a_2 \cdot \deg(j_{\mathcal{E}}) \\ \beta_1 & = & b_2 \cdot \deg(j_{\mathcal{E}}). \end{array}$$

Definition 6.3. We call T_Γ *saturated* if all vertices of T_Γ^u are trivalent and a *tree* if it is contractible.

Remark 6.4. For saturated graphs $\Delta(\Gamma) = 12\,\mathrm{rk}\,\pi_1(T_\Gamma)$, where

$$\mathrm{rk}\,\pi_1(T_\Gamma) = \mathrm{rk}\,H_1(T_\Gamma)$$

is the number of independent closed loops of $T_\Gamma \subset M_\Gamma$.

The following simple procedures produce new graphs:

- If T_1 and T_2 are (unmarked) trivalent graphs we can join T_1 and T_2 along two edges. For the resulting graph T' we have

  $$\mathrm{ET}(T') = \mathrm{ET}(T_1) + \mathrm{ET}(T_2) + 12.$$

  If T_i are marked and the marking of the ends of T' is induced from the marking of the corresponding ends of T_1 and T_2 then

  $$\Delta(T') = \Delta(T_1) + \Delta(T_2) + 12.$$

- We can glue an end p of T_1 to an edge of T_2. In this case

  $$\mathrm{ET}(T') = \mathrm{ET}(T_1) + \mathrm{ET}(T_2).$$

  The change of Δ depends on the marking of the end:

  $$\Delta(T') = \begin{cases} \Delta(T_1) + \Delta(T_2) + 6 & \text{if } p = B_2 \\ \Delta(T_1) + \Delta(T_2) + 4 & \text{if } p = A_2. \end{cases}$$

Remark 6.5. Any connected graph T can be uniquely decomposed into a union of a saturated graph and a union of trees.

Lemma 6.6. $\mathrm{ET}(\Gamma)$ *is divisible by* 12.

Proof. Every vertex of T_Γ has either one or three incoming edges. Therefore, the number of edges

$$\tau^1 = \frac{1}{2}(\tau_1^0 + \tau_3^0),$$

(τ_i^0 is the number of vertices with i-edges). Thus $\tau^0 = \tau_1^0 + \tau_3^0$ is even and since $\mathrm{ET}(\Gamma) = 6\tau^0$ we are done. □

Example 6.7. If T_Γ is a tree with $k+2$ vertices then

$$\begin{array}{ll} \mathrm{ET}(\Gamma) & = 12k + 12 \\ \Delta(\Gamma) & \begin{cases} = 6k & \text{if all ends are } B_2, \\ > 6k & \text{otherwise.} \end{cases} \end{array}$$

Lemma 6.8. *For all Γ one has*

$$\Delta(\Gamma) \geq \mathrm{ET}(\Gamma)/2 + 6(\mathrm{rk}\, H_1(T_\Gamma) - 1).$$

Proof. A direct computation shows that for saturated graphs one has an equality. Suppose that T_Γ is a concatenation of a saturated graph T_{sat} and a tree T_{tree}. The number of ends drops by one and the number of A_6 vertices increases by 1. Thus the tree will add $12k+12$ to $\mathrm{ET}(\Gamma)$ but $\Delta(\Gamma)$ will change by $6k+6$. Finally, the ratio $\Delta(\Gamma)/\mathrm{ET}(\Gamma)$ only increases if we change B_2- to A_2-markings for some ends. Indeed, $\Delta(\Gamma)$ increases without changing $\mathrm{ET}(\Gamma)$. □

Corollary 6.9. *If $\Delta(\Gamma) = \mathrm{ET}(\Gamma)/2$ then T_Γ is a concatenation of a loop L and some trees. Moreover, all the ends of T_Γ are of type B_2.*

Proposition 6.10. *Let $\mathcal{E} \to C$ be an elliptic fibration with $\mathrm{ET}(\mathcal{E}) < \mathrm{ET}(\Gamma)$. Then:*

- $M_\Gamma = \mathbb{P}^1$ *and T_Γ is a tree without A_2-ends and with* $\mathrm{ET}(\Gamma) > 24$;
- $\deg(j_{\mathcal{E}}) = 2$ *and it is ramified in all (B_2) ends of T_Γ (and, possibly, some other points);*
- $\mathcal{E}$ *has* 1 *or* 2 *singular fibers of type* I_n.

Proof. From 6.1 and 6.8 we conclude that $\mathrm{rk}\, H_1(T_\Gamma) = 0$ which implies that T_Γ is a tree and $M_\Gamma = \mathbb{P}^1$. By Lemma 2.1 and our assumption, $\mathrm{ET}(\Gamma) > 24$, which implies that $\deg(j_{\mathcal{E}}) \leq 2$. If $\deg(j_{\mathcal{E}}) = 1$, we apply Corollary 6.2 and get a contradiction to the assumption. For $\deg(j_{\mathcal{E}}) = 2$ combine Definition 1.7 and (6.1):

$$\mathrm{ET}(\mathcal{E}) = \mathrm{ET}(\Gamma) + 4a_2 + 4\alpha_1 + 8\alpha_2 + 6\beta_1 - 12.$$

Since α_1, resp. β_1 is twice the number of unramified A_2, resp. B_2-ends, and α_2 is the number of ramified A_2-ends we see that if at least one of them is not zero, then $\mathrm{ET}(\mathcal{E}) \geq \mathrm{ET}(\Gamma)$. The claim follows. □

Corollary 6.11. *For every elliptic fibration $\mathcal{E} \to \mathbb{P}^1$ one has*

$$\mathrm{ET}(\mathcal{E}) \geq \mathrm{ET}(\Gamma).$$

Further, if $\deg(j_{\mathcal{E}}) = 2$ *and* $j_{\mathcal{E}}$ *is ramified over only one* B_2*-point then*

$$\mathrm{ET}(\mathcal{E}) \geq 2\,\mathrm{ET}(\Gamma) - 12.$$

Proof. If $\deg(j_{\mathcal{E}}) = 2$ and $\mathbb{C} = \mathbb{P}^1$ then $j_{\mathcal{E}}$ ramifies in two points. If neither of these points is B_2 then, by Lemma 6.1, $\mathrm{ET}(\mathcal{E}) \geq 2\,\mathrm{ET}(\Gamma)$. If both of these points are B_2-points then the covering $j_{\mathcal{E}}$ corresponds to a subgroup Γ' of index 2 in Γ and $C = M_{\Gamma'}$, contradiction. Otherwise, the claimed inequality follows from Lemma 6.1. □

7. Elliptic K3 surfaces with $\deg(j_{\mathcal{E}}) > 1$

In this section we assume that $C = \mathbb{P}^1$, that $j_{\mathcal{E}} > 1$ and that Γ is a proper subgroup of $\mathrm{PSL}_2(\mathbb{Z})$. We consider

$$\begin{array}{rrcl} \text{general families:} & \mathrm{ET}(\mathcal{E}) - 12\ell & = & \deg(j_{\mathcal{E}})\,\mathrm{ET}(\Gamma), \\ \text{special families:} & \mathrm{ET}(\mathcal{E}) - 12\ell & < & \deg(j_{\mathcal{E}})\,\mathrm{ET}(\Gamma). \end{array}$$

In Section 3 we showed that the main building block in the construction of moduli space of elliptic surfaces with fixed Γ is the space of rational maps $j_{\mathcal{E}} : C \to M_\Gamma$ of fixed degree and ramification restrictions over certain points. For a *general* family there are no such restrictions and the corresponding moduli spaces are rational by classical results of invariant theory for actions of PGL_2 and its algebraic subgroups (see Section 4). For *special* families the corresponding space of rational maps is more complicated.

Lemma 7.1. *There are no special families of elliptic* K3 *surfaces with* $\mathrm{ET}(\Gamma) = 48, 36$.

Proof.

- If $\mathrm{ET}(\Gamma) = 48$ then $\Delta(\Gamma) \geq 18$ and $\deg(j_{\mathcal{E}}) \leq 2$. However, $\deg(j_{\mathcal{E}}) = 2$ contradicts Corollary 6.11 ($\mathrm{ET}(\mathcal{E}) \geq 96 - 24 > 48$).
- If $\mathrm{ET}(\Gamma) = 36$ and $\Delta(\Gamma) > 16$ then $\deg(j_{\mathcal{E}}) = 2$, contradicting to 6.11. We are left with $\Delta(\Gamma) = 16, 14, 12$ for $\deg(j_{\mathcal{E}}) = 3$ and $\Delta(\Gamma) = 12$ for $\deg(j_{\mathcal{E}}) = 4$.
- If $\deg(j_{\mathcal{E}}) = 4$ then T_Γ is a tree with $\mathrm{GD}(\Gamma) = [2A_6 + 4B_2]$. By Lemma 6.1, all ramifications over the B_2-ends are even, which contradicts $C = \mathbb{P}^1$ (compute $\chi(C)$).
- If $\deg(j_{\mathcal{E}}) = 3$ then T_Γ is a tree (by 6.8) and

$$\mathrm{GD}(\Gamma) = [2A_6 + a_2A_2 + (4 - a_2)B_2]$$

with $a_2 \leq 2$. We have

$$48 \geq \mathrm{ET}(\mathcal{E}) \geq 3(12 + 2a_2) + 4\alpha_1 + 8\alpha_2 + 6\beta_1,$$

where $\beta_1 \geq 2$ (since $\deg(j_{\mathcal{E}})$ is odd there is odd ramification over some B_2-end). Therefore, $a_2 = 0$ and consequently, $\beta_1 \geq 4$, contradiction.

□

Lemma 7.2. *If T_Γ is not a tree and $j_{\mathcal{E}}$ is special (and generic for the corresponding irreducible component of $\mathcal{F}_{2,\tilde{\Gamma}}$) then*

ET(Γ)	$\deg(j_{\mathcal{E}})$	GD(Γ)	RD($j_{\mathcal{E}}$)
24	4	$[2A_6 + 2B_2]$	$[(2,2)_B, (2,2)_B, (2), (2)]$
24	3	$[2A_6 + 2B_2]$	$[(2,1)_B, (2,1)_B, (2), (2)]$
24	3	$[2A_6 + A_2 + B_2]$	$[(3)_A, (2,1)_B, (2)]$
12	6	$[A_6 + A_2]$	$[(3,3)_A, (3,3)_A, (2), (2)]$
12	5	$[A_6 + A_2]$	$[(3,1,1)_A]$
12	$5 \leq d \leq 8$	$[A_6 + B_2]$	$[\beta = (\beta_i)_B, (2)^{d'}_B]$,

where

$$\beta_i \in \mathbb{N}, \ \sum \beta_i = d, \ \#\text{odd } \beta_i \leq 8 - d$$

and

$$d' = 2d - \#\text{nonzero } \beta_i.$$

Proof. Follows from Lemma 6.1. First observe that $\Delta(\Gamma) \leq 16$, which implies that $a_6 = 2$ and $a_2 \leq 2$. If $a_2 = 2$ then $\Delta(\Gamma) = 16$ and

$$\alpha_1 = \alpha_2 = \beta_1 = 0.$$

Hence both A_2-ends have a 3-cyclic ramification, which implies that the cover corresponds to a subgroup $\Gamma' \subset \Gamma$ of index 3. This excludes $\mathrm{GD}(\Gamma) = [2A_6 + 2A_2]$. If $\deg(j_{\mathcal{E}}) = 4$ then $\Delta(\Gamma) = 12$ which implies that all preimages of B_2-ends have even ramification. The description of all other ramification data follows similarly from Lemma 6.1. Notice that the (omitted) possibilities

ET(Γ)	$\deg(j_{\mathcal{E}})$	GD(Γ)	RD($j_{\mathcal{E}}$)
12	6	$[A_6 + A_2]$	$[(6)_A, (3,3)_A]$
12	5	$[A_6 + A_2]$	$[(3,2)_A]$

are degenerations of the listed cases (see Remark 3.2). □

Lemma 7.3. *If T_Γ is a tree and $j_{\mathcal{E}}$ is special (and generic for the corresponding irreducible component of $\mathcal{F}_{2,\tilde{\Gamma}}$) then*

	$\deg(j_{\mathcal{E}})$	$\mathrm{GD}(\Gamma)$	$\mathrm{RD}(j_{\mathcal{E}})$
j_1	4	$[A_6+A_2+2B_2]$	$[(1,1,1,1)_A,(2,2)_B,(2,2)_B,(2),(2)]$
j_2	4	$[A_6+A_2+2B_2]$	$[(3,1)_A,(2,2)_B,(2,2)_B]+*$
j_3	4	$[A_6+A_2+2B_2]$	$[(3,1)_A,(2,2)_B,(2,1,1)_B,(2)]$
j_4	3	$[A_6+2A_2+B_2]$	$[(3)_A,(1,1,1)_A,(2,1)_B]$
j_5	3	$[A_6+A_2+2B_2]$	$[(1,1,1)_A,(2,1)_B,(2,1)_B,(2),(2)]$
j_6	3	$[A_6+A_2+2B_2]$	$[(3)_A,(1,1,1)_B,(2,1)_B,(2)]$

or $\mathrm{GD}(\Gamma)=[A_6+3B_2]$ *and*

	$\deg(j_{\mathcal{E}})$	$\mathrm{RD}(j_{\mathcal{E}})$
j_7	8	$[(2,2,2,2)_B,(2,2,2,2)_B,(2,2,2,2)_B,(2),(2)]$
j_8	6	$[(2,2,2)_B,(2,2,2)_B,(2,2,1,1)_B,(2),(2)]$
j_9	6	$[(2,2,2)_B,(2,2,2)_B,(2,2,2)_B,(2)]+*$
j_{10}	5	$[(2,2,1)_B,(2,2,1)_B,(2,2,1)_B,(2),(2)]$
j_{11}	4	$[(2,1,1)_B,(2,1,1)_B,(2,2)_B,(2),(2)]$
j_{12}	4	$[(2,1,1)_B,(2,2)_B,(2,2)_B,(2)]+*$
j_{13}	3	$[(1,1,1)_B,(2,1)_B,(2,1)_B]$
j_{14}	3	$[(2,1)_B,(2,1)_B,(2,1)_B,(2)]+*$

or $\mathrm{ET}(\Gamma)=12$ *and* $\mathrm{GD}(\Gamma)=[2A_2]$ *with* $\deg(j_{\mathcal{E}})=4-10,12$.

(In the above tables, $+*$ means that there exists a moduli space of elliptic surfaces with the same $\mathrm{RD}(j_{\mathcal{E}})$ *and with an additional $*$-fiber over an arbitrary point)*

Proof. Assume that $\mathrm{ET}(\Gamma)=24$ and T_Γ is a tree with

$$\mathrm{GD}(\Gamma)\neq[A_6+3B_2].$$

First observe that $\deg(j_{\mathcal{E}})\leq 6$, since $\Delta(\Gamma)\geq 8$. If $\deg(j_{\mathcal{E}})\geq 5$ then, by 6.1, $\mathrm{GD}(\Gamma)=[A_6+A_2+2B_2]$. If $\deg(j_{\mathcal{E}})=6$ then $j_{\mathcal{E}}$ has to be completely ramified over all ends and no other ramifications are allowed by Euler characteristic computation. Therefore, it is a group-covering and can't be $j_{\mathcal{E}}$. If $\deg(j_{\mathcal{E}})=5$ then there are two odd ramifications over B_2-ends, and by 6.1, $\mathrm{ET}(\Gamma)>48$.

We are left with

$$\begin{aligned}\mathrm{GD}(\Gamma) &= [A_6+3A_2],\\ &= [A_6+2A_2+B_2],\\ &= [A_6+A_2+2B_2]\end{aligned}$$

and $3\leq\deg(j_{\mathcal{E}})\leq 4$. If there are at least two A_2-ends without 3-cyclic ramification points over them then $\mathrm{ET}(\mathcal{E})>48$ (see 6.1). The first case is impossible: $\deg(j_{\mathcal{E}})=4$ does not occur (the degree is not divisible by

3), if $\deg(j_{\mathcal{E}}) = 3$ and there is at most one 3-cyclic ramification over an A_2-end then, by 6.1, $\mathrm{ET}(\mathcal{E}) > 48$, contradiction. Consider the second case and $\deg(j_{\mathcal{E}}) = 4$. Then $\Delta(\Gamma) = 10$ and $4\alpha_1 + 8\alpha_2 + 6\beta_1 \leq 8$. Since $\alpha_1 \geq 2$ we have $\alpha_2 = \beta_1 = 0$ and $\alpha_1 = 2$. The only possible

$$\mathrm{RD}(j_{\mathcal{E}}) = [(3,1)_A, (3,1)_A, (2,2)_B],$$

which corresponds to a group covering, contradiction.

Similarly, if $\mathrm{GD}(\Gamma) = [A_6 + A_2 + 2B_2]$ and $\deg(j_{\mathcal{E}}) = 4$ then $\Delta(\Gamma) = 8$ and $4\alpha_1 + 8\alpha_2 + 6\beta_1 \leq 16$. We have $\alpha_1 \geq 1$ and $8\alpha_2 + 4\alpha_1 = 16$ or 4. In the first case, both B_2-ends are completely ramified, and we get j_1. The second case splits into subcases: $\beta_1 = 0$ or 2, leading to j_2, resp. j_3. If $\deg(j_{\mathcal{E}}) = 3$, then if $\mathrm{GD}(\Gamma) = [A_6 + 2A_2 + B_2]$ then exactly one of the A_2-ends has cyclic ramification. It follows that $\beta_1 = 1$, which leads to j_4. If $\mathrm{GD}(\Gamma) = [A_6 + A_2 + 2B_2]$ there are two subcases: there is cyclic ramification over the A_2-end or not. In the first subcase, possible $\mathrm{RD}(j_{\mathcal{E}})$ include $[(2,1)_B, (2,1)_B]$, which is excluded as it gives a group covering. The other case leads to j_6. In the second subcase, we get j_5.

Consider the case $T_\Gamma = A_6 + 3B_2$. Here $\Delta(\Gamma) = 6$ and

$$\mathrm{ET}(\mathcal{E}) \geq 6\deg(j_{\mathcal{E}}) + 6n,$$

where n is a number of points with odd ramification over B_2-vertices. It follows that

$$48 \geq 6\deg(j_{\mathcal{E}}) + 6\beta_1$$

and $\beta_1 \geq 3$ if $\deg(j_{\mathcal{E}})$ is odd and the number of odd ramifications over *each* B_2-end is congruent to $\deg(j_{\mathcal{E}})$ modulo 2.

If $\deg(j_{\mathcal{E}}) = 8$ then all preimages of B_2-vertices are $2n$-ramified. If $\deg(j_\Gamma)$ is odd then $\mathrm{ET}(\mathcal{E}) \geq 6\deg(j_\Gamma) + 18$, which excludes $\deg(j_\Gamma) = 7$. Now assume $\deg(j_\Gamma) = 6$. The number of possible odd ramifications over any B_2-end is even by 6.1 and it cannot exceed 2. There are two possibilities listed above. Assume that $\deg(j_{\mathcal{E}}) = 5$. The minimal possible ramifications are $(2,2,1)$ over all B_2-ends. Since $10 - 6 = 4$ we can add two more points.

In $\deg(j_{\mathcal{E}}) = 4$ we could have further RD:

$$\begin{aligned} \mathrm{RD}(j_{\mathcal{E}}) &= [(2,2)_B, (2,1,1)_B, (2,2)_B, (2)], \\ &= [(2,2)_B, (2,1,1)_B, (2,1,1)_B, (2), (2)], \\ &= [(2,2)_B, (2,1,1)_B, (2,1,1)_B, (3)] \end{aligned}$$

but they are obtained as degenerations of j_{12} and j_{13}.

The only $\mathrm{GD}(\Gamma)$ which allow $\deg(j_{\mathcal{E}}) \geq 12$ are $[A_2 + B_2]$ and $[2A_2]$. The first case corresponds to $\mathrm{PSL}_2(\mathbb{Z})$ (which we don't consider). The second case corresponds to subgroups $\Gamma \subset \mathrm{PSL}_2(\mathbb{Z})$ of index 2. For a generic $\mathcal{E}$ in each moduli space the ramification datum $\mathrm{RD}(j_{\mathcal{E}})$ is one of the following:

$$\mathrm{RD}(j_{\mathcal{E}}) = [(3,\ldots,3_{n_1},1,\ldots,1)_A,(3,\ldots,3_{n_1},1,\ldots,1)_A,(2)^d]+*,$$

where n_1, n_2, d are non-negative integers such that

$$\deg(j_{\mathcal{E}}) - (n_1+n_2) \leq 4,$$

$$3n_1, 3n_2 \leq \deg(j_{\mathcal{E}}) \text{ and, } d \leq 2(\deg(j_{\mathcal{E}} - (n_1+n_2+1))).$$

(In particular, $d \leq 4$). □

8. Rational elliptic surfaces with $\deg(j_{\mathcal{E}}) > 1$

Lemma 8.1. *There are no special families of rational elliptic surfaces with* $\mathrm{ET}(\Gamma) = 24$.

Proof. If $\deg(j_{\mathcal{E}}) = 2$ then $j_{\mathcal{E}}$ cannot be ramified over more than one B_2-end (otherwise it is a group covering). Therefore, we can apply Corollary 6.11 and get $\mathrm{ET}(\mathcal{E}) > 2\cdot 24 - 12 > 24$, contradiction (to 2.1). Thus $\deg(j_{\mathcal{E}}) = 3$ or 4 and $a_6 = 1$. Moreover, $\Delta(\Gamma) \leq 8$. This leaves the cases:

$$\begin{aligned}\mathrm{GD}(\Gamma) &= [A_6 + 3B_2],\\ &= [A_6 + A_2 + 2B_2].\end{aligned}$$

In the first case $\deg(j_{\mathcal{E}}) = 3$ is impossible, and $\deg(j_{\mathcal{E}}) = 4$ leads to

$$\mathrm{RD}(j_{\mathcal{E}}) = [(2,2)_B,(2,2)_B,(2,2)_B,(2,2)_B]$$

which corresponds to a group covering. In the second case $\deg(j_{\mathcal{E}}) \neq 4$ (since $\Delta(\Gamma) = 8$) and $\deg(j_{\mathcal{E}}) = 3$ implies that $\beta_1 \geq 2$ and $\mathrm{ET}(\mathcal{E}) \geq 36$, contradiction. □

Lemma 8.2. *If $j_{\mathcal{E}}$ is special (and generic for the corresponding irreducible component of $\mathcal{F}_{1,\tilde{\Gamma}}$) then*

	$\deg(j_{\mathcal{E}})$	$\mathrm{GD}(\Gamma)$	$\mathrm{RD}(j_{\mathcal{E}})$
j_{15}	6	$[2A_2]$	$[(3,3)_A,(3,3)_A,(2),(2)]$
j_{16}	4	$[2A_2]$	$[(3,1)_A,(3,1)_A,(2),(2)]$
j_{17}	3	$[2A_2]$	$[(3)_A,(1,1,1)_A,(2),(2)]$
j_{18}	4	$[A_6+B_2]$	$[(2,2)_B,(2),(2),(2),(2)]$
j_{19}	3	$[A_6+B_2]$	$[(2,1)_B,(2),(2)]$
j_{20}	3	$[A_6+A_2]$	$[(3)_A,(2),(2)]$

Proof. If $a_6 \geq 1$ then $\deg(j_{\mathcal{E}}) = 4$ or 3. In the first case $a_2 = 0$ and $\mathrm{GD}(\Gamma) = [A_6+B_2]$ and we have complete ramification over the B_2-end. This gives j_{18}. In the second case the ramification over B_2 is $(2,1)_B$ and we get j_{19}. If $\mathrm{GD}(\Gamma) = [A_6+A_2]$ then $\deg(j_{\mathcal{E}}) = 3$ and $\alpha_1 = \alpha_2 = 0$, leading j_{20}.

It remains to consider $\mathrm{GD}(\Gamma) = [2A_2]$. We apply the same formulas in the proof of Lemma 7.3, with the inequality

$$\deg(j_{\mathcal{E}}) - n_1 - n_2 \leq 2.$$

We have $\deg(j_{\mathcal{E}}) \leq 6$ and $\alpha_1 = \alpha_2 = 0$. Notice that $\deg(j_{\mathcal{E}}) = 5$ is impossible. □

9. Rationality of moduli

Theorem 9.1. *Any connected component of a moduli space of rational or* K3 *elliptic surfaces with fixed monodromy group is rational.*

Proof. First we consider *general families*:

$$\mathrm{ET}(\mathcal{E}) - 12\ell = \deg(j_{\mathcal{E}})\,\mathrm{ET}(\Gamma).$$

We have identified the corresponding moduli spaces as quotients of $(\mathrm{Sym}^{\ell}(\mathbb{P}^1)$-fibrations over)

$$\mathrm{PGL}_2 \backslash \mathcal{R}_\Gamma / H_\Gamma$$

where H_Γ is a subgroup of $\mathrm{Aut}(T_\Gamma)$ and $\mathcal{R}_\Gamma$ is an appropriate space of rational maps (3.11). If H_Γ is cyclic then the quotient $\mathrm{PGL}_2 \backslash \mathcal{R}_\Gamma / H_\Gamma$ is rational by Proposition 4.2. If not, then $\deg(j_{\mathcal{E}}) \leq 2$ and the quotient $\mathrm{PGL}_2 \backslash \mathcal{R}_\Gamma$ is a rational variety of dimension ≤ 2. Rationality follows by Lemma 4.5. The presence of $*$-fibers simplifies the situation, since the action of PGL_2 on the total space of the fibration has a natural section.

Now we discuss the *special families*:

$$\mathrm{ET}(\mathcal{E}) - 12\ell < \deg(j_{\mathcal{E}})\,\mathrm{ET}(\Gamma).$$

We use the classification of these families established in Section 7. All families listed in Lemma 7.2 are covered by Propositions 4.1 and 4.2. Consider the families listed in Lemma 7.3: Proposition 4.1 covers the cases: $j_1, j_4, j_5, j_6, j_{13}$ and j_{14}. Proposition 5.1 and its Corollary 5.2 cover the cases: $j_2, j_7, j_8, j_9, j_{12}$. Lemma 5.4 covers the case j_{10}, Lemma 5.5 the case j_{11} and Lemma 5.6 the case j_3.

Finally, the families j_{15} and j_{16} (listed in Lemma 8.2) are covered by Proposition 4.2 and the remaining families $j_{17} - j_{20}$ by Proposition 4.1. □

10. Pictures

In this section we give a combinatorial description of all monodromy groups of elliptic K3 surfaces. More precisely, we describe a simple procedure which allows to enumerate all the possible graphs Γ with given $\mathrm{ET}(\Gamma)$. Let $\mathcal{E} \to \mathbb{P}^1$ be an elliptic K3 surface. We have shown in Section 6 that

$$48 = \mathrm{ET}(\mathcal{E}) \geq \mathrm{ET}(\Gamma)$$

and that $\mathrm{ET}(\Gamma)$ is divisible by 12. Thus $\mathrm{ET}(\Gamma)$ equals $12, 24, 36$ or 48 and all possible $\Gamma \subset \mathrm{PSL}_2(\mathbb{Z})$ are described by connected trivalent graphs T_Γ with ≤ 8 edges embedded into S^2, with an arbitrary bicoloring of the ends.

Case $\mathrm{ET}(\Gamma) = 12$: There is only one tree T_{12} with $\mathrm{ET}(T_{12}) = 12$

FIGURE 1. The tree T_{12}.

The ends of T_{12} can be either A or B-vertices. To obtain all possible graphs T_Γ with $\mathrm{ET}(\Gamma) = 12$ we just need to attach to T_{12} a single loop L.

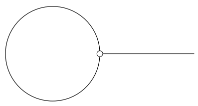


FIGURE 2. The loop L.

This gives the following list of graphs:

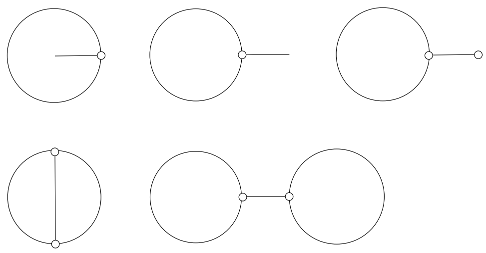


FIGURE 3. The case $\mathrm{ET}(\Gamma) = 12$.

There is only one saturated graph from the list above which has no outer loops (Figure 4).

This graph will be a basic building block in the construction of graphs with $\mathrm{ET}(\Gamma) > 12$ - we will attach trees and loops to its edges. The edges are numbered to simplify the count of all possible outcomes.

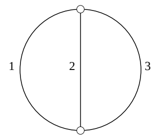


FIGURE 4

Case $\mathrm{ET}(\Gamma) = 24$: Again, we have only one topological tree T_{24} with $\mathrm{ET}(T) = 24$:

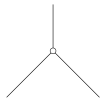


FIGURE 5. The tree T_{24}.

Case $\mathrm{ET}(\Gamma) = 36$: There are only 3 saturated graphs without end-loops (modulo equivalent embedding into the sphere):

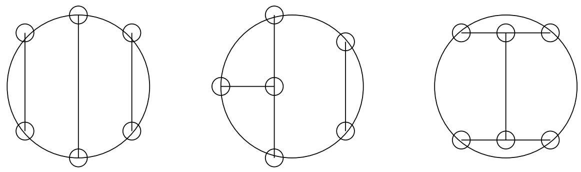


FIGURE 6. The case $\mathrm{ET}(\Gamma) = 36$.

Any other graph is either a tree or a sum of a saturated graph T' with $\mathrm{ET}(T') = 0, 12, 24$ with trees (with complementary ET). There is only one topological tree T_{36} with $\mathrm{ET}(T_{36}) = 36$.

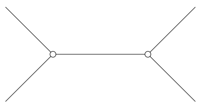
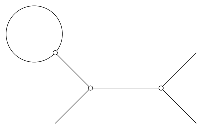


FIGURE 7

The number of possible markings of the tree or loops at the ends is 81 but due to the symmetry of the graph the actual number of graphs T_Γ corresponding to different placement of loops at the end and markings is smaller: there are 34 different T_Γ of this type.

The number of markings of T_{36} is 16 but due to its symmetry the number of different graphs T_Γ is 7. (Recall that two graphs T_Γ give the same Γ modulo conjugation if they are isotopic in a S^2).

The graphs of tree type with one end loop are topologically equivalent to:



There are 8 possible markings of the above graph and they all give different T_Γ with $\mathrm{ET}(\Gamma) = 36$. We have 12 different T_Γ with 2 end-loops, 6 with 3 end-loops and one with 4 end-loops.

All topological graphs which are sums of a loop and a tree can be obtained by placing a loop into a tree. Thus there are two types:

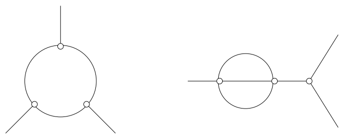


FIGURE 8

This gives 8 graphs T_Γ in the first case and 4 in the second case.

Case $\mathrm{ET}(\Gamma) = 48$: We have one tree T_{48} with $\mathrm{ET}(\Gamma) = 48$:

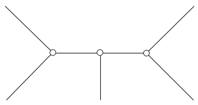


FIGURE 9. The tree T_{48}.

Here is the list of all saturated graphs with $\mathrm{ET}(\Gamma) = 48$.

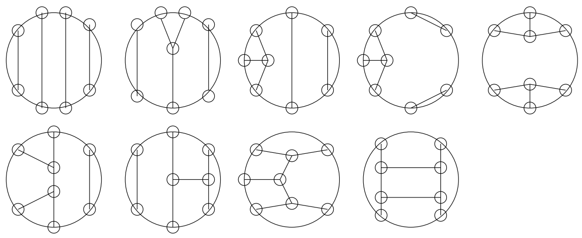


FIGURE 10. Saturated graphs in the case $\mathrm{ET}(\Gamma) = 48$.

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