Proof of the Black-Scholes formula

We search for a probability measure $Q$:

- That makes $S_t^* = e^{-rt}S_t$ a martingale.
- Is equivalent to $P$ the real world probability measure.
- Equivalent means that $P$ and $Q$ have the same probability zero events.
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Proof of the Black-Scholes Formula

We can choose $a$ so that $P^a$ is an equivalent martingale measure for $S^*_t$.

It is a consequence of Girsanov's theorem that $P^a$ is equivalent to $P^{a_1}$ for any two real numbers $a_1$ and $a_2$.

Suppose $a \in \mathbb{R}$ is the set of probability measures of the geometric Brownian motions $S^t = S^0 \exp(a t + \sigma W^t)$.

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Proof of the Black-Scholes formula

• $\mathbb{E}^\mathbb{Q}\left[ e^{-rT} S_T \right] = e^{-rT} S_0 e^{\left( \frac{1}{2} \sigma^2 T + \mu T \right)}$

• The SDE is

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

making $S_t = e^{\int_0^t \mu - \frac{1}{2} \sigma^2 d\tau}$ a martingale.

Choosing $a = \mu - \frac{1}{2} \sigma^2$ gives $e^{\int_0^t (\mu - \frac{1}{2} \sigma^2) d\tau} - \mu = e^{W_t}$ making $S_t = e^{W_t}$ a martingale.
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Proof of the Black-Scholes Formula

- We have a RNP M
- The Black-Scholes market is complete.

Use risk-neutral valuation $e^{-r(T-t)}E_q\left[g\left(S_T\right)\mid F_t\right]$. 

$\frac{\partial \mathcal{V}}{\partial \sigma} p = \mathcal{O}$
The Black-Scholes PDE

\[ 0 = \frac{\partial \Phi}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 \Phi}{\partial S^2} + rS \frac{\partial \Phi}{\partial S} - r\Phi + \frac{\partial \Phi}{\partial t} \]

Since SDE's are unique we can match coefficients to get

\[ \frac{\partial \phi}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 \phi}{\partial S^2} + rS \frac{\partial \phi}{\partial S} = (\frac{\partial}{\partial S} (\mathcal{L}S))^t \frac{\partial}{\partial S} \]

Apply Ito's formula

\[ \mathcal{L}S = \frac{1}{2} \sigma^2 S^2 \frac{\partial^2}{\partial S^2} + rS \frac{\partial}{\partial S} + \frac{\partial}{\partial t} \]

The risk neutral process is

\[ [\frac{\partial}{\partial S} (\mathcal{L}S)]_0 \mathcal{F}(t - \mathcal{L}) = \exp(\frac{\partial}{\partial S} (\mathcal{L}S)) \]

Hedging and the Black-Scholes PDE

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American Options (no dividends)

• An American option with maturity $T$ can be exercised at any time between 0 and $T$. So we may in fact exercise the put at time $t$ then we get $Y - S_t$ which can be greater or less than $e^{-r(T-t)}S$.

  If we exercise the put at time $t$ then we get $Y - S_t$, so we never exercise early.

• An American call $C$ on a non-dividend paying stock has the same value as a European call $C$.

  If we exercise the call at time $t$ then we get $S_t - K$ which can be greater than $e^{-r(T-t)}(S_t - K)$.

  If we exercise early we may in fact exercise early.

An American put $P$ on a non-dividend paying stock may have a higher value than a European put $P$.

• An American call $C$ on a non-dividend paying stock has the corresponding European option.

  An American option is worth at least as much as its time between 0 and $T$.

  An American option with maturity $T$ can be exercised at any time between 0 and $T$.

American Options (no dividends)
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American Options (no dividends)

Optimal exercise time $\tau$ is a random variable called a stopping time.

Risk neutral valuation of non-dividend paying American option

$E_\mathbb{Q} \left[ e^{-r\tau} \max \{b(S_T) \ b_{\tau \downarrow} \} \right]$ where the max is taken over all exercise rules $\tau$.

This is like solving backwards in the binomial tree.

PDE approach leads to a "free-boundary problem".

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Foreign Exchange and Continuous Dividends

European options on FX or Stocks that pay continuous dividends are essentially the same. The foreign currency earns interest while the stock pays dividends.

The important point is that the RNPM should make the asset price plus dividends a martingale.

From our earlier work if $S$ is a geometric Brownian motion we know that $\hat{Q}$ must be the measure associated with the process.

Assume the risky asset $S$ pays interest or dividends at a constant fixed rate $n$.

$\hat{Q}$ must be a martingale.

$S_{t}^{\ast}(u-n)e^{(u-r)t} = S_{t}^{\ast}e^{(u-r)t}$ this means that $\hat{Q}$ must be a martingale.

$\hat{Q}$ is associated with the process

$\hat{Q}[\int_{0}^{t}(r - u)S_{s}ds + \int_{0}^{t}\sigma S_{s}dW_{s}] = \hat{Q}[0]$.
It is not hard to see that if we replace $S_0$ by $e^{-\frac{\sigma^2 T}{2}} L_{n-\varepsilon}$ in the Black-Scholes formula we will arrive at the correct formulas for European calls and puts on stocks that pay continuous dividends.

Foreign Exchange and Continuous Dividends
Substituting the forward price \( F_0 \) gives us the risk neutral valuation formula for a claim with payoff \( (L_{n} - \theta) \mathbb{E}_{\mathcal{L}^{n}} 0^0 S \). Substituting the forward price \( F_0 \) into the risk \( L_{n} (\theta) \mathbb{E}_{\mathcal{L}^{n}} 0^0 S \) gives us Black 76.

\[
\begin{align*}
\frac{\mathcal{L} \theta^0 F_0}{\mathcal{L} F_0^0 Z_{\bar{T}}^0 T} - (\mathcal{X}/^0 \mathcal{H}) \theta^0 I_{1} &= \zeta \rho \\
\frac{\mathcal{L} \theta^0 F_0}{\mathcal{L} F_0^0 Z_{\bar{T}}^0 T} + (\mathcal{X}/^0 \mathcal{H}) \theta^0 I_{1} &= \iota \rho
\end{align*}
\]

\[
\begin{align*}
[(\iota \rho - \zeta \rho - \zeta \rho - \zeta \rho) N 0^0 \mathcal{H} - (\zeta \rho - \zeta \rho - \zeta \rho) N \mathcal{X}]_{L_{n} - \theta} = (0^0 S) \delta \\
[(\zeta \rho) N \mathcal{X} - (\iota \rho) N 0^0 \mathcal{H}]_{L_{n} - \theta} = (0^0 S) \omega
\end{align*}
\]

where \( \mathcal{X} \) is normal with mean \( \frac{\mathcal{L} \theta^0 F_0}{\mathcal{L} F_0^0 Z_{\bar{T}}^0 T} \) and variance \( \frac{\mathcal{L} \theta^0 F_0}{\mathcal{L} F_0^0 Z_{\bar{T}}^0 T} - \rho - \iota \) and \( \mathcal{Z} \) is normal with mean \( \frac{\mathcal{L} \theta^0 F_0}{\mathcal{L} F_0^0 Z_{\bar{T}}^0 T} - \rho - \iota \) and variance \( \frac{\mathcal{L} \theta^0 F_0}{\mathcal{L} F_0^0 Z_{\bar{T}}^0 T} - \rho - \iota \).

\[
\begin{align*}
\theta^0 (S_{0}) &= e^{-rT} \mathbb{E}_{\mathcal{L}^{n}_{\infty}} \left[ F_0 \mathbb{E}_{\mathcal{L}^{n}} (\mathcal{Y}) \right] \\
\iota^0 (S_{0}) &= e^{-rT} \mathbb{E}_{\mathcal{L}^{n}_{\infty}} \left[ F_0 \mathbb{E}_{\mathcal{L}^{n}} (\mathcal{Y}) \right]
\end{align*}
\]
Discrete Dividends

• Assumed dividends are known in advance with certainty and are to be paid at fixed discrete times.

• For American options, we must still check all the possible exercise possibilities.

• For American puts, we still solve a free boundary problem if using Black-Scholes.

• It is easy to apply the binomial model or even Black-Scholes to obtain the dividend.

• American calls are only optimally exercised immediately before the stock goes ex-dividend. So that the option holder can

\[
\sum_{i=1}^n \left[ S_i - D_i \right] e^{-rT} \leq \sum_{i=1}^n S_i e^{-rT} - \sum_{i=1}^n D_i e^{-rT} = 0
\]

For a European option, we can apply Black 76 by using the following formula for the forward price:

• For a European option, we can apply Black 76 by using the forward price formula for the discrete dividend case.