1. We wish to compute

\[ P(X \geq 1) = \int_1^\infty \rho(x)dx = \frac{\int_{\mathbb{R}} 1_{x \geq 1} e^{-\frac{1}{2}x^2 + \sin x}}{\int_{\mathbb{R}} e^{-\frac{1}{2}x^2 + \sin x}} \]

\[ = \frac{\int_{\mathbb{R}} (1_{x \geq 1} e^{\sin x}) e^{-\frac{1}{2}x^2} dx}{\int_{\mathbb{R}} e^{-\frac{1}{2}x^2} dx} \frac{1}{\int_{\mathbb{R}} (e^{\sin x}) e^{-\frac{1}{2}x^2} dx} \]

\[ = \frac{E_Z \left[ 1_{x \geq 1} e^{\sin x} \right]}{E_Z \left[ e^{\sin x} \right]}, \]

where \( E_Z[\cdot] \) denotes averaging with respect to a Normally distributed random variable, \( Z \sim N(0, 1) \).

2. (a) Let \( Y_t = f(t)(b - X_t) \). Then

\[ dY_t = \dot{f}(t)(b - X_t)dt - f(t)dX_t \]

\[ = \dot{f}(t)(b - X_t)dt - f(t) \left[ \frac{b - X_t}{1 - t} + dW_t \right] \]

\[ = (b - X_t) \left[ \dot{f}(t) - \frac{f(t)}{1 - t} \right] dt - f(t)dW_t. \]

Choosing an \( f(t) \) such that

\[ \dot{f}(t) - \frac{f(t)}{1 - t} = 0, \]

We get

\[ dY_t = -\frac{1}{1 - t} dW_t \quad \Rightarrow \quad Y_t = Y_0 - \int_0^t \frac{dW_s}{1 - s}. \]

A possible solution for the ODE is \( f(t) = 1/(1 - t) \). Substituting into \( Y_t \),

\[ Y_0 = f(0)(b - X_0) = b - a \]

and

\[ X_t = b - \frac{Y_t}{f(t)} = b - (1 - t) \left( b - a - \int_0^t \frac{dW_s}{1 - s} \right) \]

\[ = a(1 - t) + bt + (1 - t) \int_0^t \frac{dW_s}{1 - s}. \]
(b) Since $X_1 = b$ almost surely, $E[X_1^2] = b^2$. Also, since $X_t$ is continuous at $t = 1$, 
$\lim_{t \to 1} E[X_t^2] = E[X_1^2] = b^2$.

It is also possible to expand the square of $X_t^2 = [a(1 - t) + bt + (1 - t) \int_0^t \frac{dW_s}{1-s}]^2$, 
use the Ito Isometries and then take the limit $t \to 1$.

3. (a) Define $Y_t = f(t)X_t$. Then

$$dY_t = \dot{f}(t)X_t dt + f(t)dX_t = \dot{f}(t)X_t dt + f(t)[(m - X_t)dt + e^{-t}dW_t]$$

$$ = \left[\dot{f}(t) - f(t)\right]X_t dt + f(t)e^{-t}dW_t.$$ 

If we choose an $f$ such that $\dot{f} - f = 0$, then 
$$dY_t = f(t)e^{-t}dW_t.$$ 

Taking $f(t) = e^t$ we get

$$dY_t = dW_t.$$ 

$$Y_0 = f(0)X_0 = a.$$ 

Hence, 
$$Y_t = Y_0 + \int_0^t dW_s = a + W_t$$ 

and 

$$X_t = e^{-t}(a + W_t).$$ 

(b) Define $Y_t = f(t)X_t$. Then

$$dY_t = \dot{f}(t)X_t dt + f(t)dX_t = \dot{f}(t)X_t dt + f(t)[(m - X_t)dt + dW_t]$$ 

$$ = \left[\dot{f}(t) - f(t)\right]X_t dt + mf(t)dt + f(t)dW_t.$$ 

If we choose an $f$ such that $\dot{f} - f = 0$, then 
$$dY_t = mf(t)dt + f(t)dW_t.$$ 

Taking $f(t) = e^t$ we get

$$dY_t = me^t dt + e^t dW_t.$$ 

$$Y_0 = f(0)X_0 = b.$$ 

Hence, 
$$Y_t = Y_0 + m \int_0^t e^s ds + \int_0^t e^s dW_s = b + m(e^t - 1) + \int_0^t e^s dW_s$$ 

and 

$$X_t = m + e^{-t}(b - m) + \int_0^t e^{s-t} dW_s.$$
4. Let
\[ Q_t = -\frac{1}{2} \int_0^t b^2(W_s)ds + \int_0^t b(W_s)dW_s. \]
Then,
\[ dQ_t = -\frac{1}{2} b^2(W_t)dt + b(W_t)dW_t. \]
Define
\[ Z_t = e^{Q_t}. \]
Using Ito’s formula
\[ dZ_t = e^{Q_t}dQ_t + \frac{1}{2} e^{Q_t}b^2(W_t)dt = e^{Q_t}b(W_t)dW_t. \]
Integrating from 0 to t:
\[ Z_t = Z_0 + \int_0^t e^{Q_t}b(W_t)dW_t, \]
with \( Z_0 = e^{Q+0} = e^0 = 1 \). Therefore, using the first Ito isometry we obtain
\[ EZ_t = 1 + E \left[ \int_0^t e^{Q_t}b(W_t)dW_t \right] = 1. \]

5. We wish to express \( E[\int_0^1 X_t^2 dt] \) in terms of a functional that involves only the Weiner measure. Using the Girsanov formula,
\[ Ef(X_t) = \frac{E \left[ f(W_t^a) \exp \left( \int_0^1 b(W_t^a)dt - \frac{1}{2} \int_0^1 b^2(W_t^a)dt \right) \right]}{E \left[ \exp \left( \int_0^1 b(W_t^a)dt - \frac{1}{2} \int_0^1 b^2(W_t^a)dt \right) \right]}, \]
where \( W_t^a = a + W_t \), i.e., BM starting at a, \( f(W_t^a) \) denotes the functional \( \int_0^1 (W_t^a)^2 dt \), \( b(x) = -x^3 \) and all expectations are with respect to the Weiner measure. Using the result of question 4 we find that the denominator is equall to one. This yields:
\[ E \left[ \int_0^1 X_t^2 dt \right] = E \left[ \left( \int_0^1 (W_t^a)^2 dt \right) \exp \left( -\int_0^1 (W_t^a)^3 dW_t - \frac{1}{2} \int_0^1 (W_t^a)^6 dt \right) \right]. \]