Hedging Poisson Jumps

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We review the theory of hedging a Poisson process which has jumps of known size.

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I Introduction to Hedging Poisson Jumps

We assume zero interest rates for simplicity. Let $F_t$ be the futures price at time $t \in [0,T]$ for maturity $T' \geq T$. We assume that $F$ is a continuous time stochastic process and that the futures contract enjoys continuous marking to market. Let $\mathbb{P}$ denote statistical probability measure.

We are going to assume that $F$ has constant proportional drift except at a set of random times of measure zero. At these random times, $F$ jumps in the opposite direction of the drift by an amount which is proportional to its level. For example, $F$ might drift up continuously at the rate of 10% per year, but it also drops occasionally by say 20% all at once. If jumps occur on average once every two years, then the expected change is zero, and $F$ would be a martingale. The jump times are given by those of a standard Poisson process $N$ with constant arrival rate $\lambda_p > 0$.

A standard Poisson process $N$ is a continuous time stochastic process taking values on the nonnegative integers. $N_t$ is the number of occurrences of a random event in the time interval $(0,t]$ with two defining properties:

1. there exists a positive parameter $\lambda_p$ such that for any time $t \geq 0$ and for arbitrarily small time intervals $\Delta t > 0$, we have:

   (a) $\mathbb{P}\{N_{t+\Delta t} - N_t = 0\} = 1 - \lambda_p \Delta t + o(\Delta t)$

   (b) $\mathbb{P}\{N_{t+\Delta t} - N_t = 1\} = \lambda_p \Delta t + o(\Delta t)$.

   The notation $o(\Delta t)$ indicates that the quantity that it replaces is of higher order (and smaller value) than $\Delta t$, i.e.:

   $$\lim_{\Delta t \downarrow 0} \frac{o(\Delta t)}{\Delta t} = 0.$$ 

   The above two conditions imply:

   $$\mathbb{P}\{N_{t+\Delta t} - N_t = k\} = o(\Delta t)$$

   for $k \geq 2$. 

2. if \( N_{\Delta t} \) is the number of occurrences in a time interval \((t, t + \Delta t)\) then \( N_{\Delta_1}, N_{\Delta_2}, \ldots, N_{\Delta_k} \) are independent whenever \( \Delta_1, \Delta_2, \ldots, \Delta_k \) are disjoint.

The above definition of a standard Poisson process implies the following two important properties:

1. For any fixed \( t > 0 \), the probability mass function of \( N_t \) is Poisson with parameter \( \lambda t \):

   \[
P\{ N_t = n \} = \begin{cases} 
   \frac{e^{-\lambda t} (\lambda t)^n}{n!}, & \text{if } n \in \mathbb{N} \\
   0, & \text{otherwise.}
   \end{cases} \tag{1}
\]

2. The inter-arrival times of jumps \( \tau_1, \tau_2, \ldots \), are IID exponential random variables with common parameter \( \lambda_p \):

   \[
P\{ \tau_i \in dt \} = \begin{cases} 
   \lambda_p e^{-\lambda_p t} dt & \text{if } t > 0 \\
   0 & \text{otherwise.}
   \end{cases} \tag{2}
\]

Under \( \mathbb{P} \), we assume that the futures price \( F \) of the underlying asset is the unique solution of the following stochastic differential equation:

\[
\frac{dF_t}{F_{t-}} = \mu dt + (e^j - 1) dN_t, \quad t \in [0, T],
\tag{3}
\]

where \( F_0 \) is a known positive constant. Here, \( N \) is a standard Poisson process under \( \mathbb{P} \), which starts at zero and jumps by one at independent exponentially distributed times. We will not need to know the arrival rate of jumps under \( \mathbb{P} \). Hence, we can actually assume that the arrival rate is some positive stochastic process, in which case \( N \) is not a standard Poisson process. The drift \( \mu \) in (3) is \textit{not} the expected rate of return on the futures contract, since \( N \) is not a \( \mathbb{P} \) martingale. Rather, \( \mu \) is the easily observed relative growth rate in the futures price in the absence of jumps. We will see that \( \mu \) matters in pricing options as it affects the geometry of the possible sample paths of \( F \). If the Poisson process \( N \) jumps up by one at some time \( t \), then the futures price jumps from its pre-jump value of \( F_{t-} \) to its post jump value \( F_{t-} e^j \). Hence, \( j \in \mathbb{R} - \{0\} \) is the jump in the log of the futures price. To prevent arbitrage, we require that \( j \) and \( \mu \) have opposite signs. Thus, if \( j \) is negative, so that the futures price drops when the standard Poisson process jumps, then \( \mu \) must positive to compensate. However, we are not requiring that \( F \) be a martingale under \( \mathbb{P} \).
To value European-style path-independent contingent claims, let \( V(F, t) : \mathbb{R} \times [0, T] \mapsto \mathbb{R} \) be a \( C^{1,1} \) function. Let \( V_t = V(F_t, t) \) be a continuous time stochastic process, which will eventually be the value process. Using ordinary calculus:

\[
V(F_T, T) = V(F_0, 0) + \int_0^T \frac{\partial V}{\partial F}(F_{t-}, t) dF_t^c + \int_0^T \frac{\partial V}{\partial t}(F_{t-}, t) dt \\
+ \int_0^T \left[ V(F_{t-}e^j, t) - V(F_{t-}, t) \right] dN_t
\]

\[
= V(F_0, 0) + \int_0^T \frac{\partial V}{\partial F}(F_{t-}, t) dF_t + \int_0^T \frac{\partial V}{\partial t}(F_{t-}, t) dt \\
+ \int_0^T \left[ V(F_{t-}e^j, t) - V(F_{t-}, t) - \frac{\partial V}{\partial F}(F_{t-}, t)F_{t-}(e^j - 1) \right] dN_t. \tag{4}
\]

Since \( j \neq 0 \), (3) implies:

\[
dN_t = -\frac{\mu}{e^j - 1} dt + \frac{1}{F_{t-}(e^j - 1)} dF_t, \quad t \in [0, T]. \tag{5}
\]

Substituting (5) in (4) implies:

\[
V(F_T, T) = V(F_0, 0) + \int_0^T \frac{V(F_{t-}e^j, t) - V(F_{t-}, t)}{F_{t-}(e^j - 1)} dF_t \\
+ \int_0^T \left\{ \frac{\partial V}{\partial t}(F_{t-}, t) - \frac{\mu}{e^j - 1} \left[ V(F_{t-}e^j, t) - V(F_{t-}, t) - \frac{\partial V}{\partial F}(F_{t-}, t)F_{t-}(e^j - 1) \right] \right\} dt. \tag{6}
\]

Suppose we now require that the function \( V(F, t) \) solves the following partial differential difference equation (PDDE):

\[
\frac{\partial V}{\partial t}(F, t) - \frac{\mu}{e^j - 1} \left[ V(Fe^j, t) - V(F, t) - \frac{\partial V}{\partial F}(F, t)F(e^j - 1) \right] = 0, \tag{7}
\]

on the domain \( F > 0, t \in [0, T] \), subject to the following terminal condition:

\[
V(F, T) = f(F), \quad F > 0. \tag{8}
\]

The solution to this Cauchy problem exists and is unique. Then (6) reduces to:

\[
f(F_T) = V(F_0, 0) + \int_0^T \frac{V(F_{t-}e^j, t) - V(F_{t-}, t)}{F_{t-}(e^j - 1)} dF_t. \tag{9}
\]

Hence, by charging \( V(F_0, 0) \) dollars initially and holding \( \frac{V(F_{t-}e^j, t) - V(F_{t-}, t)}{F_{t-}(e^j - 1)} \) futures at each \( t \in [0, T] \), the final payoff \( f(F_T) \) is achieved at \( T \). Note that the number of futures to hold at each \( t \in [0, T] \) is not given
by the tangent \(\frac{\partial V}{\partial F}(F_t, t)\), but is instead given by the secant \(\frac{V(F_t - e^j, t) - V(F_{t-}, t)}{F_{t-}(e^j - 1)}\). This is the same result as in the binomial model if we treat one of \(u\) and \(d\) as \(1 + \mu dt\) (depending on the sign of \(j\) and hence \(\mu\)).

Also note that pricing is achieved without knowledge of the arrival rate of jumps. This is fortunate since it is not trivial to determine this parameter from the historical sample path of the futures price, even if it is constant. Also note that if the futures price behaved as in (3), then it would be trivial to estimate \(\mu\) and \(j\). As a result, this model has all of the econometric advantages of the standard Black model in that the parameters which one needs to price are easily determined from the sample path, while the quantities which are difficult to estimate from the sample path are not needed for pricing.

To find the unique solution of the terminal value problem (7) and (8), note that (3) can be rewritten as:

\[
\frac{dF_t}{F_{t-}} = (e^j - 1)[dN_t - \lambda dt], \quad t \in [0, T],
\]

(10)

where \(\lambda \equiv -\frac{\mu e^j - 1}{\sigma - 1}\). If we now switch to an equivalent probability measure \(Q\) for which \(N\) has arrival rate \(\lambda\), then the conditional mean of \(dF\) under \(Q\) is zero. Hence, we can refer to \(\lambda\) as the risk-neutral arrival rate of jumps. Letting:

\[
X_t \equiv \ln(F_t/F_0),
\]

(11)

be the continuously compounded return, Itô’s lemma implies that the dynamics of \(X\) under \(Q\) are given by:

\[
dX_t = -\lambda(e^j - 1)dt + jdN_t, \quad t \in [0, T].
\]

(12)

Integrating both sides and using (11) implies:

\[
F_T = F_t e^{-\lambda(e^j - 1)(T-t) + j(N_T - N_t)}, \quad t \in [0, T].
\]

(13)

Thus, the futures price is a geometric Poisson process, compensated into a positive \(Q\) martingale. Hence at any \(t \in [0, T]\), the payoff \(f(F_T)\) is the same as the payoff \(g(N_T - N_t)\) where \(g(n) \equiv f(F_t e^{-\lambda(e^j - 1)(T-t) + jn})\). Under \(Q\), the conditional distribution of \(N_T - N_t\) is Poisson:

\[
Q\{N_T - N_t = n\} = \frac{e^{-\lambda(T-t)}(\lambda(T-t))^n}{n!}, \quad n \in \mathbb{N}.
\]

(14)
Hence, the unique solution to the Cauchy problem consisting of the PDDE:

$$\frac{\partial V}{\partial t}(F,t) + \lambda \left[ V(Fe^j,t) - V(F,t) - \frac{\partial V}{\partial F}(F,t)F(e^j - 1) \right] = 0,$$

(15)
on the domain $F > 0, t \in [0,T]$, subject to the terminal condition:

$$V(F,T) = f(F), \quad F > 0,$$

(16)
is given by:

$$V(F,t) = E^Q[f(F_T)|F_t = F] = \sum_{n=0}^\infty f(Fe^{-\lambda(e^j-1)(T-t)+jn})e^{-\lambda(T-t)\frac{(\lambda(T-t))^n}{n!}}, \quad F > 0, t \in [0,T].$$

(17)

Note that if we square both sides of (10) and take conditional expectations, then we get:

$$E^Q_t \left( \frac{dF_t}{F_t} \right)^2 = (e^j - 1)^2 \lambda dt, \quad t \in [0,T].$$

(18)

Thus, if we set $e^j - 1 = \frac{\sigma}{\sqrt{\lambda}}$ in (10), then the conditional mean of $\frac{dF_t}{F_t}$ is still zero under $Q$, while its conditional variance becomes $\sigma^2 dt$, independent of $\lambda$. If we now send the risk-neutral arrival rate of jumps $\lambda$ to infinity, then the percentage jump size $e^j - 1$ shrinks in absolute value down to zero. Hence, the limiting process is just scaled Brownian motion with volatility $\sigma$:

$$\frac{dF_t}{F_t} = \sigma dW_t, \quad t \in [0,T].$$

(19)

Thus, the Black model is just a special case of our analysis.

We can also obtain the binomial model as a limiting case by letting the arrival rate $\lambda$ be a deterministic function of time. If this function is zero off the integers and infinity on them, then binomial jumps arise at integer times. As is well known, one can also go from this binomial model to the Black pricing model. One can even go from the Black model to either the binomial model or the Poisson model. For example, one can represent a compensated Poisson process as the limiting value of a stochastic integral w.r.t. a Brownian motion. In particular, consider the limit of a trading strategy which sells an increasing number of digital puts as the time to maturity and strike both approach zero\(^1\). Since a Poisson process is a semi-martingale

\(^1\)A similar strategy is actually employed by some hedge funds as it results in high Sharpe ratios until a put expires in-the-money.
and a binomial process can be interpreted as one, a theorem of Monroe states that both processes can be represented as time-changed Brownian motions.

These observations are meant only to pique your interest. Most billionaires are not conversant with the basics of stochastic processes. That said, our mayor (my boss) does know what volatility is from his days as a bond trader.