We review the theory of hedging and pricing a European-style contingent claim when the futures price follows a multiplicative binomial process.

I thank the members of Bloomberg’s QFR and QFD groups for their comments. I am solely responsible for any errors.
I  Introduction to Hedging in the Binomial Model

I-A  Single Period

We assume zero interest rates for simplicity. We also assume that the underlying security has no payouts over the option’s life. Suppose that the underlying’s spot price either jumps up by a factor $u$ or down by a factor $d$, where $0 < d < 1 < u$. So, if $S$ is the initial spot price of the underlying asset, then the spot price at the end of the period is either $Su$ or $Sd$.

Example: If $S = $40, $u = 2$, $d = \frac{1}{2}$, then:

\[
\begin{array}{c}
40 \\
\downarrow \\
20
\end{array}
\quad \begin{array}{c}
80 \\
\downarrow \\
20
\end{array}
\]

We assume that there is positive probability of either return. Consider a European call maturing at the end of the period and with strike $K$. The call value at the end of the period is either $C_u \equiv \max[0, S \cdot u - K]$ (if the underlyer’s return is $u$) or $C_d \equiv \max[0, S \cdot d - K]$ (if the underlyer’s return is $d$).

Example: If $S = $40, $u = 2$, $d = \frac{1}{2}$, and $K = $50, then the call value is either $\max[0, 40 \cdot 2 - 50] = $30 or $\max[0, 40 \cdot \frac{1}{2} - 50] = $0:
Suppose that a European call option struck at $50 matures in one year. The stock underlying the option pays no dividends over the year and its current price of $40 will either double or halve over the year. Assuming frictionless markets, zero interest rates, and no arbitrage, what must be the current price of the call, $C$?

I-B Spanning the Payoffs

Consider a portfolio of $N^s$ shares of the underlying stock and $N^b$ bonds, where each bond pays the strike price of $50 in one year. The portfolio has two possible values next year, depending on whether the stock price goes up or down:

- **Up**: $N^s80 + N^b50$
- **Down**: $N^s20 + N^b50$

Similarly, the call value has two possible values:

- **Up**: 30
- **Down**: 0

We can choose the number of shares, $N^s$, and the number of bonds, $N^b$, today, so that the value of the stock-bond portfolio equates to the value of the call next year:
Up: \( N^s80 + N^b50 = 30 \)

Down: \( N^s20 + N^b50 = 0 \)

Solving the two equations for the two unknowns implies that \( N^s = \frac{1}{2} \) and \( N^b = -\frac{1}{5} \).

Remarks:

- The required number of shares, \( N^s = \frac{1}{2} \), is the difference in next year’s possible call values, expressed as a proportion of the difference in next year’s possible stock prices:

  \[
  N^s = \frac{30 - 0}{80 - 20} = \frac{1}{2}.
  \]

  Alternatively, one can pretend that the stock price will definitely go down over the year. The required number of shares to hold today is then the gain from owning the call expressed as a fraction of the gain from owning the stock, if the stock price had instead gone up. For a European call option, the required number of shares is always between 0 and 1. In a graph of the value of the call’s replicating portfolio against stock prices, the required number of shares is the slope of the graph (See Figure 1 next page). For this reason, the required number of shares is often called the \textit{delta} of the call.

- The number of bonds required is negative (\( N^b = -\frac{1}{5} \)). Consequently, the bonds must be shorted. For a European call option, the number of bonds shorted is always between 0 and 1. Short selling bonds is equivalent to borrowing. Since each bond has face value $50, the amount to be repaid next year is \((\frac{1}{5} \cdot 50 =)\) $10. In the graph of the value of the call’s replicating portfolio against stock prices, \(-10\) is the vertical intercept of the graph.

- By varying the number of shares (slope) and the amount to be repaid (intercept), one can generate any straight line. Consequently, this approach can be used in a binomial framework to value \textit{any} claim whose payoff is contingent on the price of the stock (eg. a put).
I-C Valuation

Since buying \( \frac{1}{2} \) of a share of the stock and shorting \( \frac{1}{5} \) of a bond duplicates the payoff of the call, avoiding arbitrage requires that the current price of the call equals the cost of duplicating it. Since the current stock price is $40 and the current bond price is $50, the current call value is $10.
I-D Arbitraging Mispriced Calls

- If the market price of the call differs from $10, then there is an arbitrage opportunity.

- Recall the Golden Rule: he who has the gold makes the rules

- If you want to own a lot of gold one day, just remember one thing:

  “Buy Low - Sell High”

- If the market price of the call is $12, then buy the duplicating portfolio for $10, and sell the overpriced call for $12. The investor pockets $2 today and there are no net cash flows at expiration.

- To buy the duplicating portfolio for $12, buy $\frac{1}{2}$ of a share for $20 and short $\frac{1}{5}$ of a bond, bringing in $10. The net cost is $10.

- To see that there are no net cash flows at expiration, note that the call either finishes out-of-the-money or in-the-money. If it finishes out-of-the-money, then both the duplicating portfolio and the written call are worthless. If it finishes in-the-money, then the cash inflow from liquidating the duplicating portfolio ($30) covers the outflow from the written call.

- Similarly, if the call sells for $9, then one dollar can be made by buying the call for $9, and selling the duplicating portfolio for $10.
I-E Spanning

Consider a 2D graph whose horizontal axis is dollars in the down state and whose vertical axis is dollars in the u state. The payoff of each security plots as a point in this space. For example, if $S = $40, $K = $50, $u = 2, d = \frac{1}{2}$, then the stock’s payoff vector has coordinates $(Sd, Su) = (20, 80)$. The bond’s payoff vector is $(50, 50)$.

With these two linearly independent vectors, we can create any payoff. In particular, we can create the call’s payoff which has coordinates $(0, 30)$.
I-F  Portfolio Theory

Consider a 2D graph whose horizontal axis is price relative in the down state and whose vertical axis is
price relative in the up state. This price relative is also called the gross return. Each security plots as
a point in this space. For example, if $S = 40, K = 50, u = 2, d = \frac{1}{2}$, then the stock’s price relative is
$(d, u) = (\frac{1}{2}, 2)$. The bond’s price relative is (1,1).

The gross return on any derivative security is just the gross return on the spanning portfolio of stocks
and bonds. As a result, the return pair plots on a line connecting the two points. In the case of a call, we
know that the return in the down state is zero. Thus, the call’s return plots at (0, 3). Since the payoff in
the up state is $30, the call’s price must be $10.

II  Multiple Periods

Consider some finite time interval $[0, T]$ and for some fixed positive integer $n$, let $\triangle t = \frac{T}{n}$. Consider a futures
contract with maturity $T' \geq T$ and which marks-to-market once every $\triangle t$. Let $F_i$ be the futures price at
time $t_i = i\triangle t, i = 0, 1, \ldots, n$. We assume that $F_0$ is a known positive constant and that for $i = 1, 2, \ldots, n,$
the $F_i$ are positive random variables. In other words, $F$ is a discrete time positive stochastic process with known intial value. Let $\mathbb{P}$ denote statistical probability measure.

Under $\mathbb{P}$, we assume that $F$ follows a multiplicative binomial process:

$$F_{i+1} = F_i \times m_{i+1}, \quad i = 0, 1, \ldots, n - 1,$$

where the multipliers $m_{i+1}$ are IID Bernoulli:

$$m_{i+1} = \begin{cases} 
    u > 1 & \text{with probability } p \in (0, 1), \\
    d \in (0, 1) & \text{with probability } 1 - p.
\end{cases}$$

Since the increments in the log price are IID, the log price is said to follow a random walk. The figure below shows the set of possible realizations when $n = 5$.

Consider the problem of valuing a European-style path-independent contingent claim which pays off a given function $f(F_n)$ at time $t_n = T$. To value this claim by no arbitrage, let $V(F, i) : \mathbb{R}^+ \times [0, 1, \ldots, n] \mapsto \mathbb{R}$ be a function of the futures price and the time index. Let $V_i = V(F_i, i)$ be a discrete time stochastic process,
which will eventually be the value process. Clearly:

\[ V(F_n, n) = V(F_0, 0) + \sum_{i=0}^{n-1} [V(F_{i+1}, i + 1) - V(F_i, i)] \]

\[ = V(F_0, 0) + \sum_{i=0}^{n-1} H(F, i) \times (F_{i+1} - F_i) + \sum_{i=0}^{n-1} [V(F_{i+1}, i + 1) - V(F_i, i) - H(F, i) \times (F_{i+1} - F_i)] . \]

where \( H(F, i) : \mathbb{R}^+ \times [0, 1, \ldots, n] \mapsto \mathbb{R} \) is a function which indicates the holdings in futures at price \( F \) and time \( t_i \equiv i \Delta t \). One of our major objectives is to determine how to set \( H(F, t) \) so that it is a hedge ratio. For each unit of the contingent claim sold, the hedge ratio \( H(F, t) \) indicates the number of futures to hold at price \( F \) and time \( t \) so that the payoff to the claim is perfectly replicated.

Suppose we now require that the functions \( V(F, t) \) and \( H(F, t) \) solve the following partial difference equation:

\[ V(F_{i+1}, i + 1) - V(F_i, i) - H(F, i)(F_{i+1} - F_i) = 0, \] (4)

on the domain \( F > 0, i = 0, 1, \ldots, n - 1 \), subject to the following terminal condition:

\[ V(F, n) = f(F), \quad F > 0. \] (5)

Then (3) reduces to:

\[ f(F_T) = V(F_0, 0) + \sum_{i=0}^{n-1} H(F, i)(F_{i+1} - F_i). \] (6)

Futures positions are costless, so it costs zero to create \( \sum_{i=0}^{n-1} H(F, i)(F_{i+1} - F_i) \). Hence, by charging \( V(F_0, 0) \) dollars initially and holding \( H(F, i) \) futures contracts at each time \( t_i, i = 0, 1, \ldots, n - 1 \), (6) indicates that the final payoff \( f(F_n) \) is achieved at time \( t_n = T \).

To solve the terminal value problem (4) and (5), suppose that we already know the value function \( V(F, i + 1) \) for \( F > 0 \) and we want to know \( V(F, i) \) for \( F > 0 \). Writing out (4) for the two states:

\[ V(F_{iu}, i + 1) - V(F_i, i) - H(F, i)F_i(u - 1) = 0. \] (7)

\[ V(F_{id}, i + 1) - V(F_i, i) - H(F, i)F_i(d - 1) = 0. \] (8)
Conditioning on $F_i$, we want to solve these two equations for the two unknowns $H(F_i, i)$ and $V(F_i, i)$. Subtracting (8) from (7) implies:

$$H(F_i, i) = \frac{V(F_iu, i + 1) - V(F_id, i + 1)}{F_i(u - d)}.$$  

(9)

Substituting (9) in (7) implies:

$$V(F_i, i) = V(F_iu, i + 1) - \frac{V(F_iu, i + 1) - V(F_id, i + 1)}{F_i(u - d)}F_i(u - 1)$$

$$= \frac{1 - d}{u - d} V(F_iu, i + 1) + \frac{u - 1}{u - d} V(F_id, i + 1).$$  

(10)

Notice from (9) and (10) that the hedge ratio $H(F_i, i)$ and the value $V(F_i, i)$ are both independent of the probability $p$ of an up jump each period. This implies that the current analysis remains intact if we let the probability of an up jump follow any discrete time process taking values in $(0, 1)$.

If we let $q \equiv \frac{1 - d}{u - d}$, then we note that $q \in (0, 1)$ from (2). The backward recursion (10) implies that we can write:

$$V(F_i, i) = E^Q[V(F_{i+1}, i + 1)|F_i],$$  

(11)

for $i = 0, 1, \ldots, n - 1$, where under the probability measure $Q$:

$$F_{i+1} = F_i \times m_{i+1}, \quad i = 0, 1, \ldots, n - 1,$$  

(12)

where the multipliers $m_{i+1}$ are IID Bernoulli:

$$m_{i+1} = \begin{cases} u > 1 & \text{with probability } q \in (0, 1), \\ d \in (0, 1) & \text{with probability } 1 - q. \end{cases}$$  

(13)

It is common to call $Q$ the risk-neutral measure and to call $q$ the risk-neutral probability of an up jump. Cox and Ross introduced this terminology in 1976 to signify that values are calculated as if investors are all risk-neutral. In such an economy, prices are given by expected value under $\mathbb{P}$. If such an economy existed, it would have $p = q$ and hence arbitrage-free values would be calculated by taking expected value under $\mathbb{P} = Q$. Whether or not investors are risk-neutral, arbitrage-free values are calculated by (11), which is called risk-neutral valuation. Beginners sometimes wrongly guess that risk-neutral valuation assumes
that investors are risk-neutral, which is false. To avoid confusion, \( \mathbb{Q} \) is also referred to as the *equivalent martingale measure.*

Evaluating (11) at \( i = n - 1 \) implies:

\[
V(F_{n-1}, n - 1) = \mathbb{E}^\mathbb{Q}[V(F_n, n)|F_{n-1}], = \mathbb{E}^\mathbb{Q}[f(F_n)|F_{n-1}],
\]

using (5). For any fixed \( i \), we can backward induct using (11) a total of \( n - i - 1 \) times to get that:

\[
V(F_i, i) = \mathbb{E}^\mathbb{Q}[f(F_n)|F_i].
\] (15)

In words, under zero interest rates, the arbitrage-free value of the European-style contingent claim is just the risk-neutral conditional expectation of its terminal payoff.

To write this expectation more explicitly, let \( \nu \) be a random variable which counts the number of up jumps in the futures price over \([t_i, T]\). For any fixed \( i \), forward induction on (12) implies:

\[
F_n = F_i u^\nu d^{n-i-\nu}.
\] (16)

Substituting (16) in (15) implies:

\[
V(F_i, i) = \mathbb{E}^\mathbb{Q}[f(F_i u^\nu d^{n-i-\nu})|F_i].
\] (17)

It is well known that \( \nu \) is binomially distributed with parameters \( n - i \in \mathbb{N} \) and \( q \in (0, 1) \). In other words:

\[
\mathbb{Q}\{\nu = j\} = \binom{n-i}{j} q^j (1-q)^{n-i-j} \quad \text{if } j = 0, 1, \ldots, n - i
\]

otherwise. (18)

As a result, (17) implies:

\[
V(F_i, i) = \sum_{j=0}^{n-i} f(F_i u^j d^{n-i-j}) \binom{n-i}{j} q^j (1-q)^{n-i-j}.
\] (19)

This is an explicit formula for the arbitrage-free value of a European-style contingent claim at time \( t_i \).

The binomial framework permits exact hedging and hence unique pricing of any claim written on the futures price path. In general, explicit formulas such as (19) are not available. Instead, one uses the analog
of the backward recursion (11). This analog keeps track of the path-dependence of the claim value, which
usually but not always can be captured by one or more additional state variables. This analog also allows
an additional complexity layer induced by features such as early exercise or callability which are defined
by reference to the continuation value. For example, American-style claims can be hedged and priced in
the binomial framework. For them, the analog of (11) is

\[ V(F_i, i) = \max\{X_i(F_i), E^Q[V(F_{i+1}, i + 1)|F_i]\}, \tag{20} \]

where \( X_i(F_i) \) is the exercise value at time \( i \) and \( E^Q[V(F_{i+1}, i + 1)|F_i] \) is the continuation value at time \( i \).

The above standard notation for the American-style claim hides the dependence of the value function on
an additional state variable keeping track of the path-dependence. This state variable is binary and keeps
track of whether the American claim has been exercised previously. Since the value of an American claim
which has been exercised previously is not of interest, the LHS of (20) updates the value of a previously
unexercised American claim.

One can also extend the binomial model to a multivariate setting, but we will not discuss this extension
here.