Continuous Time Finance

Homework 1 - Outline of Solutions to Selected Problems

Part 1: The Binomial Model

1. (a) The real-world probabilities are of course irrelevant here - given that the risk-neutral probabilities are 50% for a move to $100 and 50% for a move to $0, the option is underpriced. Thus, one would sell stocks and buy options - no shares would be bought.

   (b) No need for borrowing to exploit a mispriced security, as one is essentially selling a portfolio at a positive price with zero payoff.

2. (a) The standard discrete time theory prices options on individual stocks in the additive binomial setting. A call on the index is not, in general, spanned by the stocks (and bond) - except in special cases, such as a strike that’s greater than the highest possible value the index can attain in our model (allowing for a trivial replicating portfolio). Note that this would not be the case if the index were equally weighted for all the stocks, in which case the number of possible values of the index after each period is equal to the number of independent contingent claims.

   (b) Unless all stocks have the same price, in which case they are all essentially the same security (or a short position if they are not in the same half of the alphabet), this model contains arbitrage, so options cannot be priced using a no-arbitrage argument.

   (c) Here the arbitrage is removed, as all the stocks are multiples of the same security.

3. The Central Limit Theorem used in going from the binomial tree to Black-Scholes skirts the use of Ito, by obtaining the risk neutral probabilities in the discrete time steps. Ito does this on the entire continuous process. In other words, the replication is done discretely, and a risk neutral drift is attained, instead of doing this on the continuous process.

4. The futures contract does not "hold on" to its gains and losses, an in that sense behaves like a money market account. This simplifies the
restriction that the portfolio be self financing: the entire change in its value is stored in the riskless asset. In the discrete case, this means that after each period, the change in value of the futures position is transferred into (or out of) the money market account, and a new position is put on at no cost. In the continuous case, the futures position just gives a continuous signed cashflow. Also, this handles the discounting of the gains and losses automatically.

Part 2: Poisson Jumps of Known Size

5. The value of a call obviously cannot be linear. This can be easily seen from Equations (7) or (15) in lecture 2.2: \( V(t, F) \) linear would imply \( V_t = 0 \) (the spacial difference term vanishes), which would in turn imply that \( V \) is always linear in \( F \), but at \( t = T \) this is obviously not true for a call option.

6. We write

\[
V(F e^j, t) = V(F, t) + \frac{\partial V}{\partial F}(F, t)F(e^j - 1) + \frac{1}{2} \frac{\partial^2 V}{\partial F^2}(e^j - 1)^2 + \ldots
\]

and plug into the PDDE (15), where both lower order terms are eliminated (and higher order terms ignored), so we are left with the PDE:

\[
V_t + \frac{\lambda}{2} (e^j - 1)^2 F^2 \frac{\partial^2 V}{\partial F^2} = 0
\]

In the case of a call option, the we have the usual Black-Scholes terminal value problem for a call with zero interest rate and \( \sigma = (e^j - 1)\sqrt{\lambda} \), and of course the Black Scholes formula gives the closed form solution.

7. The PDDE for \( V_F \) is given by:

\[
\frac{\partial V_F}{\partial t}(F, t) + \lambda \left[ V_F(F e^j, t) - V_F(F, t) - \frac{\partial V_F}{\partial F}(F, t)F(e^j - 1) - V_F(F, t)(e^j - 1) \right] = 0,
\]

and now the lower order terms don’t all vanish after substituting the first three terms of Taylor’s expansion:

\[
V_F(F e^j, t) = V_F(F, t) + \frac{\partial V_F}{\partial F}(F, t)F(e^j - 1) + \frac{1}{2} \frac{\partial^2 V_F}{\partial F^2}(F, t)F^2(e^j - 1)^2 + \ldots
\]
We are left with:

\[ \frac{\partial V}{\partial t} + \frac{\lambda}{2} (e^j - 1)^2 F^2 \frac{\partial^2 V}{\partial F^2} = \lambda (e^j - 1)V(F, t) \]

This is the same Black-Scholes PDE for a stock paying a dividend equal to the risk free rate - here \( \lambda (e^j - 1) \). The terminal value is the Heavyside (step) function.

8. Since \( F(e^j - 1) \) does not depend on \( t \), and is in fact fixed at any given level of the futures contract \( F \), there is no convergence of the solution to the PDE to that of the PDDE. However, when the given jump size \( j \) is quite small, the solution could be a (more or less rough) approximation.

**Part 3: Geometric Brownian motion**

9. Discounted asset prices are martingales (and volatility does not change under Girsanov): \( dS = \sigma S dW_t^Q \)

10. Ito: \( d\ln S = -\frac{\sigma^2}{2} dt + \sigma dW_t^Q \).

11. \( \ln S \) Converges a.s. to \(-\infty\) as \( T \to \infty \).

12. \( S \to 0 \) a.s. when \( T \to \infty \).

13. As the maturity goes to infinity, the probability that the stock’s path would approach zero converges to 1, and so does the probability that the option would end out of the money.

14. \( C_{ATM} \to S_0 \).

15. On the surface there seems to be a conflict - the first result suggests a very long maturity ATM call is practically worthless, whereas the second implies it is worth as much as the stock. This is the result of the fact that the limit of the expectation (taking the limit in Black Scholes) is very different from the pathwise (a.s.) limit. While (almost) all paths eventually go to zero, at any given time, some will stray to very large values, as the variance grows. Remember: one can drown in a pond whose average depth is one inch.