Forward Equations (1)

- BWD Equation:
  price of one option $C(K_0, T_0)$ for different $(S, t)$
- FWD Equation:
  price of all options $C(K, T)$ for current $(S_0, t_0)$
- Advantage of FWD equation:
  - If local volatilities known, fast computation of implied volatility surface,
  - If current implied volatility surface known, extraction of local volatilities,
  - Understanding of forward volatilities and how to lock them.
Forward Equations (2)

• Several ways to obtain them:
  – Fokker-Planck equation:
    • Integrate twice Kolmogorov Forward Equation
  – Tanaka formula:
    • Expectation of local time
  – Replication
    • Replication portfolio gives a much more financial insight
Fokker-Planck

- If \( dx = b(x, t) \, dW \)
- Fokker-Planck Equation:
  \[
  \frac{\partial \varphi}{\partial t} = \frac{1}{2} \frac{\partial^2 (b^2 \varphi)}{\partial x^2}
  \]
- Where \( \varphi \) is the Risk Neutral density. As \( \varphi = \frac{\partial^2 C}{\partial K^2} \)

\[
\frac{\partial^2 \left( \frac{\partial C}{\partial t} \right)}{\partial x^2} = \frac{\partial \left( \frac{\partial^2 C}{\partial K^2} \right)}{\partial t} = \frac{1}{2} \frac{\partial^2 \left( b^2 \frac{\partial^2 C}{\partial K^2} \right)}{\partial x^2}
\]

- Integrating twice w.r.t. \( x \):
  \[
  \frac{\partial C}{\partial t} = \frac{b^2}{2} \frac{\partial^2 C}{\partial K^2}
  \]
FWD Equation: \( \frac{dS}{S} = \sigma(S,t) \, dW \)

Define \( CS_{K,T}^{\delta T} \equiv \frac{C_{K,T+\delta T} - C_{K,T}}{\delta T} \)

Equating prices at \( t_0 \):
\[
\frac{\partial C}{\partial T} = \frac{\sigma^2(K,T)}{2} \, K^2 \, \frac{\partial^2 C}{\partial K^2}
\]
FWD Equation: \( \frac{dS}{S} = r \, dt + \sigma(S,t) \, dW \)

\[ CS_{K,T}^{\delta T} \text{ at } T = \text{Time Value} + \text{Intrinsic Value} \]

(Strike Convexity) (Interest on Strike)

Equating prices at \( t_0 \):

\[
\frac{\partial C}{\partial T} = \frac{\sigma^2(K,T)}{2} K^2 \frac{\partial^2 C}{\partial K^2} - rK \frac{\partial C}{\partial K}
\]
FWD Equation: \[ \frac{dS}{S} = (r-d) \, dt + \sigma(S,t) \, dW \]

Equating prices at \( t_0 \):
\[
\frac{\partial C}{\partial T} = \frac{\sigma^2(K,T)}{2} K^2 \frac{\partial^2 C}{\partial K^2} - (r-d)K \frac{\partial C}{\partial K} - d \cdot C
\]

\( \delta T \to 0 \):

\[ CS_{K,T} \text{ at } T = TV + \text{Interests on } K \]
\[ - \text{Dividends on } S \]

Bruno Dupire
Stripping Formula

\[
\frac{\partial C}{\partial T} = \frac{\sigma^2(K,T)K^2}{2} \frac{\partial^2 C}{\partial K^2} - (r-d)K \frac{\partial C}{\partial K} - d \cdot C
\]

– If \( \sigma(K,T) \) known, quick computation of all \( C_{K,T}(S_0,t_0) \) today,

– If all \( C_{K,T}(S_0,t_0) \) known:

\[
\sigma(K,T) = \sqrt{\frac{2 \frac{\partial C}{\partial T} + (r-d)K \frac{\partial C}{\partial K} + dC}{K^2 \frac{\partial^2 C}{\partial K^2}}}
\]

Local volatilities extracted from vanilla prices and used to price exotics.
Smile dynamics: Local Vol Model (1)

• Consider, for one maturity, the smiles associated to 3 initial spot values

Skew case

– ATM short term implied follows the local vols
– Similar skews
Smile dynamics: Local Vol Model (2)

- Pure Smile case

- ATM short term implied follows the local vols
- Skew can change sign
Summary of LVM Properties

\[ \Sigma_0 \text{ is the initial volatility surface} \]

- \[ \sigma(S,t) \] compatible with \( \Sigma_0 \leftrightarrow \sigma = \text{local vol} \)
- \[ \sigma(\omega) \] compatible with \( \Sigma_0 \leftrightarrow E[\sigma^2|S_T=K] = (\text{local vol})^2 \)
- \[ \hat{\sigma}_{k,T} \] deterministic function of \( (S,t) \)

\[ \leftrightarrow \text{future smile} = \text{FWD smile from local vol} \]
Volatility Replication
Volatility Replication

\[ \frac{dS}{S} = \sigma \, dW \] \quad \text{Apply Ito to } f(S,t).

\[ df = f_s dS + f_t dt + \frac{1}{2} f_{ss} \sigma^2 S^2 dt \]

\[ \Rightarrow \int_0^T f_{ss}(S_t, t) \sigma^2 S^2 dt = 2 \left[ f(S_T, T) - f(S_0, 0) - \int_0^T f_t(S_t, t) dt - \int_0^T f_s(S_t, t) dS_t \right] \]

\begin{align*}
\text{European PF} & \quad \Delta \text{-hedge} \\
\text{To replicate } \int_0^T g(S, t) \sigma^2 dt, \text{find } f : \quad & g(S, t) = f_{ss}(S, t) S^2 \\
& f = \iint \frac{g}{S^2}
\end{align*}
### Examples

<table>
<thead>
<tr>
<th>Variance Swap</th>
<th>$g(S,t) = 1$</th>
<th>$f(S,t) = - \ln\left( \frac{S}{S_0} \right)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Corridor Variance Swap</td>
<td>$g(S,t) = 1_{[a,b]}(S_t)$</td>
<td>$f(S,t) = - \ln\left( \frac{S}{S_0} \right)$ on $[a,b]$ + linear extrapolation</td>
</tr>
<tr>
<td>FWD Variance Swap</td>
<td>$g(S,t) = 1_{[T_1,T_2]}(t)$</td>
<td>$f(S,t) = - \ln\left( \frac{S}{S_0} \right) \times 1_{[T_1,T_2]}(t)$</td>
</tr>
<tr>
<td>Absolute Variance Swap</td>
<td>$g(S,t) = S^2$</td>
<td>$f(S,t) = \frac{(S - S_0)^2}{2}$</td>
</tr>
<tr>
<td>Local Time at level K</td>
<td>$g(S,t) = \delta_K(S)$</td>
<td>$f(S,t) = \frac{(S - K)^+}{K^2}$</td>
</tr>
</tbody>
</table>
Conditional Instantaneous FWD Variance

From local time:

\[ E \left[ \int_{0}^{T} \sigma_{t}^{2} \delta_{K} (S) dt \right] = 2 \times \frac{C(K, T)}{K^{2}} \]

Differentiating wrt T:

\[ E \left[ \sigma_{T}^{2} \delta_{K} (S_{T}) \right] = E \left[ \sigma_{T}^{2} | S_{T} = K \right] \cdot E [\delta_{K} (S_{T})] = \frac{2}{K^{2}} \times \frac{\partial C}{\partial T} (K, T) \]

And, as:

\[ E [\delta_{K} (S_{T})] = \frac{\partial^{2} C}{\partial K^{2}} (K, T) \]

\[ E \left[ \sigma_{T}^{2} | S_{T} = K \right] = \frac{2}{K^{2}} \times \frac{\partial C (K, T)}{\partial T} \cdot \frac{\partial^{2} C}{\partial K^{2}} (K, T) = \sigma_{loc}^{2} (K, T) \]
Deterministic future smiles

It is not possible to prescribe just any future smile

If deterministic, one must have

\[ C_{K,T_2}(S_0, t_0) = \int \varphi(S_0, t_0, S, T_1) C_{K,T_2}(S, T_1) dS \]

Not satisfied in general
Det. Fut. smiles & no jumps
=> = FWD smile

If \( \exists (S, t, K, T) / V_{K,T}(S, t) \neq \bar{\sigma}^2(K, T) = \lim_{\delta K \to 0, \delta T \to 0} \sigma^2_{imp}(K, T, K + \delta K, T + \delta T) \)

stripped from SmileS.t

Then, there exists a 2 step arbitrage:

Define
\[
PL_t = \left(\bar{\sigma}^2(K, T) - V_{K,T}(S, t)\right) \frac{\partial^2 C}{\partial K^2}(S, t, K, T)
\]

At t0 : Sell \( PL_t \cdot \left( \text{Dig}_{S-\varepsilon,t} - \text{Dig}_{S+\varepsilon,t} \right) \)

At t: if \( S_t \in [S - \varepsilon, S + \varepsilon] \) buy \( \frac{2}{K^2} \text{CS}_{K,T} \), sell \( \bar{\sigma}^2(K, T) \delta_{K,T} \)
gives a premium = PLt at t, no loss at T

Conclusion: \( V_{K,T}(S, t) \) independent of \( (S, t) = V_{K,T}(S_0, t_0) = \sigma^2(K, T) \)
from initial smile
Consequence of det. future smiles

- Sticky Strike assumption: Each \((K,T)\) has a fixed \(\sigma_{impl}(K,T)\) independent of \((S,t)\)
- Sticky Delta assumption: \(\sigma_{impl}(K,T)\) depends only on moneyness and residual maturity

- In the absence of jumps,
  - Sticky Strike is arbitrageable
  - Sticky \(\Delta\) is (even more) arbitrageable
Example of arbitrage with Sticky Strike

Each CK,T lives in its Black-Scholes ($\sigma_{impl}(K,T)$) world

$C_1 \equiv C_{K_1,T_1}$, $C_2 \equiv C_{K_2,T_2}$, assume $\sigma_1 > \sigma_2$

P&L of Delta hedge position over $dt$:

$\delta PL(C_1) = \frac{1}{2} \left( (\delta S)^2 - \sigma_1 S^2 \delta t \right) \Gamma_1$

$\delta PL(C_2) = \frac{1}{2} \left( (\delta S)^2 - \sigma_2 S^2 \delta t \right) \Gamma_2$

$\delta PL(\Gamma_1 C_2 - \Gamma_2 C_1) = \frac{\Gamma_1 \Gamma_2}{2} S^2 \left( \sigma_1^2 - \sigma_2^2 \right) \delta t > 0$

(no $\Gamma$, free $\Theta$)

⚠️ If no jump
Arbitraging Skew Dynamics

• In the absence of jumps, Sticky-K is arbitrageable and Sticky-Δ even more so.
• However, it seems that quiet trending market (no jumps!) are Sticky-Δ.

In trending markets, buy Calls, sell Puts and Δ-hedge.

Example:

\[ PF \equiv C_{K_2} - P_{K_1} \]

\[ S \xrightarrow{\text{↑}} \begin{cases} \sigma_1, \sigma_2 \xrightarrow{\text{↑}} \\ \text{Vega}_{K_2} > \text{Vega}_{K_1} \end{cases} \rightarrow PF \]

\[ S \xrightarrow{\text{↓}} \begin{cases} \sigma_1, \sigma_2 \xrightarrow{\text{↓}} \\ \text{Vega}_{K_2} < \text{Vega}_{K_1} \end{cases} \rightarrow PF \]

Δ-hedged PF gains from S induced volatility moves.
Skew from Historical Prices
Problem: How to compute option prices on an underlying without options?
For instance: compute 3 month 5% OTM Call from price history only.

1) Discounted average of the historical Intrinsic Values.
   Bad: depends on bull/bear, no call/put parity.

2) Generate paths by sampling 1 day return recentered histogram.
   Problem: CLT converges quickly to same volatility for all strike/maturity;
   breaks autocorrelation and vol/spot dependency.
3) Discounted average of the Intrinsic Value from recentered 3 month histogram.

4) $\Delta$-Hedging: compute the implied volatility which makes the $\Delta$-hedging a fair game.

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Theoretical Skew from historical prices

How to get a theoretical Skew just from spot price history?

Example:

3 month daily data

1 strike \( K = k S_{T_i} \)

- a) price and delta hedge for a given \( \sigma \) within Black-Scholes model
- b) compute the associated final Profit & Loss: \( PL(\sigma) \)
- c) solve for \( \sigma(k)/PL(\sigma(k)) = 0 \)
- d) repeat a) b) c) for general time period and average
- e) repeat a) b) c) and d) to get the “theoretical Skew”
IV. Volatility Expansion
Introduction

• This talk aims at providing a better understanding of:
  – How local volatilities contribute to the value of an option
  – How P&L is impacted when volatility is misspecified
  – Link between implied and local volatility
  – Smile dynamics
  – Vega/gamma hedging relationship
Framework & definitions

• In the following, we specify the dynamics of the spot in absolute convention (as opposed to proportional in Black-Scholes) and assume no rates:

\[ dS_t = \sigma_t \, dW_t \]

• \( \sigma \): local (instantaneous) volatility (possibly stochastic)

• Implied volatility will be denoted by \( \hat{\sigma} \)
P&L of a delta hedged option

Option Value

- $C_t$
- $C_{t+\Delta t}$
- Delta hedge

P&L

- Break-even points
- $-\sigma\sqrt{\Delta t}$
- $\sigma\sqrt{\Delta t}$

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P&L of a delta hedged option (2)

Correct

Volatility higher than

Ito: When $\Delta t \to 0$, spot dependency disappears
Black-Scholes PDE

P&L is a balance between gain from $\Gamma$ and

$$P&L_{(t,t+dt)} = \left( \frac{\sigma^2}{2} \Gamma_0 + \Theta_0 \right) dt$$

From Black-Scholes PDE:

$$\Theta_0 = -\frac{\sigma_0^2}{2} \Gamma_0$$

$=>$ discrepancy if $\sigma$ different from

$$\text{gain over } dt = \frac{1}{2} \left( \sigma^2 - \sigma_0^2 \right) \Gamma_0 dt$$

- $\sigma > \sigma_0$: Profit
- $\sigma < \sigma_0$: Loss

Magnified by $\Gamma_0$
Total P&L over a path

= Sum of P&L over all small time intervals

\[ P\&L = \frac{1}{2} \int_0^T \left( \sigma^2 - \sigma_0^2 \right) \Gamma_0 dt \]

No assumption is made on volatility so far
General case

• Terminal wealth on each path is:

\[
\text{wealth}_T = X(\Sigma_0) + \frac{1}{2} \int_0^T (\sigma^2 - \sigma_0^2) \Gamma_0 dt
\]

(\(X(\Sigma_0)\) is the initial price of the option)

• Taking the expectation, we get:

\[
E^{\phi}[\text{wealth}_T] = X(\Sigma_0) + \frac{1}{2} \int_0^T \int_0^\infty E[\Gamma_0 (\sigma^2 - \sigma_0^2) | S] \varphi dS dt
\]

• The probability density \(\varphi\) may correspond to the density of a NON risk-neutral process (with some drift) with volatility \(\sigma\).
Non Risk-Neutral world

• In a complete model (like Black-Scholes), the drift does not affect option prices but alternative hedging strategies lead to different expectations.

Example: mean reverting process towards $L$ with high volatility around $L$.
We then want to choose $K$ (close to $L$), $T$ and $\sigma_0$ (small) to take advantage of it.

In summary: gamma is a volatility collector and it can be shaped by:

• a choice of strike and maturity,
• a choice of $\sigma_0$, our hedging volatility.
From now on, $\phi$ will designate the risk neutral density associated with $dS_t = \sigma dW_t$.

In this case, $\mathbb{E}[\text{wealth}_T]$ is also $X(\Sigma)$ and we have:

$$X(\Sigma) = X(\Sigma_0) + \frac{1}{2} \int_0^T \int_0^\infty E[\Gamma_0 (\sigma^2 - \sigma_0^2)|S] \phi \, dS \, dt$$

Path dependent option & deterministic vol:

$$X(\Sigma) = X(\Sigma_0) + \frac{1}{2} \int \int (\sigma^2 - \sigma_0^2) E[\Gamma_0|S] \phi \, dS \, dt$$

European option & stochastic vol:

$$C(\Sigma) = C(\Sigma_0) + \frac{1}{2} \int \int (E[\sigma^2|S] - \sigma_0^2) \Gamma_0 \phi \, dS \, dt$$
Quiz

- Buy a European option at 20% implied vol
- Realised historical vol is 25%
- Have you made money?

Not necessarily!
High vol with low gamma, low vol with high gamma
Expansion in volatility

- An important case is a European option with deterministic vol:

\[ C(\Sigma) = C(\Sigma_0) + \frac{1}{2} \int_0^T \int_0^T (\sigma^2 - \sigma_0^2) \Gamma_0 \varphi dS \, dt \]

- The corrective term is a weighted average of the volatility differences

- This double integral can be approximated numerically
P&L: Stop Loss Start Gain

• Extreme case: $\sigma_0 = 0 \Rightarrow \Gamma_0 = \delta_K$

$$C(\Sigma) = (S_0 - K)^+ + \frac{1}{2} \int_0^T \sigma(K,t)^2 \phi(K,t)dt$$

• This is known as Tanaka’s formula

Bruno Dupire
Local / Implied volatility relationship

Differentiation

Implied volatility
strike maturity

Local volatility
spot time

Aggregation
• Stripping local vols from implied vols is the inverse operation:

\[
\sigma^2(S, T) = 2 \frac{\partial C}{\partial^2 C} \frac{\partial T}{\partial K^2} \tag{Dupire 93}
\]

• Involves differentiations
Let us assume that local volatility is a deterministic function of time only:

$$dS_t = \sigma(t) \, dW_t$$

In this model, we know how to combine local vols to compute implied vol:

$$\hat{\sigma}(T) = \sqrt{\frac{\int_0^T \sigma^2(t) \, dt}{T}}$$

Question: can we get a formula with $\sigma(S, t)$?
From local to implied volatility

• When $\sigma_0 = \text{implied vol}$

\[
\frac{1}{2} \iint (\sigma^2 - \sigma_0^2) \Gamma_0 \phi dSdt = 0 \quad \Rightarrow \quad \sigma_0^2 = \frac{\iint \sigma^2 \Gamma_0 \phi dSdt}{\iint \Gamma_0 \phi dSdt}
\]

• $\Gamma_0$ depends on $\sigma_0 \implies$ solve by iterations

• Implied Vol is a weighted average of Local Vols
  (as a swap rate is a weighted average of FRA)
Weighting scheme

- Weighting Scheme: proportional to $\Gamma_0 \varphi$

At the money case:

$S_0 = 100$
$K = 100$

Out of the money case:

$S_0 = 100$
$K = 110$
Weighting scheme (2)

- Weighting scheme is roughly proportional to the brownian bridge density

Brownian bridge density:

$$BB\varphi_{K,T}(x,t) = P[S_t = x | S_T = K]$$
Time homogeneous case

\[ \hat{\sigma}^2 = \int \alpha(S) \sigma^2(S) dS \]

\[ \alpha(S) = \frac{\int \Gamma_0 \varphi dt}{\int \int \Gamma_0 \varphi dS dt} \]

\[ \Gamma_0 \]

ATM (K=S_0)

\[ \sigma \sqrt{T} \]

small

\[ \alpha(S) \]

\[ S_0 \]

\[ S \]

OTM (K>S_0)

\[ \sigma \sqrt{T} \]

large

\[ \alpha(S) \]

\[ S_0 \]

\[ K \]

\[ S \]
Link with smile

\( \hat{\sigma}_{K_1} \) and \( \hat{\sigma}_{K_2} \) are averages of the same local vols with different weighting schemes

\[ \Rightarrow \text{New approach gives us a direct expression for the smile from the knowledge of local volatilities} \]

But can we say something about its dynamics?
Weighting scheme imposes some dynamics of the smile for a move of the spot:

For a given strike $K$,

$$S^\uparrow \Rightarrow \hat{\sigma}_K \downarrow$$

(we average lower volatilities)

Smile today (Spot $S_t$)

&

Smile tomorrow (Spot $S_{t+dt}$) in sticky strike model

Smile tomorrow (Spot $S_{t+dt}$) if $\sigma_{\text{ATM}}=\text{constant}$

Smile tomorrow (Spot $S_{t+dt}$) in the smile model
A sticky strike model ($\hat{\sigma}_K(t) = \hat{\sigma}_K$) is arbitrageable.

Let us consider two strikes $K_1 < K_2$

The model assumes constant vols $\sigma_1 > \sigma_2$ for example

By combining $K_1$ and $K_2$ options, we build a position with no gamma and positive theta (sell 1 $K_1$ call, buy $\Gamma_1/\Gamma_2$ $K_2$ calls)
Vega analysis

• If $\sigma$ & $\sigma_0$ are constant

$$C(\Sigma) = C(\Sigma_0) + \frac{1}{2} (\sigma^2 - \sigma_0^2) \iiint \Gamma_0 \varphi dS dt$$

• $\sigma^2 = \sigma_0^2 + \varepsilon$

$$C(\sigma_0^2 + \varepsilon) = C(\sigma_0^2) + \varepsilon \frac{1}{2} \iiint \Gamma_0 \varphi dS dt$$

$$Vega = \frac{\partial C}{\partial \sigma} = \frac{\partial C}{\partial \sigma^2} \cdot \frac{\partial \sigma^2}{\partial \sigma} = \frac{\partial C}{\partial \sigma^2} \cdot 2\sigma$$
Gamma hedging vs Vega hedging

- Hedge in $\Gamma$ insensitive to realised historical vol
- If $\Gamma=0$ everywhere, no sensitivity to historical vol => no need to Vega hedge
- Problem: impossible to cancel $\Gamma$ now for the future
- Need to roll option hedge
- How to lock this future cost?
- Answer: by vega hedging
Superbuckets: local change in local vol

For any option, in the deterministic vol case:

\[ X(\Sigma) = X(\Sigma_0) + \frac{1}{2} \int_0^2 \int_0^2 (\sigma^2 - \sigma_0^2) E[\Gamma_0|S] \varphi \, dS \, dt \]

For a small shift \( \varepsilon \) in local variance around \((S,t)\), we have:

\[ X(\Sigma) = X(\Sigma_0) + \frac{1}{2} \varepsilon E[\Gamma_0|S] \varphi \]

\[ \Rightarrow \frac{dX}{d\left(\sigma_{(S,t)}^2\right)} = \frac{1}{2} E[\Gamma_0(S,t)|S] \varphi(S,t) \]

For a european option:

\[ \frac{dC}{d\left(\sigma_{(S,t)}^2\right)} = \frac{1}{2} \Gamma_0(S,t) \varphi(S,t) \]
Superbuckets: local change in implied vol

Local change of implied volatility is obtained by combining local changes in local volatility according a certain weighting

$$\frac{dC}{d(\hat{\sigma}^2)} = \int \frac{dC}{d(\sigma^2)} \frac{d(\sigma^2)}{d(\hat{\sigma}^2)}$$

Thus:

cancel sensitivity to any move of implied vol

$$\iff$$
cancel sensitivity to any move of local vol

$$\iff$$
cancel all future gamma in expectation
Conclusion

- This analysis shows that option prices are based on how they capture local volatility

- It reveals the link between local vol and implied vol

- It sheds some light on the equivalence between full Vega hedge (superbuckets) and average future gamma hedge
Delta Hedging

- We assume no interest rates, no dividends, and absolute (as opposed to proportional) definition of volatility
- Extend $f(x)$ to $f(x,v)$ as the Bachelier (normal BS) price of $f$ for start price $x$ and variance $v$:

$$f(x,v) \equiv E^{x,v}[f(X)] \equiv \frac{1}{\sqrt{2\pi v}} \int f(y)e^{-\frac{(y-x)^2}{2v}} dy$$

with $f(x,0) = f(x)$

- Then,
- We explore various delta hedging strategies

$$f_v(x,v) = \frac{1}{2} f_{xx}(x,v)$$
Calendar Time Delta Hedging

- Delta hedging with constant vol: P&L depends on the path of the volatility and on the path of the spot price.
- Calendar time delta hedge: replication cost of

\[ f(X_t, \sigma^2(T - t)) \]

- In particular, for \( \sigma = 0 \), replication cost of

\[ f(X_t) \]

\[ f(X_0) + \frac{1}{2} \int_0^t f_{xx} dQV_{0,u} \]
Business Time Delta Hedging

- Delta hedging according to the quadratic variation: P&L that depends only on quadratic variation and spot price
  \[ df(X_t, L - QV_{0,t}) = f_x dX_t - f_v dQV_{0,t} + \frac{1}{2} f_{xx} dQV_{0,t} = f_x dX_t \]
  \[ QV_{0,T} \leq L, \]
- Hence, for
  \[ f(X_t, L - QV_{0,t}) = f(X_0, L) + \int_0^t f_x(X_u, L - QV_{0,u}) dX_t \]
  And the replicating cost of \( f \) is
  \[ \tau : QV_{0,\tau} = L \]
Daily P&L Variation

P&L from delta hedging

Change in underlying

Calendar Time
Business time
Tracking Error Comparison
V. Stochastic Volatility Models
Hull & White

• Stochastic volatility model **Hull&White (87)**

\[
\frac{dS_t}{S_t} = rdt + \sigma_t dW_t^P
\]

\[
d\sigma_t = \alpha dt + \beta dZ_t^P
\]

• Incomplete model, depends on risk premium
• Does not fit market smile

\[\rho_{Z,W} = 0\]

\[\rho_{Z,W} < 0\]
Role of parameters

- Correlation gives the short term skew
- Mean reversion level determines the long term value of volatility
- Mean reversion strength
  - Determine the term structure of volatility
  - Dampens the skew for longer maturities
- Volvol gives convexity to implied vol
- Functional dependency on S has a similar effect to correlation
Heston Model

\[
\begin{aligned}
\frac{dS}{S} &= \mu \, dt + \sqrt{v} \, dW \\
rv &= \lambda (\bar{v} - v) dt + \eta \sqrt{v} dZ \quad \langle dW, dZ \rangle = \rho \, dt
\end{aligned}
\]

Solved by Fourier transform:

\[
x \equiv \ln \frac{FWD}{K} \quad \tau = T - t
\]

\[
C_{K,T}(x, v, \tau) = e^x P_1(x, v, \tau) - P_0(x, v, \tau)
\]
Spot dependency

2 ways to generate skew in a stochastic vol model

1) $\sigma_t = x_t f(S, t), \rho(W, Z) = 0$

2) $\sigma \rho(W, Z) \neq 0$

- Mostly equivalent: similar $(S_t, \sigma_t)$ patterns, similar future evolutions
- 1) more flexible (and arbitrary!) than 2)
- For short horizons: stoch vol model $\Leftrightarrow$ local vol model
  + independent noise on vol.
Convexity Bias

\[
\begin{aligned}
\frac{dS}{dt} &= \sigma_t dW \\
\frac{d\sigma_t^2}{dt} &= \alpha dZ \\
\rho(W, Z) &= 0
\end{aligned}
\]

\[\Rightarrow \quad E[\sigma_t^2 | S_t = K] = \sigma_0^2?\]

NO! only \( E[\sigma_t^2] = \sigma_0^2 \)

\( \sigma_t \) likely to be high if \( S_t >> S_0 \) or \( S_t << S_0 \)
Impact on Models

• Risk Neutral drift for instantaneous forward variance

• Markov Model:

\[
\frac{dS}{S} = f(S,t) \sigma_t dW
\]

fits initial smile with local vols \( \sigma(S,t) \)

\[\iff \quad f(S,t) = \frac{\sigma^2(S,t)}{E[\sigma_t^2 \mid S_t = S]}\]
Smile dynamics: Stoch Vol Model (1)

Skew case (r<0)

- ATM short term implied still follows the local vols

\[
\left( E \left[ \sigma_T^2 \middle| S_T = K \right] = \sigma^2(K, T) \right)
\]

- Similar skews as local vol model for short horizons
- Common mistake when computing the smile for another spot: just change \( S_0 \) forgetting the conditioning on \( \sigma \):
  if \( S : S_0 \rightarrow S^+ \) where is the new \( \sigma \)?
Smile dynamics: Stoch Vol Model (2)

- Pure smile case \((r=0)\)

\[
\begin{align*}
\sigma &\quad S^- &\quad S_0 &\quad S^+ \\
\text{Smile } S^- &\quad \text{Smile } S^+ &\quad \text{Local vols} \\
\text{Smile } S_0 &
\end{align*}
\]

- ATM short term implied follows the local vols
- Future skews quite flat, different from local vol model
- Again, do not forget conditioning of vol by \(S\)
Forward Skew
Forward Skews

In the absence of jump:

model fits market \( \iff \forall K, T \quad E[\sigma_T^2 | S_T = K] = \sigma_{loc}^2 (K, T) \)

This constrains

a) the sensitivity of the ATM short term volatility wrt S;

b) the average level of the volatility conditioned to \( S_T = K \).

a) tells that the sensitivity and the hedge ratio of vanillas depend on the calibration to the vanilla, not on local volatility/stochastic volatility.

To change them, jumps are needed.

But b) does not say anything on the conditional forward skews.
At \( t \), short term ATM implied volatility ~ \( \sigma_t \).

As \( \sigma_t \) is random, the sensitivity \( \frac{\partial \sigma^2}{\partial S} \) is defined only in average:

\[
E_t[\sigma^2_{t+\delta t} - \sigma^2_t | S_{\delta t} = S_t + \delta S] = \sigma^2_{loc} (S_t + \delta S, t + \delta t) - \sigma^2_{loc} (S_t - t) \approx \frac{\partial \sigma^2_{loc} (S, t)}{\partial S} \cdot dS
\]

In average, \( \sigma^2_{ATM} \) follows \( \sigma^2_{loc} \).

Optimal hedge of vanilla under calibrated stochastic volatility corresponds to perfect hedge ratio under LVM.