I. Generalities
Dominating fact since 1987 crash: strong negative skew on Equity Markets

Not a general phenomenon

Gold: 

FX:

We focus on Equity Markets
Skews

- Volatility Skew: slope of implied volatility as a function of Strike
- Link with Skewness (asymmetry) of the Risk Neutral density function $\phi$ ?

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Why Volatility Skews?

• Market prices governed by
  – a) Anticipated dynamics (future behavior of volatility or jumps)
  – b) Supply and Demand

• To “arbitrage” European options, estimate a) to capture risk premium b)

• To “arbitrage” (or correctly price) exotics, find Risk Neutral dynamics calibrated to the market
Modeling Uncertainty

Main ingredients for spot modeling

• Many small shocks: Brownian Motion (continuous prices)

• A few big shocks: Poisson process (jumps)
2 mechanisms to produce Skews (1)

- To obtain downward sloping implied volatilities
  - a) Negative link between prices and volatility
    - Deterministic dependency (Local Volatility Model)
    - Or negative correlation (Stochastic volatility Model)
  - b) Downward jumps
2 mechanisms to produce Skews (2)

- a) Negative link between prices and volatility

- b) Downward jumps
Model Requirements

• Has to fit static/current data:
  – Spot Price
  – Interest Rate Structure
  – Implied Volatility Surface

• Should fit dynamics of:
  – Spot Price (Realistic Dynamics)
  – Volatility surface when prices move
  – Interest Rates (possibly)

• Has to be
  – Understandable
  – In line with the actual hedge
  – Easy to implement
Beyond initial vol surface fitting

• Need to have proper dynamics of implied volatility
  – Future skews determine the price of Barriers and OTM Cliquet
  – Moves of the ATM implied vol determine the $\Delta$ of European options

• Calibrating to the current vol surface do not impose these dynamics
Barrier options as Skew trades

- In Black-Scholes, a Call option of strike $K$ extinguished at $L$ can be statically replicated by a Risk Reversal.

- Value of Risk Reversal at $L$ is 0 for any level of (flat) vol.
- $Pb$: In the real world, value of Risk Reversal at $L$ depends on the Skew.
II. A Brief History of Volatility
A Brief History of Volatility (1)

- \( dS_t = \sigma \, dW_t^Q \) : Bachelier 1900

- \( \frac{dS_t}{S_t} = r \, dt + \sigma \, dW_t^Q \) : Black-Scholes 1973

- \( \frac{dS_t}{S_t} = r(t) \, dt + \sigma(t) \, dW_t^Q \) : Merton 1973

- \( \frac{dS_t}{S_t} = (r - \lambda k) \, dt + \sigma \, dW_t^Q + dq \) : Merton 1976

\[
\begin{cases}
  dS_t = r \, dt + \sigma_t \, dW_t^Q \\
  d\sigma_t^2 = a(V_L - V)dt + \xi \, \sigma^\alpha \, dZ_t
\end{cases}
\] : Hull&White 1987
A Brief History of Volatility (2)

\[
\frac{dS_t}{S_t} = \sigma_t \, dW_t^Q
\]

\[
d\sigma_t^2 = 2 \frac{\partial^2 L_T(t)}{\partial T^2} \, dt + \alpha \, dZ_t^Q
\]

Dupire 1992, arbitrage model which fits term structure of volatility given by log contracts.

\[
\frac{dS_t}{S_t} = r(t) \, dt + \sigma(S, t) \, dW_t^Q
\]

\[
\sigma^2(K, T) = 2 \frac{\partial C}{\partial T} + rK \frac{\partial C}{\partial K}
\]

\[
= \frac{K^2}{K^2} \frac{\partial^2 C_{K,T}}{\partial K^2}
\]

Dupire 1993, minimal model to fit current volatility surface.
Heston 1993, semi-analytical formulae.


\[
\begin{align*}
\frac{dS_t}{S_t} &= r \, dt + \sigma_t \, dW_t \\
\sigma_t^2 &= b(\sigma_\infty^2 - \sigma_t^2)dt + \beta \sigma_t \, dZ_t \\

\end{align*}
\]

\[dV_{K,T} = \alpha_{K,T} \, dt + b_{K,T} \, dZ^Q_t\]

\(V_{K,T}\): instantaneous forward variance conditional to \(S_T = K\)
A Brief History of Volatility (4)

– Bates 1996, Heston + Jumps:

\[
\begin{align*}
\frac{dS_t}{S_t} &= r \, dt + \sigma_t \, dZ_t + dq \\
\sigma_t^2 &= b(\sigma_\infty^2 - \sigma_t^2) dt + \beta \sigma_t \, dW_t
\end{align*}
\]

– Local volatility + stochastic volatility:

• Markov specification of UTV
• Reech Capital Model: f is quadratic
• SABR: f is a power function

\[
\frac{dS_t}{S_t} = r \, dt + \sigma_t \, f(S_t, t) \, dZ^0_t
\]
A Brief History of Volatility (5)

- Lévy Processes
- Stochastic clock:
  - VG (Variance Gamma) Model:
    - BM taken at random time \( g(t) \)
  - CGMY model:
    - same, with integrated square root process
- Jumps in volatility (Duffie, Pan & Singleton)
- Path dependent volatility
- Implied volatility modelling
- Incorporate stochastic interest rates
- \( n \) dimensional dynamics of \( s \)
- \( n \) assets stochastic correlation
III. Local Volatility Model
From Simple to Complex

• How to extend Black-Scholes to make it compatible with market option prices?
  – Exotics are hedged with Europeans.
  – A model for pricing complex options has to price simple options correctly.
Black-Scholes assumption

- BS assumes constant volatility
  
  => same implied vols for all options.

\[ \frac{dS}{S} = \mu \, dt + \sigma \, dW \]

(instantaneous vol)
Black-Scholes assumption

• In practice, highly varying.

Nikkei

Japanese Government Bonds
Modeling Problems

- Problem: one model per option.
  - for C1 (strike 130) $\sigma = 10\%$
  - for C2 (strike 80) $\sigma = 20\%$
One Single Model

- We know that a model with $\sigma(S,t)$ would generate smiles.
  - Can we find $\sigma(S,t)$ which fits market smiles?
  - Are there several solutions?

ANSWER: One and only one way to do it.
Interest rate analogy

- From the current Yield Curve, one can compute an Instantaneous Forward Rate.
  - Would be realized in a world of certainty,
  - Are not realized in real world,
  - Have to be taken into account for pricing.
Volatility

Dream: from Implied Vols read Local (Instantaneous Forward) Vols

How to make it real?
Discretization

• Two approaches:
  
  – to build a tree that matches European options,
  
  – to seek the continuous time process that matches European options and discretize it.
Tree Geometry

To discretize $\sigma(S,t)$ TRINOMIAL is more adapted

Example:
Tango Tree

• Rules to compute connections
  – price correctly Arrow-Debreu associated with nodes
  – respect local risk-neutral drift

• Example

![Diagram of a Tango Tree with labels and connections]
Continuous Time Approach

Call Prices ← Exotics

Distributions ? Diffusion

Bruno Dupire
Distributions - Diffusion

Distributions

Diffusion
Distributions - Diffusion

- Two different diffusions may generate the same distributions

\[ dx = -\lambda x \, dt + \sigma \, dW_t \]

\[ dx = b(t) \, dW_t \]
The Risk-Neutral Solution

But if drift imposed (by risk-neutrality), uniqueness of the solution

Diffusions

Risk Neutral Processes

Compatible with Smile

sought diffusion (obtained by integrating twice Fokker-Planck equation)
Continuous Time Analysis

- Implied Volatility
- Local Volatility

- Call Prices
- Densities
Implication: risk management

- Implied volatility
- Black box
- Price
- Perturbation
- Sensitivity
Forward Equations (1)

- BWD Equation:
  price of one option $C(K_0, T_0)$ for different $(S, t)$
- FWD Equation:
  price of all options $C(K, T)$ for current $(S_0, t_0)$
- Advantage of FWD equation:
  - If local volatilities known, fast computation of implied volatility surface,
  - If current implied volatility surface known, extraction of local volatilities,
  - Understanding of forward volatilities and how to lock them.
Forward Equations (2)

• Several ways to obtain them:
  – Fokker-Planck equation:
    • Integrate twice Kolmogorov Forward Equation
  – Tanaka formula:
    • Expectation of local time
  – Replication
    • Replication portfolio gives a much more financial insight
Fokker-Planck

- If \( dx = b(x, t)dW \)
- Fokker-Planck Equation: \( \frac{\partial \varphi}{\partial t} = \frac{1}{2} \frac{\partial^2 (b^2 \varphi)}{\partial x^2} \)
- Where \( \varphi \) is the Risk Neutral density. As \( \varphi = \frac{\partial^2 C}{\partial K^2} \)

\[
\frac{\partial^2 
\left( \frac{\partial C}{\partial t} \right)}{\partial x^2} = \frac{\partial \left( \frac{\partial^2 C}{\partial K^2} \right)}{\partial t} = \frac{1}{2} \frac{\partial^2 \left( b^2 \frac{\partial^2 C}{\partial K^2} \right)}{\partial x^2}
\]

- Integrating twice w.r.t. \( x \): \( \frac{\partial C}{\partial t} = \frac{b^2}{2} \frac{\partial^2 C}{\partial K^2} \)
FWD Equation: \( \frac{dS}{S} = \sigma(S,t) \, dW \)

Define \( CS_{K,T}^{\delta T} \equiv \frac{C_{K,T+\delta T} - C_{K,T}}{\delta T} \)

Equating prices at \( t_0 \):

\[
\frac{\partial C}{\partial T} = \frac{\sigma^2(K,T)}{2} K^2 \frac{\partial^2 C}{\partial K^2}
\]
FWD Equation: \( \frac{dS}{S} = r \, dt + \sigma(S,t) \, dW \)

\[
CS_{K,T}^{\delta T} \quad \text{at } T = \text{Time Value} \quad + \quad \text{Intrinsic Value}
\]

(Strike Convexity) \quad (Interest on Strike)

Equating prices at \( t_0 \):

\[
\frac{\partial C}{\partial T} = \frac{\sigma^2(K,T)}{2} K^2 \frac{\partial^2 C}{\partial K^2} - rK \frac{\partial C}{\partial K}
\]
FWD Equation: \( \frac{dS}{S} = (r-d) \, dt + \sigma(S,t) \, dW \)

Equating prices at \( t_0 \):

\[
\frac{\partial C}{\partial T} = \frac{\sigma^2(K,T)}{2} K^2 \frac{\partial^2 C}{\partial K^2} - (r-d)K \frac{\partial C}{\partial K} - d \cdot C
\]