Effects Of The Choice Of Numeraire

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Can you compare apples and oranges?

- Assume $\mathbb{P}(\text{Rain}) = \mathbb{P}(\text{Sun}) = 0.5$
- Choose oranges for numeraire
- Take expectation to get $1\text{Apple} = \frac{1}{2}0.5\text{Orange} + \frac{1}{2}2\text{Orange} = 1.25\text{Orange}$
- Now switch numeraire to apples: $1\text{Orange} = 1.25\text{Apple}$ !!!
• Conclusion: A probability measure cannot be risk neutral w.r.t. two numeraires.

• FTAP:
  – Choice of asset $A$ as numeraire
  – Denote price of any other asset $X$ in units of $A$ as $X_A$
  – $Q_A$ is a risk neutral measure if $X_A$ is a martingale $\forall X$.
  – Theorem: No arbitrage + Completeness $\iff \exists! Q_A$
• Writing the price of an asset in a different numeraire:

\[
X_A(0) = \mathbb{E}_A^{Q_A}[X_A(T)]
\]

\[
\frac{X_0}{A_0} = \mathbb{E}_A^{Q_A}\left[\frac{X(T)}{A(T)}\right]
\]

\[
X(0) = A(0)\mathbb{E}_A^{Q_A}\left[\frac{X(T)}{A(T)}\right]
\]

• Correlation between assets also changes when changing numeraire
Examples of Numeraires

- $\beta_t = e^{\int_0^t r_s ds}$: The Money Market Account

- $X(0) = \mathbb{E}_Q^{\beta} [X(T)e^{-\int_0^t r_s ds}]$

- $ZC_T$: The Zero coupon Bond

- $X(0) = B(0, T)\mathbb{E}_Q^T[X(T)]$

- Roll Over The Bond: "Jumping Numeraire" $\prod_{i=1}^{p} \frac{1}{B_{t_{i-1}, t_i}}, T = t_p$
\[ X(0) = \mathbb{E}^{\mathbb{Q}_{\pi}} [X(T) \prod_{i=1}^{p} B_{t_{i-1}, t_i}] \]
Statistical as Risk Neutral Measure

- Use the Radon Nikodym derivative to change between numeraires:
  
  \[ X(0) = A(0)\mathbb{E}^{Q_A} \left[ \frac{X(T)}{A(T)} \right] \]
  
  \[ X(0) = B(0)\mathbb{E}^{Q_B} \left[ \frac{X(T)}{B(T)} \right] \]
  
  \[ X(0) = B(0)\mathbb{E}^{Q_A} \left[ \frac{X(T)}{B(T)} \frac{dQ_B}{dQ_A} \right] \]
• Comparing the first and last $\mathbb{E}^{Q_A}$-expectations, we have:

\[
\frac{dQ_B}{dQ_A} = \frac{B(T)A_0}{A(T)B_0}
\]

• $B$ chosen s.t. $\mathbb{P} = Q_B$ (physical/statistical measure)

•

\[
B(T) = A(T)\frac{d\mathbb{P}}{dQ_A}
\]

• It is possible to choose a numeraire (asset) for which the price of all assets are martingales under physical measure.

• Example:

\[-\]

\[
\frac{dS}{S} = \mu dt + \sigma dW^\mathbb{P}_t
\]

\[-\]

$B = S^\alpha$ is a numeraire in which assets are $\mathbb{P}$-martingales.
The above implies, that with appropriate choice of units of account, the market price of risk is 0.

One can represent the risk/return relationship of assets as two dimensional vectors:

- Projection is expected return
- Orthogonal is diversifiable risk
• The risk premium is the length of the projection from the numeraire A

• Efficient frontier is the set of assets on the line between from the asset to $A_T$, "Atlantis".

• The frontier changes with change of numeraire.