Merton’s Jump Diffusion Model

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Continuous Time Finance

Lecture 5

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Introduction

- Merton’s 1976 JFE article “Option pricing when underlying stock returns are discontinuous” was the first to explore jump diffusion models.

- Jump diffusion models address the issue of "fat tails".


- When the underlying can jump to any level, the market is not complete, since there are many more states than assets.

- How to come up with a unique price for options in this setting?

- Merton’s novel proposal: Assume that the extra randomness due to jumps can be diversified away.
Mathematical Impact of Jumps

- Black-Scholes PDE becomes a partial *integrodifferential* equation.
- Fourier transform is a convenient tool for solving PIDE (especially in a constant coefficient setting).
- Contrast a one-dimensional diffusion for returns $y = \ln S$:
  \[ dy = \mu \, dt + \sigma \, dw, \]
  where $\mu$ and $\sigma$ can be functions of $y$ and $t$, with the SDE in a jump-diffusion setting:
  \[ dy = \mu \, dt + \sigma \, dw + J \, dN \]
- Here, the jump magnitudes $J$ are i.i.d. r.v.’s, i.e. the jump-size $J$ is selected by drawing from a pre-specified probability distribution.
- Ito’s Lemma: if $v(x, t)$ is smooth enough, $v(y(t), t)$ is again a jump-diffusion, with
  \[ d[v(y(t), t)] = (v_t + \mu v_x + \frac{1}{2} \sigma^2 v_{xx}) dt + \sigma v_x dw + [v(y(t) + J, t) - v(y(t), t)] dN. \]
Dynamics of Conditional Expectations

• Now consider the expected final-time payoff

\[ u(x, t) = E_{y(t)=x}[w(y(T))] \]

• Here \( w(x) \) is an arbitrary “payoff” (later it will be the payoff of an option).

• Solves a backward Kolmogorov equation

\[ u_t + \mathcal{L}u = 0 \text{ for } t < T, \quad \text{with } u(x, T) = w(x) \text{ at } t = T. \tag{1} \]

• The operator \( \mathcal{L} \) is

\[ \mathcal{L}u = \mu u_x + \frac{1}{2}\sigma^2 u_{xx} + \lambda E\left[u(x + J, t) - u(x, t)\right]. \]

• The expectation in the last term is over the probability distribution of jumps (in the log price).
• Let $u$ solve (1), and apply Itô’s formula:

$$u(y(T), T) - u(x, t) = \int_0^T (\sigma u_x)(y(s), s) \, dw + \int_0^T (u_s + \mu u_x + \frac{1}{2} \sigma^2 u_{xx})(y(s), s) \, ds$$

$$+ \int_0^T [u(y(s) + J, s) - u(y(s), s)] \, dN.$$  

• Take the expectation, noting that the jump magnitudes, $J$, are independent of the Poisson jump occurrence process, $N$:

$$E([u(y(s) + J, s) - u(y(s), s)] \, dN) = E ([u(y(s) + J, s) - u(y(s), s)]) \lambda ds.$$  

• Thus when $u$ solves (1), we get:

$$E[u(y(T), T)] - u(x, t) = 0.$$  

• This gives the result, since $u(y(T), T) = w(y(T))$. 

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Adding Interest Rates (Finally)

• Assume a (nonzero) constant interest rate $r$

• A similar argument shows

$$u(x, t) = E_{y(t)=x} \left[ e^{-r(T-t)}w(y(T)) \right]$$

solves

$$u_t + Lu - ru = 0 \text{ for } t < T, \quad \text{with } \quad u(x, T) = w(x) \text{ at } t = T,$$

using the same operator $L$.

• The probability distribution solves the forward Kolmogorov equation,

$$p_s - L^* p = 0 \text{ for } s > 0, \quad \text{with } \quad p(z, 0) = p_0(z)$$

• $p_0$ is the initial probability distribution, $L^*$ is the adjoint of $L$. 
• What is the adjoint of the new jump term?

• For any functions $\xi(z), \eta(z)$ we have

$$\int_{-\infty}^{\infty} E[[\xi(z + J) - \xi(z)] \eta(z) \, dz = \int_{-\infty}^{\infty} \xi(z) E[[\eta(z - J) - \xi(z)] \, dz =$$

since $\int_{-\infty}^{\infty} E[\xi(z + J)] \eta(z) \, dz = \int_{-\infty}^{\infty} \xi(z) E[\eta(z - J)] \, dz.$

• We have:

$$\mathcal{L}^*p = \frac{1}{2}(\sigma^2 p)_{zz} - (\mu p)_z + \lambda E[p(z - J) - p(z)],$$

• And so:

$$p_s - \frac{1}{2}(\sigma^2 p)_{zz} + (\mu p)_z - \lambda E[p(z - J, s) - p(z, s)] = 0.$$
Hedging and the Risk-Neutral Process

- Assume that one can only trade the underlying stock and a riskfree asset. Further assume that Merton’s jump diffusion process governs log prices.
- Without further assumptions, no-arbitrage cannot be used to give a unique price.
- Still, the payoff $w(S)$ should be the discounted final-time payoff under “the risk-neutral dynamics.”
- Stock price dynamics under statistical measure $\mathbb{P}$ are:

$$dS = (\mu + \frac{1}{2}\sigma^2)Sdt + \sigma Sdw + (e^J - 1)SdN.$$
Hedging and the Risk-Neutral Process

- Recall that the stock dynamics under statistical measure $\mathbb{P}$ are:
  \[ dS = (\mu + \frac{1}{2}\sigma^2)Sdt + \sigma Sdw + (e^J - 1)SdN. \]

- Merton proposed that the risk-neutral process be determined by two considerations:
  (a) it has the same volatility and jump statistics – i.e. it differs from the subjective process only by having a different drift; and
  (b) under the risk-neutral process $e^{-rt}S$ is a martingale, i.e. $dS - rSdt$ has mean value 0.

- Risk-neutral process is
  \[ dS = (r - \lambda E[e^J - 1])Sdt + \sigma Sdw + (e^J - 1)SdN. \] (2)
Hedging and the Risk-Neutral Process

• Applying Itô’s formula once more, we see that under the risk-neutral dynamics \( y = \log S \) satisfies

\[
dy = (r - \frac{1}{2}\sigma^2 - \lambda E[e^J - 1])dt + \sigma dw + JdN
\]

• Can we use \( \mu = r - \frac{1}{2}\sigma^2 - \lambda E[e^J - 1] \) to price options?

• Need to be able to hedge.

• Try hedging a long position in the option by a short position of \( \Delta \) units of stock:

\[
d[u(S(t), t)] - \Delta dS = u_t dt + u_S([\mu + \frac{1}{2}\sigma^2]Sdt + \sigma Sdw) + \frac{1}{2}u_{SS}\sigma^2 S^2 dt \\
+ [u(e^J S(t), t) - u(S(t), t)]dN \\
- \Delta([\mu + \frac{1}{2}\sigma^2]Sdt + \sigma Sdw) - \Delta(e^J - 1)SdN.
\]

• Market is incomplete, no choice of \( \Delta \) makes this portfolio risk-free.
• Choose $\Delta = u_S(S(t), t)$

• Randomness due to $dw$ cancels, leaving only the uncertainty due to jumps:

  \[
  \text{portfolio gain} = \left( u_t + \frac{1}{2} \sigma^2 S^2 u_{SS} \right) dt + \left\{ \left[ u(e^J S(t), t) - u(S(t), t) \right] - u_S(e^J S - S) \right\} dN.
  \]

• Merton: assume jumps uncorrelated with the marketplace.

• Impact of such randomness can be eliminated by diversification.

• According the the Capital Asset Pricing Model, for such an investment (whose $\beta$ is zero) only the mean return is relevant to pricing.

• So the mean return on our hedge portfolio should be the risk-free rate:

  \[
  (u_t + \frac{1}{2} \sigma^2 S^2 u_{SS}) dt + \lambda E[u(e^J S(t), t) - u(S(t), t) - (e^J S - S)u_S] dt = (u - Su_S) dt. \tag{3}
  \]

• Obtain the backward Kolmogorov equation describing the discounted final-time payoff under the risk-neutral dynamics (2):

  \[
  u_t + (r - \lambda E[e^J - 1]) Su_S + \frac{1}{2} \sigma^2 S^2 u_{SS} - ru + \lambda E[u(e^J S, t) - u(S, t)] = 0.
  \]
• Calls and puts have convex payoffs in $S$

• As a result, the jump term in (3) is positive:

\[ E[u(e^J S(t), t) - u(S(t), t) - (e^J S - S)u_S] \geq 0. \]

• Between jumps, the hedge portfolio rises slower than risk free rate.

• Jumps work in favour of the option holder.
Why Model Jumps?

- Better able to fit smile.
- There exists a consistent theoretical framework.
- Can experiment with adapting the stock hedge or hedging with options.
- The model can be calibrated to plain vanilla options and used to price and (partially) hedge exotic options.