Black’s Model With Default

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Assumptions

- Zero interest rates.
- Futures price $F_t$ at time $t \in [0, T]$ for maturity $T' \geq T$
- $F$: continuous time stochastic process.
- Futures: continuous marking to market.
- $\mathbb{P}$: Statistical probability measure
- $W$: Standard Brownian motion under $\mathbb{P}$.
- $N$: Standard Poisson process under $\mathbb{P}$.
• Black model:

\[ \frac{dF_t}{F_t} = \alpha dt + \sigma dW_t, \quad t \in [0, T], \quad (1) \]

- \( F_0 \) and \( \sigma \) known positive constants.

• Cox Ross single jump Poisson model:

\[ \frac{dF_t}{F_{t-}} = \mu dt + (e^j - 1) dN_t, \quad t \in [0, T], \quad (2) \]

- \( F_0 \) known positive constant, \( \mu \) and \( j \) real numbers of opposite sign.

- In (1) no need to know risk premium \( \alpha \)

- In (2) no need to know arrival rate \( \lambda_p \)
• Both models give complete market ⇒ Unique RN $\mathbb{Q}$

• After Measure Change $\mathbb{P} \rightarrow \mathbb{Q}$
  
  – Black’s model: Volatility $\sigma$ unchanged.
  – Cox Ross model: $\mu$ and $j$ unchanged.
  – Black model: risk premium $\alpha = 0$ ($\mathbb{Q}$ risk-neutral measure).
  – Cox Ross model: risk-neutral arrival rate of a jump is $\lambda_q \equiv -\frac{\mu}{e^j - 1}$.

• Intuition: no need to know $\alpha$ in Black’s model, $\lambda_p$ in jump model because info contained in futures price (known).

• $\alpha$ in Black’s model changes, $\mu$ in pure jump model does not.

• Fundamental Rules:
  
  1. $\mathbb{Q}$ is defined so that $F$ is a $\mathbb{Q}$ martingale
  2. A change of measure cannot change the numerical value of a parameter that can be estimated with certainty by continuous observation of a (segment of) a single path.
• More realistic stochastic process for $F$ combines both processes.

• Waiting time $\tau$ to the first jump of $N$: exponentially distributed r.v. with constant parameter $\lambda_p > 0$.

• Let $\tau$ be the default time of the limited liability asset underlying the futures.

• $t < \tau$: $F_t$ follows geometric Brownian motion with constant drift $\alpha$, constant volatility $\sigma$.

• At $\tau$, $F$ drops to zero and remains there afterwards.

• Put it all together (under $\mathbb{P}$):

$$\frac{dF_t}{F_{t-}} = \alpha dt + \sigma dW_t - dN_t, \quad t \in [0, T],$$

(3)
• $F_0$ and $\sigma$ are known positive constants.

• Comparing (3) with (2), jump size $j$ set to negative infinity, and B.M. has been introduced.

• Once $F$ hits zero, it absorbs there: increments multiplied by $F_{t-} = 0$.

• No need to know $\alpha$ or $\lambda_p$ - can actually assume $\alpha$ real-valued stochastic process, $\lambda_p$ positive stochastic process.

• Default Indicator Process $D$: defined by $D_t = 1(N_t > 0)$ gives:

$$\frac{dF_t}{F_{t-}} = \alpha dt + \sigma dW_t - dD_t, \quad t \in [0, T],$$

(4)

• (4) is a continuous time trinomial model up to $\tau$:
  
  – Brownian increments generate moves up and down of order $\sqrt{dt}$
  
  – The Poisson process generates an $O(dt)$ probability of a large down move in the price of order one.
• Perfect replication of every derivative on futures price path, requires ability to dynamically trade three assets.

• Dynamic trading in just the money market account and the futures contract may work for some payoffs, but it will not suffice for all payoffs.

• Assume futures written on a stock, and introduce a credit default swap (CDS) written on a bond issued by the stock issuer.

• Assume zero recovery rate for the bond for simplicity ⇒ Default event causes both the bond price and the stock price to vanish (think Enron).

• For simplicity, assume that the CDS rate is constant and observable at $\lambda_q > 0$.

• Further assume that the CDS rate is paid continuously, rather than periodically.

• As a result, prior to default, an investor can access the payoff $dD_t - \lambda_q dt$ at zero cost.
Analysis

• Let $V(F, t) : \mathbb{R} \times [0, T] \rightarrow \mathbb{R}$ be a $C^{2,1}$ function. Itô’s lemma for semi-martingales implies:

$$V(F_T, T) = V(F_0, 0) + \int_0^T \frac{\partial V}{\partial F}(F_{t-}, t) dF_t + \int_0^T \left[ \frac{\partial V}{\partial t}(F_{t-}, t) + \frac{\sigma^2 F^2_{t-}}{2} \frac{\partial^2 V}{\partial F^2}(F_{t-}, t) \right] dt$$

$$+ \int_0^T \left[ V(0, t) - V(F_{t-}, t) - \frac{\partial V}{\partial F}(F_{t-}, t)(0 - F_{t-}) \right] dD_t.$$  \hspace{1cm} (5)

• Add and subtract so that last term in (5) is gain from dynamically trading a CDS:

$$V(F_T, T)$$

$$= V(F_0, 0) + \int_0^T \frac{\partial V}{\partial F}(F_{t-}, t) dF_t$$

$$+ \int_0^T \left\{ \frac{\partial V}{\partial t}(F_{t-}, t) + \frac{\sigma^2 F^2_{t-}}{2} \frac{\partial^2 V}{\partial F^2}(F_{t-}, t) + \lambda_q \left[ V(0, t) - V(F_{t-}, t) + \frac{\partial V}{\partial F}(F_{t-}, t)F_{t-} \right] \right\} dt$$

$$+ \int_0^T \left[ V(0, t) - V(F_{t-}, t) + \frac{\partial V}{\partial F}(F_{t-}, t)F_{t-} \right] (dD_t - \lambda_q dt).$$  \hspace{1cm} (6)
• Suppose \( V(F, t) \) solves the following partial differential difference equation (PDDE):

\[
\frac{\partial V}{\partial t}(F, t) + \frac{\sigma^2 F^2}{2} \frac{\partial^2 V}{\partial F^2}(F, t) + \lambda_q \left[ V(0, t) - V(F, t) + \frac{\partial V}{\partial F}(F, t)F \right] = 0,
\]

on the domain: \( F > 0, t \in [0, T] \) and with terminal condition:

\[
V(F, T) = f(F), \quad F > 0.
\]

• The solution to this Cauchy problem exists and is unique.

• (6) reduces to:

\[
f(F_T) = V(F_0, 0) + \int_0^T \frac{\partial V}{\partial F}(F_{t-}, t) dF_t
\]

\[
+ \int_0^T \left[ V(0, t) - V(F_{t-}, t) + \frac{\partial V}{\partial F}(F_{t-}, t)F_{t-} \right] (dD_t - \lambda_q dt).
\]

• Charge \( V(F_0, 0) \) dollars initially hold \( \frac{\partial V}{\partial F}(F_{t-}, t) \) futures and \( V(0, t) - V(F_{t-}, t) + \frac{\partial V}{\partial F}(F_{t-}, t)F_{t-} \) CDS at each \( t \in [0, T] \)

• Achieve final payoff \( f(F_T) \).
• Recall:

\[ f(F_T) = V(F_0, 0) + \int_0^T \frac{\partial V}{\partial F}(F_{t-}, t)dF_t \]

\[ + \int_0^T \left[ V(0, t) - V(F_{t-}, t) + \frac{\partial V}{\partial F}(F_{t-}, t)F_{t-} \right] (dD_t - \lambda_q dt). \]

• Positions in the two risky hedge instruments will vanish after the default time.

• Replication is achieved without knowledge of the drift or the arrival rate of jumps (under \( \mathbb{P} \)).

• If the futures price behaved as in (4), it would be trivial to estimate \( \sigma \).

• This model has all of the econometric advantages of the simpler Black model: parameters needed to price are easily determined from sample path, quantities which are difficult to estimate from the path are not needed for pricing.
Pricing a Call

- $C(F, t) = V(F, t)$ value function when terminal payoff is $f(F) = (F - K)^+.$
- $f(0) = 0 \implies C(0, t) = 0.$
- PDDE (7) simplifies to the following PDE:
  \[
  \frac{\partial C}{\partial t}(F, t) + \frac{\sigma^2 F^2}{2} \frac{\partial^2 C}{\partial F^2}(F, t) - \lambda_q C(F, t) + \lambda_q F \frac{\partial C}{\partial F}(F, t) = 0
  \] (10)
  on domain $F > 0, t \in [0, T],$ w. terminal condition $C(F, T) = (F - K)^+, F > 0.$
- This is Black Scholes boundary value problem with $F$ replacing $S$ & $\lambda_q$ replacing $r.$
- Defaultable call value is thus:
  \[
  C(F, t) = FN(d_1) - Ke^{-\lambda_q(T-t)}N(d_2),
  \] (11)
  \[
  d_1 \equiv \frac{\ln(F/K) + (\lambda_q + \sigma^2/2)(T - t)}{\sigma \sqrt{T - t}} \quad d_2 \equiv d_1 - \sigma \sqrt{T - t}.
  \] (12)
• Suppose that there is no credit default swap.
• Assume (unrealistically) that investors can trade only the futures and the money market account.
• In such a setting, the market is incomplete and the parameter \( \lambda_q \) is not known.
• Call payoff cannot be perfectly replicated \( \Leftrightarrow \) There exists an infinite number of martingale measures \( \mathbb{Q} \), all consistent with the initial observed futures price \( F_0 \).
• For pricing calls, there is a one to one correspondence between martingale measures \( \mathbb{Q} \) and the parameter \( \lambda_q \) appearing in (11).
• Each martingale measure produces a call value \( C(F, t; \lambda_q) \) obtained by evaluating (11) at the associated \( \lambda_q \).
Call Value Range in Incomplete Market

• Recall that the Black Scholes call value increases in $r$, so $C$ increases in $\lambda q$.
• As $\lambda q$ approaches zero, the call value approaches the Black model value with volatility $\sigma$.
• As $\lambda q$ approaches infinity, the call value approaches $F$.
• The range of arbitrage-free call values is between the Black model value and $F$.
• This range reduces to a single point, once the market price of the CDS or the market price of another option becomes known.