Replicating a Defaultable Bond

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• Setup:
  – *Money Market Account* \( \beta_t \equiv e^{\int_0^t r_u du} \).
  – *Zero Coupon Bond* \( P_t \) - No Default.
  – Defaultable Bond: Recovery Rate \( R \in [0, 1) \) at Default Time \( \tau \in [0, T] \)
  – If \( \tau > T \), Bond Pays Full
  – CDS Protection: Pays \( 1 - R \) at Default Time
  – Assume \( P_t = 1 \) (0 Risk Free Rate)
  – Assume \( R = 0 \) (0 Recovery)
  – Payoff of Defaultable Bond: \( 1(\tau > T) \) at \( T \).
• The Model
  
  – Assume that default time $\tau$ is exponentially distributed

  \[
  \mathbb{P}\{\tau \in (t, t + dt)\} = \begin{cases} 
  \lambda_p e^{-\lambda_p t} dt, & \text{if } t \in \mathbb{R}^+ \\
  0 & \text{otherwise.}
  \end{cases}
  \]

  \hspace{1cm} (1)

  – Easier: $\mathbb{P}(\tau > t)$ is just $e^{-\lambda_p t}$

  – under $\mathbb{P}, \lambda_p$ can be a positive r.v. with unknown distribution
Credit Default Swaps (CDS)

- For a CDS with unit notional and maturity $T$, the protection buyer:
  - pays a fixed amount periodically until $\tau \wedge T \equiv \min(\tau, T)$, where $\tau$ is the default time.
  - receives 0 if $\tau > T$ and $1 - R$ at $\tau$ otherwise, where $R \in [0, 1)$ is the recovery rate.

- We assume continuous payment and $R = 0$, so the protection buyer:
  - pays a fixed amount $\lambda q_0 dt$ continuously until $\tau \wedge T$
  - receives 0 if $\tau > T$ and one dollar at $\tau$ otherwise.

- In general, the CDS rate $\lambda qt$ is stochastic, but we assume $\lambda qt = \lambda q > 0$ is the constant CDS rate.
• We will replicate a defaultable bond with CDS and money market account
• We assume no arbitrage, so there exists a risk-neutral measure $\mathbb{Q}$
• Let $B_t$ be the value of defaultable bond at time $t$ maturing at $T > t$.
• We have zero interest rate so $\{B_t, t \in [0, T]\}$ is a $\mathbb{Q}$ martingale
• Let $D_t$ be the default indicator process, i.e. 0 before $\tau$ and 1 after
• Guess that $B_t = B(t)1(D_t = 0)$, where $B(t)$ is a function of just time.
• $B(t)$ is the value at time $t$ of the defaultable bond, conditional on no default up to time $t$
• $\{B(t), t \in [0, T]\}$ is not a $\mathbb{Q}$ martingale
Recall our guess that $B_t = B(t)1(D_t = 0)$, where $B_t$ is the price of the defaultable bond and $D$ is the default indicator.

Suppose there has been no default up to time $t$, so $B_t = B(t)$.

Suppose an investor borrows $B(t)$ dollars riskfree and buys the defaultable bond.

Then gain over $[t, t + dt]$ is $B'(t)dt(1 - dD_t) - B(t)dD_t$.

Suppose instead that an investor goes long 1 CDS maturing an instant later.

Then gain over $[t, t + dt]$ is $dD_t - \lambda_q dt$.

$\lambda_q$ is R.N. Arrival Rate under $\mathbb{Q}$.

Can also write the gain on the CDS as $dD_t - \lambda_q dt(1 - dD_t)$.

Both gains are accessed at zero cost.
Consider combining strategies, i.e. the investor is:

- long one defaultable bond completely financed by borrowing at the riskfree rate of zero, and
- long $N$ credit default swaps.

Gain over $[t, t + dt]$ is: $B'(t)dt(1 - dD_t) - B(t)dD_t + N[dD_t - \lambda_q dt(1 - dD_t)]$.

Choose $N = B(t)$

Now, the gain over $[t, t + dt]$ is $[B'(t)dt - \lambda_q B(t)dt](1 - dD_t)$.

No Arbitrage $\Rightarrow B'(t) = \lambda_q B(t), \quad t \in [0, T]$.

$B(T) = 1 \Rightarrow B(t) = e^{-\lambda_q (T-t)}, \quad t \in [0, T]$.

We have found the pricing formula!
• The martingale representation of the defaultable bond payoff is:

\[ 1(\tau > T) = e^{-\lambda_q T} - \int_0^{\tau \land T} e^{-\lambda_q (T-t)} [dD_t - \lambda_q dt]. \]

(2)

• Exercise: Check that above holds when \(1(\tau > T) = 1\) and when \(1(\tau > T) = 0\).

• So if charge \(e^{-\lambda_q T}\) initially and then short \(e^{-\lambda_q (T-t)}\) CDS’s up to \(\tau \land T\), then replicate defaultable bond payoff \(1(\tau > T)\).

• Taking risk-neutral expectations on both sides of the top equation recovers the pricing result:

\[ B_0 = e^{-\lambda_q T}. \]

(3)

• In this model, the R-N arrival rate of default \(\lambda_q\) enters the defaultable bond valuation formula in the same way that the riskfree interest rate enters the default-free bond valuation formula.
Extensions

1. Add positive deterministic interest rates $r(t)$.
2. Make the CDS rate $\lambda_q$ a deterministic function of time.
3. Price a defaultable bond at an arbitrary time $t$ rather than at time 0.
4. Give the defaultable bond a positive deterministic recovery rate $R(t)$.
5. Show how to replicate a (long position in one) CDS by dynamic trading in defaultable bonds and the money market account.
6. Do all of the above together.
7. Think about what trades you would do if the actual market price of the defaultable bond differed from its arbitrage-free value.