Hedging Poisson Jumps

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• Model Setup:
  – $F_t$ is futures price at time $t \in [0, T]$ for maturity $T' \geq T$.
  – $\mathbb{P}$ is the statistical probability measure.
  – $F_t$ has constant proportional drift.
  – Jumps at random times in opposite direction to drift.
  – Jump times correspond to Poisson process.
• What is a Poisson process?
  – Poisson process $N$ is a continuous time stochastic process.
  – Range in the positive integers
  – There exists a positive parameter $\lambda_p > 0$, s.t.
    1. $\mathbb{P}\{N_{t+\Delta t} - N_t = 0\} = 1 - \lambda_p \Delta t + o(\Delta t)$
    2. $\mathbb{P}\{N_{t+\Delta t} - N_t = 1\} = \lambda_p \Delta t + o(\Delta t)$.
  – Reminder:
    $$\lim_{\Delta t \to 0} \frac{o(\Delta t)}{\Delta t} = 0.$$  
  – Obviously we have:
    $$\mathbb{P}\{N_{t+\Delta t} - N_t = k\} = o(\Delta t)$$
    for $k \geq 2$.
  – Number of occurrences in a time interval $(t, t + \Delta t)$, $N_{\Delta t}$, independent of occurrences in other intervals.

• Poisson process has the following properties:
  –
    $$\mathbb{P}\{N_t = n\} = \frac{e^{-\lambda t}(\lambda t)^n}{n!}, \quad n \in \mathbb{N}.$$  

– Inter-arrival times of jumps $\tau_1, \tau_2, \ldots$, IID exponential r.v.s w/parameter $\lambda_p$:

$$\mathbb{P}\{\tau_i \in dt\} = \lambda_p e^{-\lambda_p t}, \quad t > 0.$$ 

• Back to the futures:

$$\frac{dF_t}{F_{t-}} = \mu dt + (e^j - 1)dN_t, \quad t \in [0, T],$$

$$\tag{2}$$

– We will not need to know the arrival rate $\lambda_p$ - can even be stochastic.
– $\mu$, o.t.h. can be observed easily from sample path.
– $j$ is the jump in log futures price - has opposite sign to $\mu$.
– $F$ is not a martingale under $\mathbb{P}$, and $\mu$ is not its expected return.

• We’ll value European-style path-independent contingent claims.

• $V(F, t) : \mathbb{R} \times [0, T] \mapsto \mathbb{R} \ C^{1,1}$ function.

• Ito’s lemma for semi-martingales gives:
\[ V(F_T, T) = V(F_0, 0) + \int_0^T \frac{\partial V}{\partial F}(F_{t-}, t) dF_t + \int_0^T \frac{\partial V}{\partial t}(F_{t-}, t) dt \]

\[ + \int_0^T \left[ V(F_{t-e^j}, t) - V(F_{t-}, t) - \frac{\partial V}{\partial F}(F_{t-}, t) F_{t-}(e^j - 1) \right] dN_t(3) \]

- \( j \neq 0 \) (2) implies:

\[ dN_t = -\frac{\mu}{e^j - 1} dt + \frac{1}{F_{t-}(e^j - 1)} dF_t, \quad t \in [0, T]. \quad (4) \]

- And so

\[ V(F_T, T) = V(F_0, 0) + \int_0^T \frac{V(F_{t-e^j}, t) - V(F_{t-}, t)}{F_{t-}(e^j - 1)} dF_t \]

\[ + \int_0^T \left\{ \frac{\partial V}{\partial t}(F_{t-}, t) - \frac{\mu}{e^j - 1} \left[ V(F_{t-e^j}, t) - V(F_{t-}, t) - \frac{\partial V}{\partial F}(F_{t-}, t) F_{t-}(e^j - 1) \right] \right\} dt. \quad (5) \]

- Require the function \( V(F, t) \) solve the PDDE:
\[ - \partial_t V(F, t) + \frac{\mu}{e^t} \left[ V(e^F, t) - V(F, t) - \partial_F V(F, t) F(e^t - 1) \right] = 0, \quad (6) \]

\[ - F > 0, t \in [0, T], \text{w/terminal condition:} \]

\[ V(F, T) = f(F), \quad F > 0. \quad (7) \]

• This Cauchy problem has a unique solution.

• (5) becomes:

\[ f(F_T) = V(F_0, 0) + \int_0^T \frac{V(F_{t-e^j, t}) - V(F_{t- \cdot}, t)}{F_{t-}(e^j - 1)} dF_t. \quad (8) \]

• Replicate payoff:

  \[ \text{Charge } V(F_0, 0) \text{ initially} \]

  \[ \text{Hold } \frac{V(F_{t-e^j, t}) - V(F_{t- \cdot}, t)}{F_{t-}(e^j - 1)} \text{ futures at time } t. \]

• Similar to binomial model, u and d like \( 1 + \mu dt \).
• Do not hedge with partial derivative $\frac{\partial V}{\partial F}(F_{t-}, t)$

• No need to know the arrival rate of jumps.

• Parameters can be backed out of sample path.

• Solving our terminal value PDDE, we write

\[
\frac{dF_t}{F_{t-}} = (e^j - 1)[dN_t - \lambda dt], \quad t \in [0, T],
\]

where $\lambda \equiv -\frac{\mu}{e^j - 1}$.

• Let $\mathbb{Q}$ be the equivalent probability with arrival rate $\lambda$.

• Let

\[
X_t \equiv \ln(F_t/F_0),
\]

\[
(10)
\]

• Under $\mathbb{Q}$,

\[
dX_t = -\lambda(e^j - 1)dt + j dN_t, \quad t \in [0, T].
\]

\[
(11)
\]
• This gives
\[ F_T = F_t e^{-\lambda(e^j - 1)(T - t) + j(N_T - N_t)}, \quad t \in [0, T]. \]  
(12)

• We obtain the solution:
\[ V(F, t) = \mathbb{E}^Q[f(F_T) | F_t = F] = \sum_{n=0}^{\infty} f(F e^{-\lambda(e^j - 1)(T - t) + jn}) e^{-\lambda(T - t)} (\lambda(T - t))^n \frac{1}{n!}. \]  
(13)

•
\[ E_t^Q \left( \frac{dF_t}{F_{t-}} \right)^2 = (e^j - 1)^2 \lambda dt, \quad t \in [0, T]. \]  
(14)
• set $e^j - 1 = \frac{\sigma}{\sqrt{\lambda}}$, get $\frac{dF_t}{F_{t-}}$ has
  
  – conditional mean 0 under $Q$
  – conditional variance $\sigma^2 dt$, independent of $\lambda$.

• In the limit get Black’s Model:

$$
\frac{dF_t}{F_t} = \sigma dW_t, \quad t \in [0, T].
$$

(15)

• binomial model can be obtained as limit letting $\lambda(t)$ be a sum of point masses at the integers.

• Can also go from binomial to Black and vice versa.

• Compensated Poisson is the limit of Brownian integral

• Monroe’s Theorem: both the binomial and Poisson are time-changed B.M.