Hedging and Pricing in the Binomial Model

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• **Initial Setup:**
  
  – 0 risk-free interest rate.
  – 0 payouts/dividends.
  – Spot price $S$ goes up by factor of $u$ or down by factor of $d$.
  – $0 < d < 1 < u$.
  – The probability of either state must be strictly positive.
  – $S_1$, the price at the end, is a r.v. with two values.
  
• **Example:**
  
  – If $S = 40, u = 2, d = \frac{1}{2}$, then:
- Now add a European call, struck at $K = $50.
- Its value at the end of the period is $C_1 = \max[S_1 - K, 0]$.
- In our example, this is $30$ if the stock goes up, $0$ otherwise.

- What is the option price $C$?
Spanning the Payoffs

• Consider the portfolio:
  – $N^s$ shares of the stock $S$.
  – $N^b$ bonds, returning $50$ each.

• The portfolio returns:
  – Up: $N^s 80 + N^b 50$
  – Down: $N^s 20 + N^b 50$

• Can choose $N^s, N^b$, so that the portfolio and the call have same value:
  – Up: $N^s 80 + N^b 50 = 30$
  – Down: $N^s 20 + N^b 50 = 0$

• Solve to get $N^s = \frac{1}{2}$ and $N^b = -\frac{1}{5}$. 
• Graph portfolio as a function of S:

Call Replicating Portfolio Values Against Stock Prices

\[ \text{Terminal Stock price} \]

• From the graph, we see:
  
  – y-intercept is value of bond holdings: \( 50N^b = -10 \)
  
  – slope is number of stocks, delta
  
  – for European call, \( 0 < \Delta < 1 \), \( N^b < 0 \)
  
  – changing the slope and the intercept, we can get any line
Valuation

- The price of buying $\frac{1}{2}$ share and shorting $\frac{1}{5}$ bond.
- No arbitrage demands the price is $10$. 
Valuation

• If the market price of the call differs from $10, then there is an arbitrage opportunity.

• Recall the Golden Rule: he who has the gold makes the rules

• If you want to own a lot of gold one day, just remember one thing:

  “Buy Low - Sell High”

• If the market price of the call is $12, then buy the duplicating portfolio for $10, and sell the overpriced call for $12. The investor pockets $2 today and there are no net cash flows at expiration.

• To buy the duplicating portfolio for $12, buy \( \frac{1}{2} \) of a share for $20 and short \( \frac{1}{5} \) of a bond, bringing in $10. The net cost is $10.
• To see that there are no net cash flows at expiration, note that the call either finishes out-of-the-money or in-the-money. If it finishes out-of-the-money, then both the duplicating portfolio and the written call are worthless. If it finishes in-the-money, then the cash inflow from liquidating the duplicating portfolio ($30) covers the outflow from the written call.

• Similarly, if the call sells for $9, then one dollar can be made by buying the call for $9, and selling the duplicating portfolio for $10.
Spanning

- Consider assets as vectors in $\mathbb{R}^2$

- In our case, the stock & bond are $(20, 80); (50, 50)$ resp.

- linearly independent iff span $\mathbb{R}^2$

- In this case, we can create any payoff

$\$ dollars in down state
• Price Relative $\equiv$ Gross Return
  
  – Bond: (1,1)
  – Stock: (0.5,2)
  – Call: (0,3)

• Gross return on the derivative security is the same as on the replicating portfolio

$\text{dollars in down state}$
Multiple Periods

- Time Interval $[0,T]$
- $n$ periods $\Delta t = \frac{T}{n}$
- Futures Contract, $F_i$ is the price at time $t_i = i\delta t$, $i = 0, 1, \ldots, n$.
- $\mathbb{P}$ is statistical probability measure
- $F_{i+1} = F_i m_{i+1}$, $i = 0 \ldots n - 1$
- $m_i$ are IID, Bernoulli

$$m_{i+1} = \begin{cases} u > 1 & \text{with probability } p \in (0, 1), \\ d \in (0, 1) & \text{with probability } 1 - p. \end{cases} \quad (1)$$

- log price follows random walk
For example, when $n=5$ we have the following:
• Contingent Claim has payoff $f(F_n)$ at time $T \equiv t_n \equiv n\triangle t$.

• $V(F_i, i)$ will denote its value at futures price $F_i$ and time index $i$.

\[
V(F_n, n) = V(F_0, 0) + \sum_{i=0}^{n-1} [V(F_{i+1}, i + 1) - V(F_i, i)]
\]

\[
= V(F_0, 0) + \sum_{i=0}^{n-1} H(F_i, i) \times (F_{i+1} - F_i)
\]

\[
+ \sum_{i=0}^{n-1} [V(F_{i+1}, i + 1) - V(F_i, i) - H(F_i, i) \times (F_{i+1} - F_i)].
\]

• $H(F, i) : \mathbb{R}^+ \times [0, 1, \ldots, n] \mapsto \mathbb{R}$ indicates the holdings in futures when the futures price is $F$ and the time index is $i$.

• $H$ will be determined as a hedge ratio
• Suppose we choose $V$ so that:

$$V(F_{i+1}, i + 1) - V(F_i, i) - H(F_i, i)(F_{i+1} - F_i) = 0,$$

for $F > 0, i = 0, 1, \ldots, n - 1$, and:

$$V(F, n) = f(F), \quad F > 0.$$  \hfill (5)

• Then substituting the top two equations in (3) implies:

$$f(F_n) = V(F_0, 0) + \sum_{i=0}^{n-1} H(F_i, i)(F_{i+1} - F_i).$$  \hfill (6)

• Entering a futures contract is free, so the cost of the last term is 0.

• Thus, if $V(F_0, 0)$ is charged up front, then holding $H(F_i, i)$ futures each period achieves $f(F_n)$. 
• To solve the partial difference equation (4), write it out for both states:

- \( V(F_{i}u, i + 1) - V(F_{i}, i) - H(F_{i}, i)F_{i}(u - 1) = 0. \) \( (7) \)

- \( V(F_{i}d, i + 1) - V(F_{i}, i) - H(F_{i}, i)F_{i}(d - 1) = 0. \) \( (8) \)

• Get:

- \( H(F_{i}, i) = \frac{V(F_{i}u, i + 1) - V(F_{i}d, i + 1)}{F_{i}(u - d)}. \) \( (9) \)

- \( V(F_{i}, i) = V(F_{i}u, i + 1) - \frac{V(F_{i}u, i + 1) - V(F_{i}d, i + 1)}{F_{i}(u - d)}F_{i}(u - 1) \)
  \( = \frac{1 - d}{u - d}V(F_{i}u, i + 1) + \frac{u - 1}{u - d}V(F_{i}d, i + 1). \) \( (10) \)

• \( p \) (and \( \mathbb{P} \)) unimportant, as long as \( 0 < p < 1. \)
• Recall from the last page that:

\[ V(F_i, i) = \frac{1 - d}{u - d} V(F_i u, i + 1) + \frac{u - 1}{u - d} V(F_i d, i + 1). \]  

(12)

• Let \( q \equiv \frac{1 - d}{u - d} \)

•

\[ V(F_i, i) = E^Q[V(F_{i+1}, i + 1)|F_i], \]

(13)

• – where:

\[ F_{i+1} = F_i \times m_{i+1}, \quad i = 0, 1, \ldots, n - 1, \]

(14)

\[ m_{i+1} = \begin{cases} 
  u > 1 & \text{with probability } q \in (0, 1), \\
  d \in (0, 1) & \text{with probability } 1 - q.
\end{cases} \]

(15)

• \( Q \) is the risk-neutral measure (or equivalent martingale measure)
• Recall the backward recursion:

\[ V(F_i, i) = E^Q[V(F_{i+1}, i+1)|F_i], \quad (16) \]

• By chaining:

\[ V(F_i, i) = E^Q[f(F_n)|F_i]. \quad (17) \]

• Expanding, we get:

\[ V(F_i, i) = E^Q[f(F_i u^\nu d^{n-i-\nu})|F_i]. \quad (18) \]

\[ \mathbb{Q}\{\nu = j\} = \begin{cases} \binom{n-i}{j} q^j (1 - q)^{n-i-j} & \text{if } j = 0, 1, \ldots, n - i \\ 0 & \text{otherwise.} \end{cases} \quad (19) \]

• And the explicit formula:

\[ V(F_i, i) = \sum_{j=0}^{n-i} f(F_i u^j d^{n-i-j}) \binom{n-i}{j} q^j (1 - q)^{n-i-j}. \quad (20) \]

• Formula gives arbitrage-free value of European claims
• For path-dependents, modify the backward recursion by adding state variables
• For American claims, the backward recursion is:

\[ V(F_i, i) = \max\{X_i(F_i), E^Q[V(F_{i+1}, i + 1)|F_i]\}, \]

(21)

• Can interpret as a hidden binary state variable indicating no exercise before time i
• Can extend the binomial model to a multivariate setting, but not tonight.