Interest Rates: More on Numeraires, and the Ho Lee Model

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Bloomberg LP
• $B_{t,T} \equiv P_t(T)$
• $D_T \equiv B_{0,T}$
• Money Market Account: $\beta_t = e^{\int_0^t r_s ds}$
• If interest rates are deterministic: $\beta_t = 1/D_t$
• Forward Rates: $B_{t,T} = e^{-\int_t^T f_{t,s} ds}$
• Solving for $f$: $f_{t,T} = -\frac{\partial \ln B_{t,T}}{\partial T}$
The Forward Martingale Measure

- Numeraire:  $Z_T \equiv B_{t,T}$

- Corresponding risk neutral measure: $\mathbb{Q}_T$.

- Forward rate is a $\mathbb{Q}_T$-martingale:
  \[
  f_{t,T} = \lim_{\delta T \to 0} \frac{B_{t,T} - B_{t,T+\delta T}}{B_{t,T} \delta T}
  \]

- For the short rate, we then have:
  \[
  \mathbb{E}^{\mathbb{Q}_T}_t[\rho_T] = \mathbb{E}_t^{\mathbb{Q}_T}[f_{T,T}] = f_{i,T}
  \]

- Conclusion: Under the forward measure, the expected value of the short rates is the initial forward curve.
The Money Market Measure

- Numeraire: $\beta_T \equiv e^{\int_0^T r_s ds}$
- Corresponding risk neutral measure: $Q_\beta$.
- For any tradeable $X$:
  $$X_0 = \frac{X_0}{\beta_0} = \mathbb{E}_{Q_\beta}\left[\frac{X_T}{\beta_T}\right]$$
- Under this measure, the discount factor doesn’t ”come out”:
  $$\mathbb{E}_{Q_\beta}\left[X_T e^{-\int_0^T r_s ds}\right] \neq Z_T(0)\mathbb{E}_{Q_\beta}\left[X_T\right]$$
- Conclusion: bond prices aren’t martingales under this measure.
- We have a convexity bias: higher interest rates are discounted at higher rates.
The Ho Lee Model

- The SDE for the short rate: $dr = \sigma dW_t$.

- From above considerations, we need: $\mathbb{E}^{Q_T}[r_t] = f_{0,t}$.

- Get $r_t = f_{0,t} + \sigma W_t$.

- Under $Q_\beta$,

$$dr_t = \frac{\partial f_{0,t}}{\partial t} dt + \sigma^2 t dt + \sigma dW^\beta$$
• Can be obtained from the HJM result for the forward rates:

\[ df_{t,T} = \sigma_{t,T}(\int_t^T \sigma_{t,s}ds)dt + \sigma_{t,T}dW^\beta \]

• Combine \( r_T = f_{T,T} \), with the constant volatility in the model, which implies:

\[ df_{t,T} = \sigma^2(T - t)dt + \sigma dW^\beta \]

• Get:

\[ r_T = f_{0,T} + \sigma^2 \int_0^T (T - s)ds + \sigma W_T^\beta \]

• Which gives the SDE we had:

\[ dr_t = \frac{\partial f_{0,t}}{\partial t}dt + \sigma^2 tdt + \sigma dW^\beta \]