Interest Rate Models: BGM

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Modeling Forward LIBOR Rates

- Brace Gatarek Musiela (1997): model for LIBOR (London Inter-Bank Offer Rate)
  - HJM framework.
  - Finite number, N of time periods.
  - LIBOR over each period lognormal.
  - Black’s Formula for caplets satisfied.
Model Setup

- $P_t(T_1, T_2)$: price at time $T_1$ of a zero coupon bond paying $1$ at time $T_2$.
- Determined at time $t$.
- $t < T_1 < T_2$.
- FRA for the time interval $[T_1, T_2]$ struck at time $t$.
- $P_t(T_2)$: $t = T_1$, i.e. spot (rather than forward).
• Forward LIBOR, $L(T_1, T_2)$:

$$P_t(T_1, T_2) = P_t(T_2)(1 + \alpha(T_1, T_2)L_t(T_1, T_2)) \quad (1)$$

• $\alpha$ is the day count convention.

• Solve for L:

$$L_t(T_1, T_2) = \frac{1}{\alpha(T_1, T_2)} \frac{P_t(T_1) - P_t(T_2)}{P_t(T_2)}. \quad (2)$$

• Divide time interval $T_i = i\Delta T, i = 1..N$.

• Get N rates:

$$L_{i,t} \equiv L_t(T_i, T_{i+1}) = \frac{1}{\alpha_i} \frac{P_{i,t} - P_{i+1,t}}{P_{i+1,t}}$$

• Denote $Q^{i+1}$ the equivalent martingale measure associated with $P_{i+1}$.

• $L_{i,t}$ is a $Q^{i+1}$-martingale: difference of two traded assets, deflated by numeraire.
• Assumptions:
  – $L > 0$
  – $L$ continuous in time
  – $L$ follows a lognormal process with deterministic vol.

• Rewrite third assumption:

\[
\frac{dL_{i,t}}{L_{i,t}} = \sigma_i(t) dW_{i+1}^t, \quad t \in [0, T_i], i = 1...N
\]  

• $W_{i+1}^t$ is B.M. under $Q^{i+1}$. 

Black’s Formula for Caplets

• Price a caplet, payoff: \( C_{i,T_{i+1}} \equiv \alpha_i (L_{i,T_i} - K)^+ \).

• Settled at time \( T_i \), payoff at time \( T_{i+1} \).

• By definition of Risk Neutral Measure:

\[
\frac{C_{i,t}}{P_{i+1,t}} = \mathbb{E}^{Q^{i+1}} \left[ \frac{C_{i,T_{i+1}}}{P_{i+1,T_{i+1}}} \bigg| \mathcal{F}_t \right]
\]

• Denominator in R.H.S. is 1, so

\[
C_{i,t} = P_{i+1,t} \mathbb{E}^{Q^{i+1}} \left[ \alpha_i (L_{i,T_i} - K)^+ \bigg| \mathcal{F}_t \right]
\]

• Equation 3 implies that under \( Q^{i+1} \), LIBOR satisfies:

\[
L_{i,T_i} = L_{i,t} e^{-\frac{1}{2} \int_t^{T_i} \sigma_i^2(u)du + \int_t^{T_i} \sigma_i dW_u^{i+1}}.
\]
• Get Black’s Formula:

\[ C_{i,t} = P_{i+1,t} \alpha_i (L_{i,t} N(d_1) - KN(d_2)) \]

\[ d_1 = \frac{\ln\left(\frac{L_{i,t}}{K}\right) + \Sigma_i^2/2}{\Sigma_i} \]

\[ d_2 = \frac{\ln\left(\frac{L_{i,t}}{K}\right) - \Sigma_i^2/2}{\Sigma_i} \]

\[ \Sigma_i^2 = \int_t^{T_i} \sigma_i^2(u) du \]
Changing the Measure

• Objective: find SDE’s for all rates under a measure, $Q^{N+1}$ (corresponding to the latest maturity).

• All the measures are equivalent, just need to find drift of each rate $L_{i,t}$.

• For any asset $A$ and time $t > 0$,

$$\frac{A_0}{P_{n,0}} = \mathbb{E}^{Q^n} \left[ \frac{A_t}{P_{n,t}} | \mathcal{F}_0 \right]$$

• Multiplying and dividing by the factors $P_{n+1,0}$ and $P_{n+1,t}$:

$$\frac{A_0}{P_{n+1,0}} = \mathbb{E}^{Q^n} \left[ \frac{P_{n,0} P_{n+1,t} A_t}{P_{n+1,0} P_{n,t} P_{n+1,t}} | \mathcal{F}_0 \right] = \mathbb{E}^{Q^{n+1}} \left[ \frac{A_t}{P_{n+1,t}} | \mathcal{F}_0 \right]$$

• Implying:

$$\frac{dQ^{n+1}}{dQ^n} = \frac{P_{n,0} P_{n+1,t}}{P_{n+1,0} P_{n,t}}$$
• Define

\[ D_t = \mathbb{E}_Q^n \left[ \frac{dQ^{n+1}}{dQ^n} | \mathcal{F}_0 \right] \]

• \( D_t \) has mean 1, is a positive \( Q^n \)-martingale.

• Assume a process \( X_t \) defined by

\[ dX_t = \mu_t dt + a_t dW_t^n \]

• Satisfies, under \( Q^{n+1} \):

\[ dX_t = \mu_t dt + \frac{dD_t}{D_t} dX_t + a_t dW_t^{n+1} \]

• Take \( X_t = L_{n-1,t} \), i.e. \( \mu = 0 \), \( a_t = \sigma_{n-1}(t)L_{n-1,t} \).

• Get:

\[ dL_{n-1,t} = \sigma_{n-1}(t)L_{n-1,t} dW_t^{n+1} + \frac{dD_t}{D_t} dL_{n-1,t} \]
• ”Trick” substitution: $R_t = \frac{1}{D_t}$.

• Preserves differential:

$$\frac{dD_t}{D_t} dL_{n-1,t} = -\frac{dR_t}{R_t} dL_{n-1,t},$$

• Since

$$D_t = \frac{P_{n,0} P_{n+1,t}}{P_{n+1,0} P_{n,t}},$$

• Get:

$$R_t = \frac{P_{n+1,0} P_{n,t}}{P_{n,0} P_{n+1,t}} = \frac{P_{n+1,0}}{P_{n,0}} (1 + \alpha_n L_{n,t})$$

• Differentiate:

$$dR_t = \frac{P_{n+1,0}}{P_{n,0}} \alpha_n dL_{n,t}$$
• Substituting the expressions above:

\[
\frac{dR_t}{R_t} = \frac{\alpha_n dL_{n,t}}{1 + \alpha_n L_{n,t}} = \frac{\alpha_n}{1 + \alpha_n L_{n,t}} \sigma_n(t) L_{n,t} dW_{t}^{n+1}
\]

• So we finally get:

\[
\frac{dD_t}{D_t} dL_{n-1,t} = -\frac{dR_t}{R_t} dL_{n-1,t} = -\sigma_{n-1}(t) L_{n-1,t} \frac{\alpha_n L_{n,t}}{1 + \alpha_n L_{n,t}} \sigma_n(t) dt.
\]

• Under \(Q^{n+1}\), the process for \(L_{n-1,t}\) thus satisfies the SDE:

\[
\frac{dL_{n-1,t}}{L_{n-1,t}} = -\sigma_{n-1}(t) \frac{\alpha_n L_{n,t}}{1 + \alpha_n L_{n,t}} \sigma_n(t) dt + \sigma_{n-1}(t) dW_{t}^{n+1}
\]

• Get drifts for previous intervals recursively:

\[
\frac{dL_{i,t}}{L_{i,t}} = -\sum_{k=i+1}^{n} \sigma_k(t) \frac{\alpha_k L_{k,t}}{1 + \alpha_k L_{k,t}} \sigma_i(t) dt + \sigma_i(t) dW_{t}^{n+1},
\]