Interest Rate Models: Hull White

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The Model

- Described by the SDE for the short rate:

\[ dr = (\theta(t) - ar) \, dt + \sigma \, dw \]  \hspace{1cm} (1)

- See also Sections 23.11-23.12 of Hull (5th edition).
- Our version simplified: \( a \) and \( \sigma \) constant.
- AKA Extended Vasicek.
- \( \theta \) determined uniquely by term structure.
Solving for $r(t)$

- \[
d(e^{at}r) = e^{at} dr + ae^{at} r dt = \theta(t)e^{at} dt + e^{at} \sigma dw,\]

- \[
e^{at}r(t) = r(0) + \int_0^t \theta(s)e^{as} ds + \sigma \int_0^t e^{as} dw(s).
\]

- Simplify:
\[
r(t) = r(0)e^{-at} + \int_0^t \theta(s)e^{-a(t-s)} ds + \sigma \int_0^t e^{-a(t-s)} dw(s). \tag{2}
\]

- Since the starting time is arbitrary:
\[
r(t) = r(s)e^{-a(t-s)} + \int_s^t \theta(\tau)e^{-a(t-\tau)} d\tau + \sigma \int_s^t e^{-a(t-\tau)} d\omega(\tau).
\]

- Note: $r(t)$ is Gaussian.
Solving for $P(t, T)$

- $P(t, T) = V(t, r(t))$ where $V$ solves the PDE
  $$V_t + (\theta(t) - ar)V_r + \frac{1}{2}\sigma^2 V_{rr} - rV = 0$$

- Final-time condition $V(T, r) = 1$ for all $r$ at $t = T$.

- Ansatz:
  $$V = A(t, T)e^{-B(t, T)r(t)}. \quad (3)$$

- $A$ and $B$ must satisfy:
  $$A_t - \theta(t)AB + \frac{1}{2}\sigma^2 AB^2 = 0 \quad \text{and} \quad B_t - aB + 1 = 0$$

- Final-time conditions
  $$A(T, T) = 1 \quad \text{and} \quad B(T, T) = 0.$$
• $B$ independent of $\theta$, so solution is same as in Vasicek:

$$B(t, T) = \frac{1}{a} \left( 1 - e^{-a(T-t)} \right).$$  

\hspace{1cm} (4)

• Solving for $A$ requires integration of $\theta$:

$$A(t, T) = \exp \left[ - \int_t^T \theta(s) B(s, T) \, ds - \frac{\sigma^2}{2a^2} (B(t, T) - T + t) - \frac{\sigma^2}{4a} B(t, T)^2 \right].$$  

\hspace{1cm} (5)
Determining $\theta$ from the term structure at time $0$

- Goal: demonstrate the relation
  \[ \theta(t) = \frac{\partial f}{\partial T}(0, t) + af(0, t) + \frac{\sigma^2}{2a}(1 - e^{-2at}). \]  
  \[ (6) \]

- Note: HJM gives a simple proof of this relation.

- For now, use explicit representation of $P(t, T)$ given by (3)-(5).

- Recall
  \[ f(t, T) = -\partial \log P(t, T)/\partial T \]
• We have

\[- \log P(0, T) = \int_0^T \theta(s)B(s, T) \, ds + \frac{\sigma^2}{2a^2} (B(0, T) - T) + \frac{\sigma^2}{4a} B(0, T)^2 + B(0, T) r_0.\]

• Differentiating and using that \(B(T, T) = 0\) and \(\partial_T B - 1 = -aB\):

\[f(0, T) = \int_0^T \theta(s)\partial_T B(s, T) \, ds - \frac{\sigma^2}{2a} B(0, T) + \frac{\sigma^2}{2a} B(0, T) \partial_T B(0, T) + \partial_T B(0, T) r_0.\]

• Differentiating again, get:

\[\partial_T f(0, T) = \theta(T) + \int_0^T \theta(s)\partial_{TT} B(s, T) \, ds - \frac{\sigma^2}{2a} \partial_T B(0, T) \]

\[+ \frac{\sigma^2}{2a} [(\partial_T B(0, T))^2 + B(0, T) \partial_{TT} B(0, T)] + \partial_{TT} B(0, T) r_0.\]
• Combine these equations, and use $a \partial_T B + \partial_{TT} B = 0$

• Get:

$$af(0, T) + \partial_T f(0, T) = \theta(T) - \frac{\sigma^2}{2a} (aB + \partial_T B) + \frac{\sigma^2}{2a} [aB\partial_T B + (\partial_T B)^2 + B\partial_{TT} B].$$

• Substitute formula for $B$ and simplify, to get

$$af(0, T) + \partial_T f(0, T) = \theta(T) - \frac{\sigma^2}{2a} (1 - e^{-2aT}),$$

• This is equivalent to (6).
• (6) seems to imply need for differentiated term structure $\partial_T f(0, T)$ for calibration.

• Problem: differentiation amplifies effect of observation-error.

• Actually, need only $f$.

• Try a representation of the form

$$r(t) = \alpha(t) + x(t)$$  \hspace{1cm} (7)

• $\alpha(t)$ deterministic, $x(t)$ solves

$$dx = -ax \, dt + \sigma \, dw \quad \text{with } x(0) = 0.$$
• Calculation gives
\[ \alpha' + a\alpha = \theta \quad \text{and} \quad \alpha(0) = r_0 \]

• \( \alpha(t) + x(t) \) solves the SDE for \( r(t) \) with initial condition.

• Uniqueness \( \Rightarrow \) equals \( r(t) \).

• The ODE for \( \alpha \): \( (e^{at}\alpha)' = e^{at}\theta \)

• Solution:
\[ \alpha(t) = r_0 e^{-at} + \int_0^t e^{-a(t-s)}\theta(s)\, ds. \]
• Substituting (6), get

\[ \alpha(t) = r_0 e^{-at} + \int_0^t \partial_s [e^{-a(t-s)} f(0, s)] + \frac{\sigma^2}{2a} e^{-a(t-s)} (1 - e^{-2as}) ds. \]

• Simplifies to

\[ \alpha(t) = f(0, t) + \frac{\sigma^2}{2a^2} (1 - e^{-at})^2. \]

• Decomposition (7) expresses \( r \) as sum of:
  
  – deterministic \( \alpha(t) \) reflecting the term structure at time 0
  – random process \( x(t) \) entirely independent of market data. Validity of Black’s formula. The situation is exactly the same as for Vasicek.
Validity of Black’s formula

- SDE for the interest rate under the forward-risk-neutral measure is
  \[ dr = [\theta(t) - \alpha r - \sigma^2 B(t, T)] dt + \sigma d\bar{w} \]

- \( d\bar{w} \) is a Brownian motion under this measure.

- This is a version of Hull-White with a different choice of \( \theta \).

- Get bond prices lognormal \( \Rightarrow \) Black’s formula is valid.