Interest Rate Models: Vasicek

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The Short Rate Dynamics

• The Vasicek model describes the short rate’s $\mathbb{Q}$ dynamics by the following SDE:

$$dr_t = (\theta - ar_t)\, dt + \sigma \, dw_t$$

(1)

where $\theta$, $a > 0$, and $\sigma$ are constants.

• An explicit formula for $r_t$: Start with:

$$d(e^{at}r_t) = e^{at} \, dr_t + ae^{at}r_t \, dt = \theta e^{at} \, dt + e^{at} \sigma \, dw_t,$$

so:

$$e^{at} r_t = r_0 + \theta \int_0^t e^{as} \, ds + \sigma \int_0^t e^{as} \, dw_s.$$

• Simplifying:

$$r_t = r_0 e^{-at} + \frac{\theta}{a} (1 - e^{-at}) + \sigma \int_0^t e^{-a(t-s)} \, dw_s.$$

(2)
• Recall

\[ r_t = r_0 e^{-at} + \frac{\theta}{a}(1 - e^{-at}) + \sigma \int_0^t e^{-a(t-s)} dw_s. \]  \hspace{1cm} (3)

• As the starting time is arbitrary:

\[ r_t = r_s e^{-a(t-s)} + \frac{\theta}{a}(1 - e^{-a(t-s)}) + \sigma \int_s^t e^{-a(t-\tau)} d\tau. \] \hspace{1cm} (4)

• (??) implies that \( r_t \) is Gaussian at each \( t \), with

  - expectation:

\[ E^Q[r_t] = r_0 e^{-at} + \frac{\theta}{a}(1 - e^{-at}), \quad \text{and} \]

  - variance:

\[ \text{Var}^Q[r_t] = \sigma^2 E \left[ \left( \int_0^t e^{-a(t-s)} dw(s) \right)^2 \right] = \sigma^2 \int_0^t e^{-2a(t-s)} ds = \frac{\sigma^2}{2a} (1 - e^{-2at}). \]
We now show that the bond price is lognormally distributed in the Vasicek model:

– By definition of the risk-neutral measure $\mathbb{Q}$, the zero coupon bond price is:

$$P_t(T) = E^Q \left[ e^{-\int_t^T r(s) \, ds} \mid \mathcal{F}_t \right].$$

(5)

– From (5), (interchanging $t, s$)

$$P_t(T) = A(t, T)e^{-B(t, T)r_t}$$

(6)

$$B(t, T) = \int_t^T e^{-a(s-t)} \, ds,$$

and

$$A(t, T) = E \left[ e^{-\int_t^T \left\{ \frac{\theta}{a}(1-e^{-a(s-t)}) + \sigma \int_t^s e^{-a(s-\tau)} \, dw(\tau) \right\} \, ds} \right].$$

– $A(t, T), B(t, T)$ deterministic, $r_t$ Gaussian $\Rightarrow P_t(T)$ lognormal.
Explicit Bond Pricing Formula

• Can evaluate $A(t, T), B(t, T)$ - see Lamberton & Lapeyre, pages 128-129.

• Alternative approach: use $P_t(T) = V(t, r_t)$, where $V(t, r)$ solves BVP consisting of PDE:

$$V_t + (\theta - ar)V_r + \frac{1}{2}\sigma^2 V_{rr} = rV$$

subject to the final-time condition $V(T, r) = 1$ for all $r$.

• Guess a solution of the form:

$$V(t, r; T) = A(t, T)e^{-B(t, T)r}.$$
• Considered as functions of $t$, $A(t, T)$ and $B(t, T)$ solve the ODE’s:

$$A_t - \theta AB + \frac{1}{2}\sigma^2 AB^2 = 0 \quad \text{and} \quad B_t - aB + 1 = 0$$

subject to:

$$A(T, T) = 1 \quad \text{and} \quad B(T, T) = 0.$$

• Get:

$$B(t, T) = \frac{1}{a}(1 - e^{-a(T-t)})$$

• and:

$$A(t, T) = \exp \left[ \left( \frac{\theta}{a} - \frac{\sigma^2}{2a^2} \right) (B(t, T) - T + t) - \frac{\sigma^2}{4a} B^2(t, T) \right].$$
Term Structure and Volatility

- Only three parameters \(\Rightarrow\) special term structure.

- By definition, the initial instantaneous forward rate curve \(f_0(T) = -\frac{\partial \ln P_0(T)}{\partial T}\).

- After some calculations, in the Vasicek model, one has:
  \[
  f_0(T) = \frac{\theta}{a} + e^{-aT} \left( r_0 - \frac{\theta}{a} \right) - \frac{\sigma^2}{2a^2} (1 - e^{-aT})^2.
  \]

- Volatility of \(f\), \(\sigma(t, T)\) is defined by
  \[
  df_t(T) = (\text{stuff}) \, dt + \sigma(t, T) \, dw_t.
  \]

- \(\ln P_t(T) = \ln A(t, T) - B(t, T) r_t\), implies
  \[
  f_t(T) = -\partial_T \ln A(t, T) + \partial_T B(t, T) r_t,
  \]

- Itô's formula gives:
  \[
  \sigma(t, T) = \sigma \partial_T B(t, T) = \sigma e^{-a(T-t)}.
  \]
Validity of Black’s Formula

• We now show that $P_t(T)$ is lognormal under the forward-risk-neutral measure $Q_T$. (The measure under which tradeables normalized by $P(t,T)$ are martingales.)

• Already know that $P_t(T)$ is lognormal under the risk-neutral measure $Q$, but here we’re interested in a different numeraire.

• Change-of-numeraire in the one-factor setting:
  – The risk-neutral measure is associated with the risk-free money-market account $\beta$ as numeraire (by definition $d\beta_t = r_t\beta_t \ dt$ with $\beta_0 = 1$).
  – Say $N$ is another numeraire, and $\overline{Q}$ is the associated equivalent martingale measure.
  – Only positive tradeables can be numeraires, so the risk-neutral process for $N$ is $dN_t = r_tN_t \ dt + \sigma_t^N N_t \ dw_t$
    where $\sigma_t^N$ is in general stochastic and $w$ is a $Q$ standard Brownian motion.
• Itô’s formula gives:

\[ d \left( \frac{\beta_t}{N_t} \right) = \beta_t \, d(N_t^{-1}) + N_t^{-1} \, d\beta_t \]

• After some algebra:

\[ d \left( \frac{\beta_t}{N_t} \right) = \frac{\beta_t}{N_t} (\sigma_t^N)^2 \, dt - \frac{\beta_t}{N_t} \sigma_t^N \, dw_t. \]

• \( \frac{\beta_t}{N_t} \) is a \( \bar{Q} \)-martingale, i.e.

\[ d \left( \frac{\beta_t}{N_t} \right) = -\frac{\beta_t}{N_t} \sigma_t^N \, d\bar{w}_t \]

where \( \bar{w} \) is a \( \bar{Q} \)-Brownian motion.

• Therefore:

\[ d\bar{w}_t = -\sigma_t^N \, dt + dw_t. \]
• What is the SDE for the short rate in the Vasicek model under the forward-risk-neutral measure $Q_T$?

  – Numeraire is $P_t(T) = A(t, T)e^{-B(t,T)r_t}$
  – Ito $\Rightarrow$ the (usual lognormal) volatility of $P_t(T)$ is $-B(t,T)\sigma$.
  – The preceding calculation gives:
    
    $$d\overline{w}_t = \sigma B(t,T) dt + dw_t.$$ 

  – Conclusion:
    
    $$dr_t = (\theta - ar_t) dt + \sigma dw_t = [\theta - ar_t - \sigma^2 B(t,T)] dt + \sigma d\overline{w}_t,$$

    where $\overline{w}$ is a $Q_T$ standard Brownian motion.

• This SDE shows that short rates are normal and bond prices are lognormal, as under the risk-neutral measure $Q$. 