Interest Rate Models: Introduction

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Basic Terminology

• Time-value of money is expressed by the discount factor:

\[ P(t, T) = \text{value at time } t \text{ of a dollar received at time } T. \]

• Interest rates stochastic ⇒ \( P(t, T) \) not known until time \( t \)

• \( P(t, T) \) is a function of two variables: initiation time \( t \) and maturity time \( T \).

• Dependence on \( T \) reflects term structure of interest rates

• \( P(t, T) \) fairly smooth as function of \( T \) at each \( t \), because of averaging.

• Convention: Present time is \( t = 0 \) ⇒ initial observable is \( P(0, T) \) for all \( T > 0 \).
Representations of the time-value of money

• The (continuously compounded annualized) *yield-to-maturity* (or just yield) $R(t, T)$ is defined implicitly by:

$$P(t, T) = e^{-R(t,T)(T-t)}.$$  

• It is the unique (continuously compounded annualized) *constant* short term interest rate implied by the market price $P(t, T)$.

• Evidently:

$$R(t, T) = -\frac{\log P(t, T)}{T - t}.$$
• The (continuously compounded annualized) instantaneous forward rate $f(t, T)$ is defined by:

$$P(t, T) = e^{-\int_t^T f(t, \tau) d\tau}.$$ 

• It is the deterministic time-varying interest rate describing all loans starting at $t$ with various maturities.

$$f(t, T) = -\frac{\partial \log P(t, T)}{\partial T}.$$
• The (continuously compounded annualized) \textit{instantaneous short term interest rate} \( r(t) \), (a.k.a. the short rate), is
\[
r(t) = f(t, t);
\]

• It is the rate earned on the shortest-term loans starting at time \( t \).

• Yields \( R(t, T) \) and instantaneous forward rates \( f(t, T) \) carry the same information as \( P(t, T) \).

• The short rate \( r(t) \) contains less information: it is a function of just one variable.
Why is $f(t, T)$ called the instantaneous forward rate?

The ratio $P(0, T)/P(0, t)$ is the time-$t$ borrowing, time-$T$ maturing discount factor locked in at time 0:

- Consider the portfolio:
  (a) at time 0, go long a zero-coupon bond paying out one dollar at time $T$ (p.v. = $P(0, T)$), and
  (b) at time 0, go short a zero-coupon bond paying out $P(0, T)/P(0, t)$ dollars at time $t$ (present value $-P(0, T)$).

- Has p.v. 0; holder pays $P(0, T)/P(0, t)$ dollars at time $t$, receives one dollar at time $T$.

- Portfolio “locks in” $P(0, T)/P(0, t)$ as the discount factor from time $T$ to $t$.

- This ratio is called the forward term rate at time 0, for borrowing at time $t$ with maturity $T$. 
• Similarly, $P(t, T_2)/P(t, T_1)$ is the forward term rate at time $t$, for borrowing at time $T_1$ with maturity $T_2$.

• The associated yield (locked in at $t$ and applying to $(T_1, T_2)$) is:

$$-rac{\log P(t, T_2) - \log P(t, T_1)}{T_2 - T_1}.$$ 

• In the limit $T_2 - T_1 \downarrow 0$, we get $-\frac{\partial \log P(t, T)}{\partial T} = f(t, T)$. 
Example 1: put option on a zero-coupon bond.

Payoff at time $t$: $(K - P(t, T))^+$. 

Example 2: caplet places a cap on the term interest rate $\mathcal{R}$ for lending between times $T_1$ and $T_2$ at the fixed rate $R_0$. 
• For a loan with a principal of one dollar, the caplet’s payoff at time $T_2$ is:

$$\Delta t \{\mathcal{R} - R_0\}^+ = \{(\mathcal{R} - R_0)\Delta t\}^+,$$

where $\Delta t = T_2 - T_1$ and $\mathcal{R}$ is the actual term interest rate in the market at time $T_1$ (defined by $P(T_1, T_2) = 1/(1 + \mathcal{R}\Delta t)$).

• The discounted value of this payoff at time $T_1$ is:

$$\frac{\{(\mathcal{R} - R_0)\Delta t\}^+}{1 + \mathcal{R}\Delta t} = \left\{\frac{(\mathcal{R} - R_0)\Delta t}{1 + \mathcal{R}\Delta t}\right\}^+ = (1 + R_0\Delta t)\left\{\frac{1}{1 + R_0\Delta t} - \frac{1}{1 + \mathcal{R}\Delta t}\right\}^+.$$

• Thus, a caplet maturing at $T_2$ has the same discounted payoff as $1 + R_0\Delta t$ put options maturing at $T_1$. The put options have strike $\frac{1}{1 + R_0\Delta t}$ and are written on a zero-coupon bond paying one dollar at maturity $T_2$.

• A cap is a collection of caplets, equivalent to a portfolio of puts on zero-coupon bonds.
How are interest rates both risk-free and random?

- For short-rate models, the standard assumption is that the short rate $r$ solves a stochastic differential equation under $Q$ of the form $dr_t = \alpha(r_t, t) \, dt + \beta(r_t, t) \, dw_t$.

- Then the value at time $t$ of a dollar received at time $T$ is:
  $$P(t, T) = E \left[ e^{-\int_t^T r(s) \, ds} \mid \mathcal{F}_t \right].$$

- $P(t, T)$ can be determined by solving the BVP consisting of the following PDE for $V(t, r)$:
  $$V_t + \alpha V_r + \frac{1}{2} \beta^2 V_{rr} - rV = 0,$$
  subject to the final-time condition $V(T, r) = 1$ for all $r$. The value of $P(t, T)$ is then $V(t, r(t))$. 
Modeling interest rates

- As in equities (Black-Scholes v Local Vol Model) - there is a tradeoff between simplicity and accuracy.

- Three basic viewpoints:
  
  (a) Simple short rate models.
  (b) Richer short-rate models.
  (c) One-factor Heath-Jarrow-Morton.
(a) Simple short rate models

- Example: Vasicek model, assumes that under r-n probability measure $Q$, the short rate solves:

$$dr_t = (\theta - ar_t) dt + \sigma dw_t$$ (1)

- $\theta$, $a$, and $\sigma$ constant and $a > 0$.
- Advantage of such a model: it leads to explicit formulas.
- For some short-rate models - including Vasicek - Black’s formula for a call on a bnd is validated, since $P(t, T)$ is lognormally distributed under the so-called forward measure.
- Disadvantage of such a model: has just a few parameters $\Rightarrow$ no hope of calibrating to the entire yield curve $P(0, T)$.
- As a result, Vasicek and similar short rate models are rarely used in practice.
(b) Richer short-rate models

– Example: Extended Vasicek model, a.k.a. the Hull-White model:

$$dr_t = [\theta(t) - ar_t] dt + \sigma dw_t.$$  \hspace{1cm} (2)

– $a$ and $\sigma$ are still constant but $\theta$ is now a function of $t$.

– Advantage: when $\theta$ satisfies

$$\theta(t) = \frac{\partial f}{\partial T}(0, t) + af(0, T) + \frac{\sigma^2}{2a}(1 - e^{-2at})$$ \hspace{1cm} (3)

the Hull-White model correctly reproduces the entire yield curve at time 0.

– Hull-White still leads to explicit formulas and is still consistent with Black’s formula.

– Can be approximated by a recombining trinomial tree (very convenient for numerical use).

– Disadvantage: gives little freedom in modeling evolution of the yield curve.
(c) One-factor Heath-Jarrow-Morton

- Theory that can be calibrated to time-0 yield curve
- Also permits many possible assumptions about the evolution of the yield curve.
- Specifies the evolution of the instantaneous forward rate $f(t, T)$:
  \[ df(t, T) = \alpha(t, T) \, dt + \sigma(t, T) \, dw_t. \]  
  (4)
- Initial data $f(0, T)$ obtained from market data.
- The volatility $\sigma(t, T)$ in (4) must be specified – it is what determines the model.
- Vasicek corresponds to the choice $\sigma(t, T) = \sigma e^{-a(T-t)}$.
- The drift $\alpha(t, T)$ determined by $\sigma$ and the requirements of no arbitrage.
- Disadvantage: no guidance on how to choose $\sigma(t, T)$.
- Difficult numerically: only a few special cases (mainly corresponding to familiar short-rate models such as Hull-White and Black-Derman-Toy) can be modelled using recombining trees.