Homework 3

April 20, 2005

Exercise 1 (Bruno Dupire)

At $t = t_0$, toss a coin. Instantaneous log-normal volatility will be 20% or 30% forever, according to Heads or Tails. The coin is not supposed to be fair.

$C$ is a Call of maturity $T$ and strike $K$ traded on the market. Its implied volatility is $\sigma_C$.

**Question 1.** Do we necessarily have $\sigma_C \in [20\%, 30\%]$?
If YES : show the arbitrage if $\sigma_C \notin [20\%, 30\%]$?
If NO : prove it

**Question 2.** Assume $\sigma_C \in [20\%, 30\%]$. Interpret it as implied probability of tails : $\Pi_C$.
   *Hint : Think of prices*

**Question 3.** $P$ is a Call of maturity $T$ and strike $L < K$ with implied volatility $\sigma_P$ and implied probability $\Pi_P$.
   Find an arbitrage if $\Pi_C \neq \Pi_P$.
   *Hint : Think of the values just after the toss*
Exercise 2 : Interest Rate Models (Peter Carr)

**Question 1.** We make all of the assumptions in the overheads called *Interest rate Models: Introduction.*
Please answer true or false (but not both) to the following statements. No explanation is required.

1. If money can be costlessly stored and there is no arbitrage, then zero coupon bond price must be (weakly) between zero and one.
2. The Vasicek model has bond prices between zero and one.
3. In all one factor spot rate models, prices of bonds of different maturities have increments which are perfectly correlated locally.
4. In the Vasicek model, the bond price depends negatively on the interest rate volatility.
5. In all one factor spot rate models, an investor can perfectly replicate the payoff on a 30 year caplet by dynamic trading in a money market account and a one year bond.

**Question 2.** Suppose that the spot interest rate is a mean reverting diffusion:

\[ dr_t = \kappa(\theta - r_t)dt + \sigma(r_t, t)dW_t \]

where \( \kappa > 0, \theta > 0, \) and \( W \) is a standard Brownian motion.

1. What is the initial (time 0) expected value of the time \( t > 0 \) spot rate \( r_t \) (for some fixed \( t \))?  
   Hint: take expected value and solve an ODE.
2. What is the limit of this expected value as \( t \uparrow \infty \)?
3. Do your answers change if the spot rate volatility at \( t \) depends on the \( r \) path up to \( t \)?
4. Do your answers change if we add a jump martingale to the \( r \) dynamics?
Question 3. Interest Rate Models: BGM  We make all of the assumptions in the overheads called Interest rate Models: BGM.

Please answer true or false (but not both) to the following statements. No explanation is required.

(a) BGM models may be inconsistent with an initial yield curve.
(b) In the BGM model, bond prices are between zero and one.
(c) In the one factor BGM model presented in class, prices of bonds of different maturities have increments which are perfectly correlated locally.
(d) In the BGM model (with $\delta t > 0$), the spot rate $r$ process is determined.
(e) In the one factor BGM model presented in class, an investor can perfectly replicate the payoff on a 30 year caplet by dynamic trading in a money market account and a one year bond.