Part 1: The Binomial Model

1. A publicly traded insurer deals exclusively in the Manhattan earthquake market. In the event of a major earthquake hitting New York City, the insurance giant will have to sell all of its assets to cover the claims from the policies it sold, all of which expire tomorrow. The company has no other liability, no employees, and no expenses. Its assets consist entirely of U.S. Government bonds, all of which are due to mature tomorrow, giving the company $100 per share. This cashflow will be passed on to the shareholders the next day - unless there is an earthquake - with no tax liabilities (due to a special exemption). The company’s share price is $50, with no bid-ask spread, perfect liquidity, and infinite availability for shorting. There is a similarly perfect market for call options on the stock, struck at $90, trading at $4 an option. You are a tax-exempt trader with a great appetite for risk, who can borrow at the risk free interest rate of 0%, with no money of your own. The color-coded federal warning system indicates the threat of an earthquake striking New York City in the next 10 days is below 10%, and you believe it is actually 1%.

   (a) How many shares of the company should you buy?
   (b) Formulate your strategy, if no one would lend you any money.

2. (a) The price of the stock of every company in the S&P500 follows a binomial model, with the CEO of each company flipping a coin to determine whether her stock will go up or down by $1 (miraculously, markets for these stocks are still perfectly liquid). Notice that the binomial process is additive, not multiplicative, i.e. negative prices are possible. Can you price a European call option on an individual stock? Can you price a call on the entire index?
(b) In order to be more efficient, the CEO’s decide to have the Chairman of the NYSE flip the coin for them. Companies starting with the letters A-M go up by $1 if heads, and down by $1 if tails. Companies starting with N-Z do the reverse. Will the option prices change? Is there arbitrage in the market?

(c) If instead of changing by $1, prices changed by 1%, how would your answer to part (b) change?

3. The multi-period binomial pricing formula (19) developed in the second lecture gives, in the continuous limit, the Black-Scholes pricing formula. What basic theorem of probability do we need to use here? Can you explain how we avoided Itô’s formula?

4. What would change in the multi-period analysis if one uses the stock instead of the futures contract?

Part 2: Poisson Jumps of Known Size

In the following, assume that the futures price process under real world/statistical probability measure $\mathbb{P}$ is given by the solution of the following SDE:

$$
\frac{dF_t}{F_t} = \mu dt + (e^j - 1) dN_t,
$$

where $F_0$ is a known positive constant. As in the notes, $N$ is a standard Poisson process $j \in \mathbb{R}$ is the constant jump size in the log futures price, and $\mu \in \mathbb{R}$ has the opposite sign of $j$.

5. Can the value $V(t, F)$ of a European call option expiring at time $T$ be linear in $F$ at a given time $t < T$?

6. Obtain a PDE for the value of an option: write the Taylor expansion for $V(Fe^j, t)$ in the PDDE, and disregard all derivatives of $V$ of order higher than 2. What is the boundary value in the case of the call option? Can you write an explicit solution?
7. Differentiate the PDDE w.r.t. \( F \), to get a PDDE for \( \frac{\partial V}{\partial F} \). Carry out an analysis similar to the previous question. What does this say about the hedge ratio?

8. Is it justified to neglect the higher order derivatives in the Taylor expansion?

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Part 3: Geometric Brownian Motion

In the following, assume that the Stock price process under the real world/statistical probability measure \( P \) follows:

\[
\frac{dS}{S} = \mu dt + \sigma dW_t,
\]

Assume 0 interest rates and dividends.

9. Write the risk-neutral dynamics of \( S \).

10. Write the risk-neutral dynamics of \( \ln S \).

11. What is the behavior of \( \ln S \) when \( t \) goes to infinity?

12. Deduce behavior of \( S \) when \( t \) goes to infinity.

13. What are the implications in terms of the price of an ATM call when the maturity goes to infinity?

14. Make the maturity go to infinity in the BS formula.

15. Is there a conflict between the previous two questions? Solve the paradox.