Inequalities  
11/11/2009

1. For $a, b, c \geq 0$, prove 
\[(a + b)(b + c)(c + a) \geq 8abc.\]

2. For all $x$, prove 
\[\frac{x^2 + 2}{\sqrt{x^2 + 1}} \geq 2.\]

3. Show that 
\[1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots + \frac{1}{\sqrt{n}} > 2\sqrt{n + 1} - 2.\]

4. If $a_i > 0$ for $i = 1, 2, \ldots, n$ and $a_1a_2\ldots a_n = 1$, then 
\[(1 + a_1)(1 + a_2)\ldots(1 + a_n) \geq 2^n.\]

5. For $a, b, c > 0$, prove 
\[\frac{a}{b + c} + \frac{b}{a + c} + \frac{c}{a + b} \geq \frac{3}{2}.\]

6. Let $a$ be a real number and $n$ a positive integer, with $a > 1$. Show that 
\[a^n - 1 \geq n \left(a^{\frac{n+1}{2}} - a^{\frac{n-1}{2}}\right).\]

7. [1982-B6] Let $K(x, y, z)$ denote the area of a triangle whose sides have lengths $x, y,$ and $z$. For any two triangles with sides $a, b, c$ and $a', b', c'$, respectively, prove that 
\[\sqrt{K(a, b, c)} + \sqrt{K(a, b, c)} \leq \sqrt{K(a + a', b + b', c + c')}\]
and determine the case of equality.

8. [1980-B1] For which real numbers $c$ is $(e^x + e^{-x})/2 \leq e^{cx^2}$ for all real $x$?

Problems from last year:

9. [2004-A2] For $i = 1, 2$ let $T_i$ be a triangle with side lengths $a_i, b_i, c_i,$ and area $A_i$. Suppose that $a_1 \leq a_2, b_1 \leq b_2, c_1 \leq c_2,$ and that $T_2$ is an acute triangle. Does it follow that $A_1 \leq A_2$?

10. [2004-B2] Let $m$ and $n$ be positive integers. Show that 
\[\frac{(m + n)!}{(m + n)^{m+n}} < \frac{m!}{m^m} \frac{n!}{n^n}.\]

11. [2003-A2] Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be nonnegative real numbers. Show that 
\[(a_1a_2\cdots a_n)^{1/n} + (b_1b_2\cdots b_n)^{1/n} \leq [(a_1 + b_1)(a_2 + b_2)\cdots (a_n + b_n)]^{1/n}.\]
12. [2002-B3] Show that, for all integers \( n > 1 \),
\[
\frac{1}{2ne} < \frac{1}{e} - \left( 1 - \frac{1}{n} \right)^n < \frac{1}{ne}.
\]

13. [1996-B2] Show that for every positive integer \( n \),
\[
\left( \frac{2n - 1}{e} \right)^{\frac{2n+1}{2}} < 1 \cdot 3 \cdot 5 \cdots (2n - 1) < \left( \frac{2n + 1}{e} \right)^{\frac{2n+1}{2}}.
\]

14. [1996-B3] Given that \( \{x_1, x_2, \ldots, x_n\} = \{1, 2, \ldots, n\} \), find, with proof, the largest possible value, as a function of \( n \) (with \( n \geq 2 \)), of
\[
x_1x_2 + x_2x_3 + \cdots + x_{n-1}x_n + x_nx_1.
\]

15. [1988-B2] Prove or disprove: If \( x \) and \( y \) are real numbers with \( y \geq 0 \) and \( y(y + 1) \leq (x + 1)^2 \), then \( y(y - 1) \leq x^2 \).

16. [1978-A5] Let \( 0 < x_i < \pi \) for \( i = 1, 2, \ldots, n \) and set
\[
x = \frac{x_1 + x_2 + \cdots + x_n}{n}.
\]
Prove that
\[
\prod_{i=1}^{n} \frac{\sin x_i}{x_i} \leq \left( \frac{\sin x}{x} \right)^n.
\]