0. Compute
\[ \sum_{n=0}^{\infty} \arctan \left( \frac{1}{1 + n + n^2} \right). \]

1. [1982-A2] For positive real \( x \), let
\[ B_n(x) = 1^x + 2^x + 3^x + \ldots + n^x. \]
Prove or disprove the convergence of
\[ \sum_{n=2}^{\infty} \frac{B_n(\log_2 n)}{(n \log_2 n)^2}. \]

2. [1985-A3] Let \( d \) be a real number. For each integer \( m \geq 0 \), define a sequence \( \{a_m(j)\} \), \( j = 0, 1, 2, \ldots \) by the condition
\[ a_m(0) = d/2^m, \]
\[ a_m(j + 1) = (a_m(j))^2 + 2a_m(j), \quad j \geq 0. \]
Evaluate \( \lim_{n \to \infty} a_n(n) \).

3. [1988-A3] Determine, with proof, the set of real numbers \( x \) for which
\[ \sum_{n=1}^{\infty} \left( \frac{1}{n} \csc \frac{1}{n} - 1 \right)^x \]
converges.

4. [1989-B3] Let \( f \) be a function on \( [0, \infty) \), differentiable and satisfying
\[ f'(x) = -3f(x) + 6f(2x) \]
for \( x > 0 \). Assume that \( |f(x)| \leq e^{-\sqrt{x}} \) for \( x \geq 0 \) (so that \( f(x) \) tends rapidly to 0 as \( x \) increases). For \( n \) a non-negative integer, define
\[ \mu_n = \int_0^\infty x^n f(x) \, dx \]
(sometimes called the \( n \)th moment of \( f \)).

a) Express \( \mu_n \) in terms of \( \mu_0 \).

b) Prove that the sequence \( \{\mu_n \frac{\mu_n}{n!}\} \) always converges, and that the limit is 0 only if \( \mu_0 = 0 \).

5. [1987-B4] Let \( (x_1, y_1) = (0.8, 0.6) \) and let \( x_{n+1} = x_n \cos y_n - y_n \sin y_n \) and \( y_{n+1} = x_n \sin y_n + y_n \cos y_n \) for \( n = 1, 2, 3, \ldots \). For each of \( \lim_{n \to \infty} x_n \) and \( \lim_{n \to \infty} y_n \), prove that the limit exists and find it or prove that the limit does not exist.

6. [1988-B4] Prove that if \( \sum_{n=1}^{\infty} a_n \) is a convergent series of positive real numbers, then so is \( \sum_{n=1}^{\infty} (a_n)^{(n+1)}/n^3 \).

7. [1987-A6] For each positive integer \( n \), let \( a(n) \) be the number of zeroes in the base 3 representation of \( n \). For which positive real numbers \( x \) does the series
\[ \sum_{n=1}^{\infty} \frac{x^{a(n)}}{n^3} \]
converge?

8. [2006-B6] Let \( k \) be an integer greater than 1. Suppose \( a_0 > 0 \), and define
\[ a_{n+1} = a_n + \frac{1}{\sqrt[n]{a_n}} \]
for \( n > 0 \). Evaluate
\[ \lim_{n \to \infty} \frac{a_n^{k+1}}{n^k}. \]
Problems with recurrent sequences:

9. A sequence $a_n$ is defined by $a_0 = 0$, $a_{n+1} = \sqrt{6 + a_n}$. Show that $a_n$ is
   (a) monotonically increasing
   (b) bounded above by 3.
   (c) Find its limit.
   (d) Find the convergence rate versus its limit.

10. The sequence $x_n$ is defined by $x_0 = 0$, $x_{n+1} = \sqrt{4 + 3x_n}$. Show that it is convergent and find its limit.
    What is the convergence rate near limiting point?

Some problems from last year:

11. (a) Sum the following:
    \[ \frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots + \frac{n-1}{n!}. \]
    (b) Evaluate $\prod_{n=2}^\infty (1 - 1/n^2)$.

12. Let $p$ and $q$ be real numbers with $1/p - 1/q = 1$, $0 < p \leq \frac{1}{2}$. Show that
    \[ p + \frac{1}{2}p^2 + \frac{1}{3}p^3 + \cdots = q - \frac{1}{2}q^2 + \frac{1}{3}q^3 - \cdots. \]

13. [2001-B3] For any positive integer $n$, let $\langle n \rangle$ denote the closest integer to $\sqrt{n}$. Evaluate
    \[ \sum_{n=1}^\infty \frac{2^{\langle n \rangle} + 2^{-\langle n \rangle}}{2^n}. \]

14. [2000-A1] Let $A$ be a positive real number. What are the possible values of $\sum_{j=0}^\infty x_j^2$, given that $x_0, x_1, \ldots$ are positive numbers for which $\sum_{j=0}^\infty x_j = A$?

15. [1999-A3] Consider the power series expansion
    \[ \frac{1}{1 - 2x - x^2} = \sum_{n=0}^\infty a_n x^n. \]
    Prove that, for each integer $n \geq 0$, there is an integer $m$ such that
    \[ a_n^2 + a_{n+1}^2 = a_m. \]

16. [1992-B3] For any pair $(x, y)$ of real numbers, a sequence $(a_n(x, y))_{n \geq 0}$ is defined as follows:
    \[ a_0(x, y) = x, \quad a_{n+1}(x, y) = \frac{(a_n(x, y))^2 + y^2}{2}, \quad \text{for } n \geq 0. \]
    Find the area of the region
    \[ \{(x, y) | (a_n(x, y))_{n \geq 0} \text{ converges}\}. \]

    \[ \sqrt{2207 - \frac{1}{2207 - \frac{1}{2207 - \cdots}}}. \]
    Express your answer in the form $\frac{a + b\sqrt{c}}{d}$, where $a, b, c, d$ are integers.

18. [2004-B5] Evaluate
    \[ \lim_{x \to 1} \prod_{n=0}^\infty \left( \frac{1 + x^{n+1}}{1 + x^n} \right)^x. \]