Number Theory
10/21/2009

Easy problems, appeared in previous years’ problem sets:

1. Find the last two digits of $7^{7^{7^{7^{7^{7^{7}}}}}}$ where the tower contains seven 7’s.

2. What are the last 2 digits of $3^{1234}$?

3. Show that $2x + 3y$ and $9x + 5y$ are divisible by 17 for the same set of $x$ and $y$.

4. Show that for any set of $n$ positive integers, there exists a subset whose sum is divisible by $n$.

5. [1955-B4] Do there exist 1,000,000 consecutive integers, each of which contains a repeated prime factor?

6. [1956-A2] Prove that every positive integer has a multiple whose decimal representation involves all ten digits.

7. [2005-A1] Show that every positive integer is a sum of one or more numbers of the form $2^r3^s$, where $r$ and $s$ are nonnegative integers and no summand divides another. (For example, $23 = 9 + 8 + 6$.)

New:

8. [2006-A2] Alice and Bob play a game in which they take turns removing stones from a heap that initially has $n$ stones. The number of stones removed at each turn must be one less than a prime number. The winner is the player who takes the last stone. Alice plays first. Prove that there are infinitely many $n$ such that Bob has a winning strategy. (For example, if $n = 17$, then Alice might take 6 leaving 11; then Bob might take 1 leaving 10; then Alice can take the remaining stones to win.)

9. [2001-A5] Prove that there are unique positive integers $a, n$ such that $a^{n+1} - (a+1)^n = 2001$.

Some more problems from last year:

10. [1996-A5] If $p$ is a prime number greater than 3 and $k = \lfloor 2p/3 \rfloor$, prove that the sum

$$\binom{p}{1} + \binom{p}{2} + \cdots + \binom{p}{k}$$

of binomial coefficients is divisible by $p^2$.

11. [2006-A3] Let $1, 2, 3, \ldots, 2005, 2006, 2007, 2009, 2012, 2016, \ldots$ be a sequence defined by $x_k = k$ for $k = 1, 2, \ldots, 2006$ and $x_{k+1} = x_k + x_{k-2005}$ for $k \geq 2006$. Show that the sequence has 2005 consecutive terms each divisible by 2006.

12. [1989-A1] How many primes among the positive integers, written in the usual base 10, are such that their digits are alternating 1’s and 0’s, beginning and ending with 1?

13. [1995-A3] The number $d_1d_2\ldots d_9$ has nine (not necessarily distinct) decimal digits. The number $e_1e_2\ldots e_9$ is such that each of the nine 9-digit numbers formed by replaying just one of the digits $d_i$ in $d_1d_2\ldots d_9$ by the corresponding digit $e_i$ ($1 \leq i \leq 9$) is divisible by 7. The number $f_1f_2\ldots f_9$ is related to $e_1e_2\ldots e_9$ in the same way: that is, each of the nine numbers formed by replaying one of the $e_i$ by the corresponding $f_i$ is divisible by 7. Show that, for each $i$, $d_i - f_i$ is divisible by 7. [For example, if $d_1d_2\ldots d_9 = 199501996$ then $e_6$ may be 2 or 9, since 199502996 and 199509996 are multiples of 7].

14. [2000-B2] Prove that the expression

$$\frac{\gcd(m, n)}{n} \binom{n}{m}$$

is an integer for all pairs of integers $n \geq m \geq 1$.

15. [2000-A2] Prove that there exist infinitely many integers $n$ such that $n, n+1, n+2$ are each the sum of two squares of integers. [Example: $0 = 0^2 + 0^2$, $1 = 0^2 + 1^2$, and $2 = 1^2 + 1^2$.]
Let $A_1 = 0$ and $A_2 = 1$. For $n > 2$, the number $A_n$ is defined by concatenating the decimal expansions of $A_{n-1}$ and $A_{n-2}$ from left to right. For example, $A_3 = A_2A_1 = 10$, $A_4 = A_3A_2 = 101$, $A_5 = A_4A_3 = 10110$, and so forth. Determine all $n$ such that 11 divides $A_n$. 