Polynomials
10/05/2009

1. Let \( x_1, x_2, x_3 \) be roots of \( x^3 + 3x^2 - 7x + 1 \). Find \( x_1^2 + x_2^2 + x_3^2 \).

2. For which \( a \in R \) is the sum of the squares of the zeroes of \( x^2 - (a - 2)x - a - 1 \) minimal?

3. Let \( P(x) \) be a polynomial of degree \( n \), so that \( P(k) = k/(k + 1) \) for \( k = 0, 1, \ldots, n \). Find \( P(n + 1) \).

4. Let \( a, b, c \) be three different integers, and let \( P \) be a polynomial with integer coefficients. Show that in this case the conditions \( P(a) = b, \ P(b) = c, \ P(c) = a \) cannot be satisfied simultaneously.

5. Solve the equation \( z^8 + 4z^6 - 10z^4 + 4z^2 + 1 = 0 \).

Recent Putnam problems:

1. [2007-B1] Let \( f \) be a polynomial with positive integer coefficients. Prove that if \( n \) is a positive integer, then \( f(n) \) divides \( f(f(n) + 1) \) if and only if \( n = 1 \). [Editor’s note: one must assume \( f \) is nonconstant.]

2. [2007-B4] Let \( n \) be a positive integer. Find the number of pairs \( P, Q \) of polynomials with real coefficients such that \( (P(X))^2 + (Q(X))^2 = X^{2n} + 1 \) and \( \deg P > \deg Q \).

3. [2007-B5] Let \( k \) be a positive integer. Prove that there exist polynomials \( P_0(n), P_1(n), \ldots, P_{k-1}(n) \) (which may depend on \( k \)) such that for any integer \( n \),

\[
\left\lfloor \frac{n}{k} \right\rfloor = P_0(n) + P_1(n) \left\lfloor \frac{n}{k} \right\rfloor + \cdots + P_{k-1}(n) \left\lfloor \frac{n}{k} \right\rfloor^{k-1}.
\]

(\( \lfloor a \rfloor \) means the largest integer \( \leq a \).)

4. [2005-A3] Let \( p(z) \) be a polynomial of degree \( n \), all of whose zeros have absolute value 1 in the complex plane. Put \( g(z) = p(z)/z^{n/2} \). Show that all zeros of \( g'(z) = 0 \) have absolute value 1.

5. [2005-B1] Find a nonzero polynomial \( P(x, y) \) such that \( P(\lfloor a \rfloor, \lfloor 2a \rfloor) = 0 \) for all real numbers \( a \). (Note: \( \lfloor \nu \rfloor \) is the greatest integer less than or equal to \( \nu \).)

6. [2005-B5] Let \( P(x_1, \ldots, x_n) \) denote a polynomial with real coefficients in the variables \( x_1, \ldots, x_n \), and suppose that

\[
\left( \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2} \right) P(x_1, \ldots, x_n) = 0 \quad (\text{identically})
\]

and that

\( x_1^2 + \cdots + x_n^2 \) divides \( P(x_1, \ldots, x_n) \).

Show that \( P = 0 \) identically.
Last year's problem set:

1.[2004-B1] Let \( P(x) = c_n x^n + c_{n-1} x^{n-1} + \cdots + c_0 \) be a polynomial with integer coefficients. Suppose that \( r \) is a rational number such that \( P(r) = 0 \). Show that the \( n \) numbers
\[
c_n r, \ c_n r^2 + c_{n-1} r, \ c_n r^3 + c_{n-1} r^2 + c_{n-2} r, \ldots, \ c_n r^n + c_{n-1} r^{n-1} + \cdots + c_1 r
\]
are integers.

2.[2004-A4] Show that for any positive integer \( n \), there is an integer \( N \) such that the product \( x_1 x_2 \cdots x_n \) can be expressed identically in the form
\[
x_1 x_2 \cdots x_n = \sum_{i=1}^{N} c_i (a_{i1} x_1 + a_{i2} x_2 + \cdots + a_{in} x_n)^n
\]
where the \( c_i \) are rational numbers and each \( a_{ij} \) is one of the numbers \(-1, 0, 1\).

3.[2003-B1] Do there exist polynomials \( a(x), b(x), c(y), d(y) \) such that
\[
1 + xy + x^2 y^2 = a(x)c(y) + b(x)d(y)
\]
holds identically?

4.[2003-B4] Let
\[
f(z) = az^4 + bz^3 + cz^2 + dz + e = a(z - r_1)(z - r_2)(z - r_3)(z - r_4)
\]
where \( a, b, c, d, e \) are integers, \( a \neq 0 \). Show that if \( r_1 + r_2 \) is a rational number and \( r_1 + r_2 \neq r_3 + r_4 \), then \( r_1 r_2 \) is a rational number.

5.[2002-A1] Let \( k \) be a fixed positive integer. The \( n \)-th derivative of \( \frac{1}{x^k} \) has the form \( \frac{P_n(x)}{(x^k-1)^{n+1}} \) where \( P_n(x) \) is a polynomial. Find \( P_n(1) \).

6.[2001-A3] For each integer \( m \), consider the polynomial
\[
P_m(x) = x^4 - (2m + 4)x^3 + (m - 2)^2.
\]
For what values of \( m \) is \( P_n(x) \) the product of two non-constant polynomials with integer coefficients?

7.[2001-B2] Find all pairs of real numbers \((x, y)\) satisfying the system of equations
\[
\frac{1}{x} + \frac{1}{2y} = (x^2 + 3y^2)(3x^2 + y^2) \\
\frac{1}{x} - \frac{1}{2y} = 2(y^4 - x^4).
\]

8.[1999-A2] Let \( p(x) \) be a polynomial that is nonnegative for all real \( x \). Prove that for some \( k \), there are polynomials \( f_1(x), \ldots, f_k(x) \) such that
\[
p(x) = \sum_{j=1}^{k} (f_j(x))^2.
\]

9.[1999-A3] Consider the power series expansion
\[
\frac{1}{1 - 2x - x^2} = \sum_{n=0}^{\infty} a_n x^n.
\]
Prove that, for each integer \( n \geq 0 \), there is an integer \( m \) such that
\[
a_n^2 + a_{n+1}^2 = a_m.
\]

10.[1999-B2] Let \( P(x) \) be a polynomial of degree \( n \) such that \( P(x) = Q(x)P''(x) \), where \( Q(x) \) is a quadratic polynomial and \( P''(x) \) is the second derivative of \( P(x) \). Show that if \( P(x) \) has at least two distinct roots then it must have \( n \) distinct roots.