1. \[ 1994 - A2 \] (169, 3, 2, 0, 0, 0, 0, 1, 3, 22, 6)

Let \( A \) be the area of the region in the first quadrant bounded by the line \( y = \frac{1}{2}x \), the \( x \)-axis, and the ellipse \( \frac{1}{2} x^2 + y^2 = 1 \). Find the positive number \( m \) such that \( A \) is equal to the area of the region in the first quadrant bounded by the line \( y = mx \), the \( y \)-axis, and the ellipse \( \frac{1}{2} x^2 + y^2 = 1 \).

**Answer.** To make the areas equal, \( m \) must be \( 2/9 \).

**Solution 1.** The linear transformation given by \( x_1 = x/3 \), \( y_1 = y \) transforms the region \( R \) bounded by \( y = x/2 \), the \( x \)-axis, and the ellipse \( x^2/9 + y^2 = 1 \) into the region \( R' \) bounded by \( y_1 = 3x_1/2 \), the \( x_1 \)-axis, and the circle \( x_1^2 + y_1^2 = 1 \); it also transforms the region \( S \) bounded by \( y = mx \), the \( y \)-axis, and \( x^2/9 + y^2 = 1 \) into the region \( S'' \) bounded by \( y_1 = 3mx_1 \), the \( y_1 \)-axis, and the circle. Since all areas are multiplied by the same (nonzero) factor under the linear transformation, \( R \) and \( S \) have the same area if and only if \( R' \) and \( S' \) have the same area. However, we can see by symmetry about the line \( y_1 = x_1 \) that this happens if and only if \( 3m = 2/3 \), that is, \( m = 2/9 \).

**Solution 2 (Noam Elkies).** Apply the linear transformation \( (x, y) \rightarrow (3y, x/3) \). This preserves area, and the ellipse \( x^2/9 + y^2 = 1 \). It switches the \( x \) and \( y \) axes, and takes \( y = x/2 \) to the desired line, \( x/3 = (3y/2) \), i.e., \( y = (2/9)x \). Thus \( m = 2/9 \).

**Remark.** There are, of course, less enlightened solutions. Setting up the integrals for the two areas yields the equation

\[
\int_0^{3/\sqrt{13}} \left( \sqrt{9-9y^2} - 2y \right) \, dy = \int_0^{3/\sqrt{1+9m^2}} \left( \sqrt{1-x^2/9} - mx \right) \, dx.
\]

At this point, one might guess that a substitution \( y = cx \) will transform one integral into the other, if \( c \) and \( m \) satisfy

\[
\frac{3}{\sqrt{13}} = c \cdot \frac{3}{\sqrt{1+9m^2}}, \quad 3c = 1, \quad 2c^2 = m,
\]

and in fact, \( c = 1/3 \) and \( m = 2/9 \) work. If this shortcut is overlooked, then as a last resort one could use trigonometric substitution to evaluate both sides: this yields

\[
\frac{3}{2} \arcsin \left( \frac{3}{\sqrt{13}} \right) = \frac{3}{2} \arcsin \left( \frac{1}{\sqrt{1+9m^2}} \right).
\]

Solving yields \( m = 2/9 \).
Evaluate
\[ \int_2^4 \frac{\sqrt{\ln(9-x)}}{\sqrt{\ln(9-x)} + \sqrt{\ln(x+3)}} \, dx. \]

**Answer.** The value of the integral is 1.

**Solution.** The integrand is continuous on [2, 4]. Let I be the value of the integral. As \( x \) goes from 2 to 4, 9 - \( x \) and \( x + 3 \) go from 7 to 5, and from 5 to 7, respectively. This symmetry suggests the substitution \( x = 6 - y \) reversing the interval [2, 4]. After interchanging the limits of integration, this yields
\[ I = \int_2^4 \frac{\sqrt{\ln(y+3)}}{\sqrt{\ln(y+3)} + \sqrt{\ln(9-y)}} \, dy. \]

Thus
\[ 2I = \int_2^4 \frac{\sqrt{\ln(x+3)} + \sqrt{\ln(9-x)}}{\sqrt{\ln(x+3)} + \sqrt{\ln(9-x)}} \, dx = \int_2^4 dx = 2, \]
and \( I = 1. \)

**Remark.** The same argument applies if \( \sqrt{\ln x} \) is replaced by any continuous function such that \( f(x+3) + f(9-x) \neq 0 \) for \( 2 \leq x \leq 4. \)

4. \([1989 - A2]\) (141, 6, 29, 0, 0, 0, 0, 0, 0, 5, 7, 4, 7)
Evaluate \( \int_0^a \int_0^b e^{\max(bx^2, a^2y^2)} \, dy \, dx, \) where \( a \) and \( b \) are positive.

**Answer.** The value of the integral is \( (e^{a^2b^2} - 1)/(ab). \)

**Solution.** Divide the rectangle into two parts by the diagonal line \( ay = bx \) to obtain
\[
\int_0^b \int_0^{\max(bx^2, a^2y^2)} e^{\max(bx^2, a^2y^2)} \, dy \, dx = \int_0^b \int_0^{bx/a} e^{bx^2} \, dy \, dx + \int_0^b \int_0^{ay/b} e^{a^2y^2} \, dx \, dy \\
= \int_0^b \frac{bx}{a} e^{bx^2} \, dx + \int_0^b \frac{ay}{b} e^{a^2y^2} \, dy \\
= \int_0^b e^{bx^2} \frac{1}{2ab} e^{bx^2} \, dx + \int_0^b e^{a^2y^2} \frac{1}{2ab} e^{a^2y^2} \, dy \\
= e^{a^2b^2} - 1/ab. \]

5. \([1990 - B1]\) (114, 2, 52, 0, 0, 0, 0, 0, 0, 11, 5, 3, 10, 4)
Find all real-valued continuously differentiable functions \( f \) on the real line such that for all \( x \)
\[ (f(x))^2 = \int_0^x ((f(t))^2 + (f'(t))^2) \, dt + 1990. \]

**Answer.** There are two such functions, namely \( f(x) = \sqrt{1990}e^x \), and \( f(x) = -\sqrt{1990}e^x. \)

**Solution.** For a given \( f \), the functions on the left- and right-hand sides are equal if and only if their values at 0 are equal, i.e., \( f(0)^2 = 1990 \), and their derivatives are equal for all \( x \), i.e.,
\[ 2f(x)f'(x) = (f(x))^2 + (f'(x))^2 \]
for all \( x. \)

The latter condition is equivalent to each of the following: \( (f(x) - f'(x))^2 = 0, \]
\[ f'(x) = f(x), f(x) = Ce^x \]
for some constant \( C. \) Combining this condition with \( f(0)^2 = 1990 \) yields \( C = \pm \sqrt{1990} \), so the desired functions are \( f(x) = \pm \sqrt{1990}e^x. \)