Solutions to "Number Theory continued" problems

7.[2005-A1] We proceed by induction, with base case $1 = 2^03^0$. Suppose all integers less than $n-1$ can be represented. If $n$ is even, then we can take a representation of $n/2$ and multiply each term by 2 to obtain a representation of $n$. If $n$ is odd, put $m = \lfloor \log_3 n \rfloor$, so that $3^m \leq n < 3^{m+1}$. If $3^m = n$, we are done. Otherwise, choose a representation $(n - 3^m)/2 = s_1 + \cdots + s_k$ in the desired form. Then

$$n = 3^m + 2s_1 + \cdots + 2s_k,$$

and clearly none of the $2s_i$ divide each other or $3^m$. Moreover, since $2s_i \leq n - 3^m < 3^{m+1} - 3^m$, we have $s_i < 3^m$, so $3^m$ cannot divide $2s_i$ either. Thus $n$ has a representation of the desired form in all cases, completing the induction.

Remarks: This problem is originally due to Paul Erdős. Note that the representations need not be unique: for instance,

$$11 = 2 + 9 = 3 + 8.$$ 

8.[2005-B2] By the arithmetic-harmonic mean inequality or the Cauchy-Schwarz inequality,

$$(k_1 + \cdots + k_n) \left( \frac{1}{k_1} + \cdots + \frac{1}{k_n} \right) \geq n^2.$$ 

We must thus have $5n - 4 \geq n^2$, so $n \leq 4$. Without loss of generality, we may suppose that $k_1 \leq \cdots \leq k_n$.

- If $n = 1$, we must have $k_1 = 1$, which works. Note that hereafter we cannot have $k_1 = 1$.
- If $n = 2$, we have $(k_1, k_2) \in \{(2, 4), (3, 3)\}$, neither of which work.
- If $n = 3$, we have $k_1 + k_2 + k_3 = 11$, so $2 \leq k_1 \leq 3$. Hence

$$(k_1, k_2, k_3) \in \{(2, 2, 7), (2, 3, 6), (2, 4, 5), (3, 3, 5), (3, 4, 4)\},$$

and only $(2, 3, 6)$ works.

- If $n = 4$, we must have equality in the AM-HM inequality, which only happens when $k_1 = k_2 = k_3 = k_4 = 4$.

Hence the solutions are $n = 1$ and $k_1 = 1$, $n = 3$ and $(k_1, k_2, k_3)$ is a permutation of $(2, 3, 6)$, and $n = 4$ and $(k_1, k_2, k_3, k_4) = (4, 4, 4, 4)$.

Remark: In the cases $n = 2, 3$, Greg Kuperberg suggests the alternate approach of enumerating the solutions of $1/k_1 + \cdots + 1/k_n = 1$ with $k_1 \leq \cdots \leq k_n$. This is easily done by proceeding in lexicographic order: one obtains $(2, 2)$ for $n = 2$, and $(2, 3, 6), (2, 4, 4), (3, 3, 3)$ for $n = 3$, and only $(2, 3, 6)$ contributes to the final answer.