1. Find a formula for the sum of the first \(n\) odd numbers.

2. Let \(f(n)\) be the number of regions which are formed by \(n\) lines in the plane, where no two lines are parallel and no three meet in a point (e. g. \(f(4) = 11\)). Find a formula for \(f(n)\).

3. (a) Prove that
\[
1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots + \frac{1}{\sqrt{n}} < 2\sqrt{n}.
\]
(b) Prove that
\[
2!4!\ldots(2n)! \geq ((n+1)!)^n.
\]

4. Define \(f\) on the positive integers by \(f(1) = 1, f(2) = f(n)\) and \(f(2n + 1) = f(n) + 1\). Prove that \(f(n)\) is the number of 1’s in the binary representation of \(n\).

5. Let \(F_i\) denote the \(i\)-th term in the Fibonacci sequence (i. e. \(F_1 = F_2 = 1, F_{n+2} = F_n + F_{n+1}\)). Prove that
\[
F_{2n+1} + F_{2n} = F_{2n+2}.
\]

6. Let \(n > 1\) and let \(P(x)\) be a polynomial with integer coefficients and degree at most \(n\). Suppose that \(|P(x)| < n\) for all \(|x| < n^2\). Show that \(P\) is constant.

7. [1978-A1] Let \(A\) be any set of 20 distinct integers chosen from the arithmetic progression \(1, 4, 7, \ldots, 100\). Prove that there must be two distinct integers in \(A\) whose sum is 104.

8. Given any \(n + 2\) integers, show that there exist two of them whose sum, or else whose difference, is divisible by \(2n\).

9. Given any \(n + 1\) integers between 1 and \(2^n\), show that one of them is divisible by another. Is this best possible, i. e. , is the conclusion still true for \(n\) integers between 1 and \(2n\)?

10. Let each of nine lines cut a square into two quadrilaterals whose areas are in the proportion 2 : 3. Prove that at least three of the lines pass through the same point.

11. [1971-A1] Let there be given nine lattice points (points with integral coordinates) in three dimensional Euclidean space. Show that there is a lattice point on the interior of one of the line segments joining two of these points.) [To test your understanding, how many lattice points does one need in four dimensions to reach the same conclusion?]

12. [2000-B1] Let \(a_j, b_j, c_j\) be integers for \(1 \leq j \leq N\). Assume for each \(j\), at least one of \(a_j, b_j, c_j\) is odd. Show that there exist integers \(r, s, t\) such that \(ra_j + sb_j + tc_j\) is odd for at least \(4N/7\) values of \(j\), \(1 \leq j \leq N\).

13. [1995-B1] For a partition \(\pi\) of \(\{1, 2, 3, 4, 5, 6, 7, 8, 9\}\), let \(\pi(x)\) be the number of elements in the part containing \(x\). Prove that for any two partitions \(\pi\) and \(\pi'\), there are two distinct numbers \(x\) and \(y\) in \(\{1, 2, 3, 4, 5, 6, 7, 8, 9\}\) such that \(\pi(x) = \pi(y)\) and \(\pi'(x) = \pi'(y)\). [A partition of a set \(S\) is a collection of disjoint subsets (parts) whose union is \(S\).]

14. [1980-B4] Let \(A_1, A_2, \ldots, A_{1066}\) be subsets of a finite set \(X\) such that \(|A_i| > \frac{1}{2}|X|\) for \(1 \leq i \leq 1066\). Prove that there exist ten elements \(x_1, \ldots, x_{10}\) of \(X\) such that every \(A_i\) contains at least one of \(x_1, \ldots, x_{10}\). (Here \(|S|\) means the number of elements in the set \(S\).)