Problem 1. Find all solutions to the following congruences
(a) $57x \equiv 1 \mod 71$
(b) $57x \equiv 1 \mod 81$
(c) $4183x \equiv 5781 \mod 15087$
(d) $91x \equiv 419 \mod 440$

Problem 2. Find all solutions to the following simultaneous congruences
(a) $x \equiv 1 \mod 3$
(b) $x \equiv -4 \mod 35$
(c) $x \equiv 9 \mod 19$
(d) $x \equiv 6 \mod 21$
(e) $x \equiv -4 \mod 35$
(f) $x \equiv 1 \mod 7$
(g) $x \equiv 33 \mod 16$
(h) $x \equiv 17 \mod 23$
(i) $3x^2 \equiv 3 \mod 5$

Problem 3. Find the number of solutions to the following (no need to find the solutions, but prove your answer)
(a) $x^2 \equiv 1 \mod 210$
(b) $x^2 \equiv -1 \mod 55$
(c) $x^2 \equiv -1 \mod 65$
(d) $x^3 + x + 1 \equiv 0 \mod 15$
(e) $x^2 + 4 \equiv 0 \mod 20$

Problem 4. Find all solutions to the following
(a) $x^2 - 2 \equiv 0 \mod 7^e$ for $e = 1, 2, 3$
(b) $x^2 + x - 1 \equiv 0 \mod 11^e$ for $e = 1, 2, 3$
(c) $x^3 + 2x + 2 \equiv 0 \mod 5^e$ for $e = 1, 2, 3$

Problem 5. Given $n \in \mathbb{N}$, we have seen that writing $n = \gamma \overline{\gamma}$ for some $\gamma \in \mathbb{Z}[i]$ yields a solution of $n = x^2 + y^2$; we consider $n = x^2 + y^2 = x'^2 + y'^2$ to be the same sum of squares decomposition if $(x, y) = (\pm x', \pm y')$ or $(x, y) = (\pm y', \pm x')$. We’ll also allow one of the squares to be 0.
(a) Prove that $n = \gamma \overline{\gamma} = \gamma' \overline{\gamma'}$ give the same sum of squares decomposition if and only if $\gamma \sim \gamma'$ or $\overline{\gamma} \sim \gamma'$.
(b) For the following values of $n$, find all distinct sums of squares decompositions.
   (i) $n = 125$
   (ii) $n = 650$
   (iii) $n = 75$
   (iv) $n = 234$
(c) Classify the numbers that appear as the hypotenuse of a primitive Pythagorean triple.
(d) Show that every odd $n \in \mathbb{N}$ is a leg of a primitive Pythagorean triple. ($\text{Hint:}$ Show that every odd number $n$ can be written as a difference of two squares $n = p^2 - q^2$ with $p, q$ coprime and of opposite parity.) What even $n \in \mathbb{N}$ are legs of primitive Pythagorean triples?