Final
Math 248

December 19, 2012
1. Recall that Wilson’s theorem states, for $p$ a prime, that $(p - 1)! \equiv -1 \mod p$. Prove Wilson’s theorem using primitive roots.
2. (a) Show that $x = 10$ is a solution to $x^8 \equiv 1 \mod 73$ and $x = 2$ is a solution to $x^9 \equiv 1 \mod 73$. (*Hint*: $10^5 \equiv -10 \mod 73$)

(b) What are the orders of $x = 10$ and $x = 2 \mod 73$?

(c) Prove that 20 is a primitive root mod 73.
3. Suppose $p$ is an odd prime with $p \equiv 1 \mod 3$.

   (a) Show using quadratic reciprocity that \( \left( \frac{-3}{p} \right) = 1 \).

   (b) Show that an element $c$ of order 3 mod $p$ exists.

   (c) Show that if $c$ has order 3, then $(2c + 1)^2 \equiv -3 \mod p$. 
4. Suppose \( q(x, y) = ax^2 + bxy + cy^2 \) is a quadratic form of discriminant \( d(q) = b^2 - 4ac \).

(a) Show that \( q(x, y) \) factors as \( q(x, y) = (\alpha_1 x + \beta_1 y)(\alpha_2 x + \beta_2 y) \) if and only if \( d(q) \) is a perfect square (possibly zero).

(b) Show that \( q(x, y) \) factors as \( q(x, y) = d(\alpha x + \beta y)^2 \) if and only if its discriminant is 0.

(c) Suppose \( d(q) = 0 \). Show that \( q \) represents all integers of the form \( dn^2 \), where \( d = \gcd(a, b, c) \).
5. (a) Find the four properly reduced forms of discriminant $-44$.

(b) For the following values of $n$, find all proper representations (possibly none) of $n$ by the forms in part (a).

(i) $n = 5$.

(ii) $n = 13$.

(iii) $n = 53$. 