Problem 1. We know from the Proposition on page 5 of Hatcher that each choice of \(1 \leq p\) gives us a triple of the form
\[
(2pq, p^2 - q^2, p^2 + q^2).
\]
It is also easy to verify that each triple comes from one and only one \((p, q)\). See Hatcher or try it yourself, “suppose \((p_1, q_1)\) and \((p_2, q_2)\) give the same triple...”

They give a primitive triple if

1. \(p\) and \(q\) have no common factors
2. \(p\) and \(q\) have opposite parity (one is odd and the other is even)

It’s clear that \(p \leq 9\), otherwise \(c = p^2 + q^2 > 100\) since \(q\) is at least 1. So just begin listing cases

1. \(q = 1, \quad p = 2, 4, 6, 8\)
2. \(q = 2, \quad p = 3, 5, 7, 9\)
3. \(q = 3, \quad p = 4, 8\)
4. \(q = 4, \quad p = 5, 7\)
5. \(q = 5, \quad p = 6, 8\)
6. \(q = 6, \quad p = 7, 9\)

This gives all 16 of them.

To get the non primitives, we find which multiples of the primitive triples still have \(c \leq 100\). There are 36 more.

Finally, for part \(c\) we only need to see in how many ways we can express \(q^2 + p^2 = 65\) to find primitive triples. Enumerate cases to find \((q, p) = (1, 8)\) or \((4, 7)\) i.e. \((16, 63, 65), (56, 33, 65)\). For non primitives, we can allow \(q^2 + p^2 = 5\) and \(q^2 + p^2 = 13\), so there are 2 more, \((13, 4, 3, 5), (5, 12, 5, 13)\).

Problem 2. If we fix the ratio of the hypotenuse and any side, it fixes the cosine of the angle between that side and the hypotenuse. Clearly, fixing one ratio fixes all the angles in the triangle (why?). We get a pythagorean triplet for every \((p, q)\).

If we wanted one angle to be 60, we would require that
\[
\frac{2pq}{p^2 + q^2} = \frac{1}{2}.
\]
This gives us the quadratic equation
\[
\frac{4p}{q} = \frac{p^2}{q^2} + 1.
\]
We discard one root since \(p/q > 1\) and get that the solution is the irrational \(1 + \sqrt{3}\). So all we have to do is ensure that we choose \(p\) and \(q\) such that \(p/q\) is close to \(2 + \sqrt{3}\). This would ensure that the angle is close to 60. So if you have this calculation and choose \(p/q\) even somewhat close to \(1 + \sqrt{3}\), you get full credit.
Note that we know that we can use rationals to approximate irrationals arbitrarily well, so we can get the angle to be as close to 60 as we’d like.

**Problem 3.** Consider the equation
\[ a^2 + b^2 = c^2. \]
Try \( a = b + 2 \) and the calculation gets messy; discard it. Try \( c = a + 2 \) and get
\[ b^2 = 4(a + 1) \]
We can find an integer pair \((a, b)\) that works if 4 divides \( b^2 \); this is always true if \( b \) is even. So for \( b \) even and \( \geq 4 \), we get what the problem asks for.

**Problem 4.** I have a messy solution. You need to know a little about congruences. Here are a few facts: for \( a, b, c, d, N \in \mathbb{Z} \),
1. \( a = b \mod N \) and \( c = d \mod N \) \( \Rightarrow \) \( a + c = b + d \mod N \)
2. \( a = b \mod N \) and \( c = d \mod N \) \( \Rightarrow \) \( a \cdot c = b \cdot d \mod N \)
Consider the equation \( \mod 3 \):
\[ a^2 + b^2 = c^2 \mod 3 \]
and suppose none of \( a, b, c \) are divisible by 3 (this is a proof by contradiction). Any number that is 1 \( \mod 3 \) or 2 \( \mod 3 \) when squared gives a number that is 1 \( \mod 3 \). So
\[ c^2 = a^2 = b^2 = 1 \mod 3 \]
But then \( a^2 + b^2 = 2 \mod 3 \) and this is a contradiction. So our supposition must be false; i.e., at least one of \( a, b \) or \( c \) must be divisible by 3. Similar arguments apply to 4 and 5.