

## Practice problems, Calculus II, Spring 2006.

### Complex numbers

1. Roots and powers.
  - (a) Find the two square roots of  $z = 1 + i$ .
  - (b) Find the three cube roots of  $z = 4(-1 + \sqrt{3}i)$ .
  - (c) Let  $z = 1 - \sqrt{3}i$ . Compute  $z^6$ .
2. Describe geometrically the regions of the complex plane determined by:
  - (a)  $|5z - 2| < 1$ .
  - (b)  $|2z + 3i| > 1$ .
  - (c)  $\operatorname{Re}[(1 + i)z] < 0$ .

### Chapter 9

1. Find the arc length of the curve  $f(x) = \frac{x^3}{6} + \frac{1}{2x}$ ,  $x \in [1, 3]$ .
2. Express the curve by an equation in  $x$  and  $y$ :  $x(t) = 3 + \cos t$ ,  $y(t) = 3 - 2 \sin t$ .
3. Find the area of the region that is common to  $r = a \cos 3\theta$  and  $r = \frac{a}{2}$ . Take  $a > 0$ .
4. Identify the curve given by  $r = \frac{10}{2 + \cos \theta}$  and write the equation in rectangular coordinates.
5. Write the equation  $x + y^2 = x - y$  in polar coordinates.
6. Find the equation in  $x$  and  $y$  for the line tangent to the curve  $x(t) = 3t$ ,  $y(t) = t^2 - 1$  at  $t = 1$ .
7. The equations  $x(t) = t^2 + 2$ ,  $y(t) = t^3 - 3$  gives the position of a particle at time  $t$  from  $t = 0$  to  $t = 1$ . Find the initial speed of the particle, the terminal speed, and the distance travelled.
8. Calculate  $\frac{d^2y}{dx^2}$  at the point  $t = 1$  without eliminating the parameter if  $x(t) = e^t - 1$  and  $y(t) = 3 + e^{2t}$ .
9. Sketch and identify the polar curve  $r = 1 + \cos \theta$ .
10. Find the area of the region enclosed by  $r = 2 \cos 3\theta$ .

## Chapter 10

1. Find the limit if the sequence converges:

(a)  $\frac{2^n}{4^n+1}$

(b)  $\frac{4n}{\sqrt{n^2+1}}$

(c)  $\frac{(2n+1)^2}{(3n-1)^2}$

(d)  $(1 + \frac{1}{n})^{n/2}$

(e)  $2\ln(3n) - \ln(n^2 + 1)$

2. Find the limits as  $n \rightarrow \infty$ :

(a)  $(1 + \frac{x}{n})^{3n}$

(b)  $(1 + \frac{1}{n^2})^n$

(c)  $(1 + \frac{1}{n})^{n^2}$

3. (a)  $\lim_{x \rightarrow 0} \frac{1+x-e^x}{x(e^x-1)}$

(b)  $\lim_{x \rightarrow 0} [\frac{1}{\sin^2 x} - \frac{1}{x^2}]$

(c)  $\lim_{n \rightarrow \infty} (\frac{1}{n} \ln \frac{1}{n})$

4. Evaluate:

(a)  $\int_e^\infty \frac{\ln x}{x} dx$

(b)  $\int_{-3}^3 \frac{dx}{x(x+1)}$

5. Determine whether (and why) the following diverge or converge:

(a)  $\int_1^\infty \frac{x}{\sqrt{1+x^5}} dx$

(b)  $\int_0^\infty \frac{dx}{(1+x^5)^{1/6}}$

(c)  $\int_1^\infty \frac{\ln x}{x^2} dx$

## Chapter 11

1. Find the sum of the series:

(a)  $\sum_{k=0}^\infty \frac{(-1)^k}{5^k}$

(b)  $\sum_{k=0}^\infty \frac{1}{(k+1)(k+3)}$

(c)  $\sum_{k=0}^\infty (-1)^k x^{2k}$

(d)  $\sum_{k=0}^{\infty} \frac{1}{k!} x^{2k+1}$ .

2. Determine if the following series converges or diverges:

(a)  $\sum_{k=0}^{\infty} \frac{\ln(k)}{k^3}$

(b)  $\sum_{k=0}^{\infty} \frac{\ln(k)}{k^{5/4}}$

(c)  $\sum_{k=0}^{\infty} k^2 2^{-k^3}$

(d)  $\sum_{k=0}^{\infty} \frac{2^k k!}{k^k}$

(e)  $\sum_{k=0}^{\infty} \frac{k^2}{e^k}$ .

3. Check the conditional and absolute convergence of:

(a)  $\sum_{k=0}^{\infty} (-1)^k \frac{(k!)^2}{(2k)!}$

(b)  $\sum_{k=0}^{\infty} \frac{(-1)^k}{k \ln k}$ .

(c)  $\sum_{k=0}^{\infty} (-1)^k \frac{k^2}{2^k}$

4. Find the interval of convergence of:

(a)  $\sum_{k=0}^{\infty} \frac{2^k}{k^2} x^k$ .

(b)  $\sum_{k=0}^{\infty} \frac{1}{\ln k} x^k$ .

5. Expand  $f(x)$  in powers of  $x$ :

(a)  $f(x) = x \cos(x^2)$ ,

(b)  $f(x) = \frac{1-x}{1+x}$ .

(c)  $f(x) = \ln(2 + 3x)$ . Expand around  $x = 4$  and specify the values of  $x$  for which the expansion is valid.

(d)  $f(x) = \frac{1}{\sqrt{1+x^2}}$ . Explicitly write out the first five terms. (Hint: binomial series).

## Chapter 18 (18.1-18.3).

For section 18.4, see suggested HW problems.

1. Solve  $y' + \frac{2}{x}y = \frac{y^3}{x^2}$

2. An object with mass  $m$  is dropped from rest and we assume that the air resistance is proportional to the speed of the object. If  $s(t)$  is the distance dropped after  $t$  seconds, then the speed is  $v = s'(t)$  and the acceleration is  $a = v'(t)$ . If  $g$  is the

acceleration due to gravity, then the downward force on the object is  $mg - cv$ , where  $c$  is a positive constant, and Newton's Second Law gives

$$m \frac{dv}{dt} = mg - cv.$$

Solve for  $v(t)$ , what is the velocity as  $t \rightarrow \infty$ , find the distance the object has fallen after  $t$  seconds.

3. Solve the IVP:  $\frac{dv}{dt} - 2tv = 3t^2e^{t^2}$ ,  $v(0) = 5$ .
4. Solve  $\frac{dy}{dx} = y^2 + 1$ ,  $y(1) = 0$
5. The Pacific halibut fishery has been modeled by the differential equation

$$\frac{dy}{dt} = ky \left(1 - \frac{y}{K}\right)$$

where  $y(t)$  is the biomass (total mass of the members of the population) in kilograms at time  $t$  (measured in years), the carrying capacity is estimated to be  $K = 8 \times 10^7$ kg, and  $k = 0.71$  per year. If  $y(0) = 2 \times 10^7$ kg, find the biomass a year later. How long will it take for the biomass to reach  $4 \times 10^7$ kg?

6. Find the integral curve  $(xe^y - e^x)y' = ye^x - e^y$
7. Solve  $(x - 2)y' + y = 5(x - 2)^2y^{1/2}$
8. Find the integral curve that satisfies  $y' = \frac{y^3 - x^3}{xy^2}$ ,  $y(1) = 2$
9. Solve  $xy' + 2y = \frac{\cos x}{x}$
10. Solve  $yy' = 4x\sqrt{y^2 + 1}$
11. Find the integral curves  $xy' = x^2(e^y)^{1/x}/y + y$

## Solutions

### Complex numbers

1. (a)  $2^{1/4}(\cos(\pi/8) + i \sin(\pi/8))$ ,  $2^{1/4}(\cos(9\pi/8) + i \sin(9\pi/8))$ .  
 (b)  $2(\cos(2\pi/9) + i \sin(2\pi/9))$ ,  $2(\cos(8\pi/9) + i \sin(8\pi/9))$ ,  $2(\cos(14\pi/9) + i \sin(14\pi/9))$ .  
 (c) 64.
2. (a) Interior of circle w radius  $1/5$  centered in  $2/5$ .  
 (b) Exterior of circle w radius  $1/2$  centered in  $-3/2i$ .  
 (c) The half complex plane s.t  $Im(z) > Re(z)$ .

## Chapter 9

1.  $\frac{14}{3}$
2. ellipse:  $(x - 3)^2 + \frac{(y-3)^2}{4} = 1$ ;
3.  $a^2 \left( \frac{\pi}{6} - \frac{\sqrt{3}}{8} \right)$
4. ellipse:  $\frac{(x+10/3)^2}{400/9} + \frac{y^2}{400/12} = 1$ ;
5.  $r = \frac{\cos \theta - \sin \theta}{1 + 2 \sin \theta \cos \theta}$ .
6.  $2x - 3y = 6$ .
7. initial speed = 0, terminal speed =  $\sqrt{13}$ , distance =  $\frac{1}{27} (13\sqrt{13} - 8)$ .
8. 2.
9.  $1 + \cos \theta$ .

## Chapter 10

1. (a) 0  
(b) 4  
(c)  $4/9$   
(d)  $\sqrt{e}$   
(e)  $\ln 9$
2. (a)  $e^{3x}$   
(b) 1  
(c)  $\infty$
3. (a)  $-1/2$   
(b)  $1/3$   
(c) 0
4. (a) diverges  
(b) diverges
5. (a) converges  
(b) diverges  
(c) converges

## Chapter 11

- Geometric series with  $x = -1/5$ ; the sum is  $5/6$ .
  - Use partial fraction decomposition. Almost all terms cancel. The sum is  $3/4$ .
  - Geometric series for  $-x^2$ . The sum is  $1/(1+x^2)$ .
  - The series is  $x \sum_{k=0}^{\infty} \frac{1}{k!} x^{2k} = x \sum_{k=0}^{\infty} \frac{1}{k!} (x^2)^k$ . This is  $x$  times the Taylor series for  $e^{x^2}$ , i.e the sum is  $xe^{x^2}$ .
- Remember: there is often more than one way to obtain the answer.
  - Converges. For example basic comparison test with  $1/k^2$ .
  - $\sum_{k=0}^{\infty} \frac{\ln(k)}{k^{5/4}}$  Converges. Integral test.
  - Converges. Integral test.
  - $\sum_{k=0}^{\infty} \frac{2^k k!}{k^k}$  Converges. Ratio test gives  $2(k/(k+1))^k$ , which has the limit  $2e^{-1} < 1$  (use L'Hopital).
  - Converges. Ratio test.
- Converges absolutely and hence also conditionally (ratio test).
  - Converges conditionally but not absolutely (integral test).
  - Converges absolutely and hence also conditionally (ratio test).
- $-1/2 \leq x \leq 1/2$ .
  - $-1 \leq x < 1$ .
- Expand  $f(x)$  in powers of  $x$ :
  - $\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{4k+1}$  (Can use the known expansion for  $\cos(y)$  to obtain this).
  - This is  $(1-x)$  times the sum of the geometric series for  $-x$ .  $\sum_{k=0}^{\infty} (-1)^k (x^k - x^{k+1})$ .
  - $f(x) = \ln 14 + \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \left(\frac{3}{14}\right)^k (x-4)^k$ . Valid for  $-2/3 < x \leq 26/3$ .
  - $(1+x^2)^{-1/2} = \sum_{k=0}^{\infty} \binom{-1/2}{k} x^{2k}$ . First five terms:  $1 - \frac{1}{2}x^2 + \frac{3}{8}x^4 - \frac{5}{16}x^6 + \frac{35}{128}x^8$ .

## Chapter 18 (18.1-18.3)

- $y = \pm[Cx^4 + 2/(5x)]^{-1/2}$
- $v(t) = \frac{mg}{c}(1 - e^{-ct/m})$ ,  
 $v(t \rightarrow \infty) = mg/c$ ,  
 $s(t) = (mg/c)[t + (m/c)e^{-ct/m}] - m^2g/c^2$

3.  $v = t^3 e^{t^2} + 5e^{t^2}$
4.  $y = \tan(x - 1)$
5.  $3.23 \times 10^7 \text{kg}$ , it will take about 1.55 years.
6.  $x e^y - y e^x = C$
7.  $y = \left[ (x - 2)^2 + \frac{C}{\sqrt{x-2}} \right]^2$
8.  $y^3 + 3x^3 \ln|x| = 8x^3$
9.  $y = \frac{\sin x}{x^2} + \frac{C}{x^2}$
10.  $y = \pm \sqrt{(2x^2 + C)^2 - 1}$
11.  $y + x = x e^{y/x} [C - \ln x]$